PROCEEDINGS

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RECORDS OF PROCEEDINGS AT MEETINGS

SESSION NOVEMBER, 1905—JUNE, 1906.

Thursday, November 9th, 1905.

ANNUAL GENERAL MEETING.

Prof. A. R. FORSYTH, President, in the Chair.

Present seventeen members.

Messrs. P. J. Anderson and J. A. H. Johnston were elected members.

The Treasurer (Prof. J. Larmor) presented his Report. On the motion of Lieut.-Col. Cunningham, seconded by Mr. C. S. Jackson, the Report was received.

Dr. J. G. Leathem was appointed Auditor.

Mr. Grace, as Secretary, reported that the number of members at the beginning of the last Session was 269. During the Session the Society had lost 1 member by death and the name of 1 member had been removed from the List. Nine new members had been elected, bringing the numbers at the beginning of this Session to 276. The Accademia Gioenia di Scienze Naturali di Catania had been added to the list of the Societies with which publications are exchanged.

The President presented the De Morgan Medal to Dr. H. F. Baker.

The Council and Officers for the ensuing Session were elected as follows:—President, Prof. A. R. Forsyth; Vice-Presidents, Prof. W. Burnside and Sir W. Niven; Treasurer, Prof. J. Larmor; Secretaries, Prof. A. E. H. Love and Mr. J. H. Grace; other members of the Council, Dr. H. F. Baker, Mr. A. Berry, Mr. J. E. Campbell, Prof. E. B. Elliott, Dr. J. W. L. Glaisher, Mr. G. H. Hardy, Dr. E. W. Hobson, Major P. A. MacMahon, Mr. A. E. Western, Mr. A. Young.

The following papers were communicated:—

*Linear Content of a Plane set of Points: Dr. W. H. Young.

* Printed in *Proceedings*, Ser. 2, Vol. 3.

*On Absolutely Convergent Improper Double Integrals: Dr. E. W. Hobson.

*On the Arithmetic Continuum: Dr. E. W. Hobson.

*On the Arithmetical Nature of the Coefficients in a Group of Linear Substitutions of Finite Order (Second Paper): Prof. W. Burnside.

On the Asymptotic Value of a Type of Finite Series: Mr. J. W. Nicholson.

On an Extension of Dirichlet's Integral: Prof. T. J. I'A. Bromwich.

*The Continuum and the Second Number-Class: Mr. G. H. Hardy.

Thursday, December 14*th*, 1905.

Prof. A. R. FORSYTH, President, in the Chair.

Present sixteen members and seven visitors.

Mr. C. V. Durell and Prof. J. Harkness were elected members.

Mr. J. A. H. Johnston was admitted into the Society.

The President presented the Auditor's Report. On the motion of the President, seconded by Prof. Elliott, the Treasurer's Report, presented in November, was adopted, and the thanks of the Society were given to the Treasurer and the Auditor.

The following papers were communicated:—

*Some Difficulties in the Theory of Transfinite Numbers and Order Types: Hon. B. A. W. Russell.

*On Well-ordered Aggregates: Prof A. C. Dixon.

*On the Representation of certain Asymptotic Series as Convergent Continued Fractions: Prof. L. J. Rogers.

*The Hessian Configuration and its connection with the Group of 360 Plane Collineations: Prof. W. Burnside.

The Imaginary in Geometry: Mr. J. L. S. Hatton.

On a new Cubic connected with the Triangle: Mr. H. L. Trachtenberg.

*The Theory of Integral Equations: Mr. H. Bateman.

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* Printed in this volume.

Thursday, January 11*th*, 1906.

Prof. A. R. FORSYTH, President, in the Chair.

Present ten members.

Miss Hilda Phoebe Hudson, Mr. W. F. S. Churchill, and the Hon. B. A. W. Russell were elected members.

The President referred to the loss sustained by the Society by the death of Prof. C. J. Joly, and gave an account of his scientific work.

The following papers were communicated:—

*On the Monogeneity of a Function defined by an Algebraic Equation: Dr. H. F. Baker.

On the Diffraction of Sound by Large Cylinders: Mr. J. W. Nicholson.

*On the Expression of the so-called Biquaternions and Triquaternions by means of Quaternary Matrices: Mr. J. Brill.

Dr. Hobson made an informal communication:

"On the Representation of Functions of Real Variables."

Thursday, February 8*th*, 1906.

SPECIAL GENERAL MEETING.

Sir W. D. NIVEN, Vice-President, in the Chair.

Present seventeen members.

The Chairman stated that the meeting was the Special General Meeting called by the Council acting in accordance with the Articles of Association.

The Articles of Association relating to Special General Meetings were read.

Dr. Hobson moved, and Prof. Elliott seconded, the resolution of which notice had been given, viz.:

That By-law II. be amended as follows:—

Clause I. of By-law II. to be struck out and the following three clauses to be substituted:

1. Any member who shall have been elected not later than January 11th, 1906, may compound for his or her annual subscriptions by the payment of ten guineas in one sum.

2. Any member who, having been elected on or after February 8th, 1906, shall have already paid not less than ten annual subscriptions may compound for subsequent annual subscriptions by the payment of ten guineas in one sum.

3. Any member, other than those specified in Clauses 1 and 2, may compound for his or her annual subscriptions by the payment of fifteen guineas in one sum.

The resolution was carried unanimously.

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* Printed in this volume.

Thursday, February 8th, 1906.

Sir W. D. NIVEN, Vice-President, in the Chair.

Present eighteen members.

The following papers were communicated:—

*The Eisenstein-Sylvester Extension of Fermat's Theorem: Dr. H. F. Baker.

A Chapter of the present state in the Historical Development of Elliptic Functions: Prof. H. Hancock.

*Reduction of the Ternary Quintic and Septimic to their Canonical Forms: Prof. A. C. Dixon and Dr. T. Stuart.

The Scattering of Sound by Spheroids and Discs: Mr. J. W. Nicholson.

Major P. A. MacMahon made a preliminary communication:
"On Partitions of Numbers in Space of Two Dimensions."

Thursday, March 8th, 1906.

Prof. W. BURNSIDE, Vice-President, and, subsequently, Sir W. NIVEN, Vice-President, in the Chair.

Present seventeen members.

The following papers were communicated:—

*On Sommerfeld's Diffraction Problem and on Reflection by a Parabolic Mirror: Prof. H. Lamb.

*On Function Sum Theorems connected with the Series $\sum_{n=1}^{\infty} x^n/n^2$: Prof. L. J. Rogers.

*Investigations on Series of Zonal Harmonics: Prof. T. J. I'A. Bromwich.

*On certain Functions defined by Taylor's Series of Finite Radius of Convergence: Rev. E. W. Barnes.

On the Relations between the p Line Determinants formable from a p by q Array: Prof. E. J. Nanson.

Lt.-Col. A. Cunningham made an informal communication:
†"On the Divisors of Numbers of certain Special Forms."

Dr. F. S. Macaulay made an informal communication:
"On the Equilibrium of Forces of given Magnitudes each passing through a given Point."

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* Printed in this volume. † See "Notes and Corrections" in this volume.

Thursday, April 26*th*, 1906.

Prof. A. R. FORSYTH, President, and, subsequently, Prof. W. BURNSIDE, Vice-President, in the Chair.

Present fourteen members and two visitors.

The President reported to the meeting the death of Mr. R. Rawson, and gave an account of his contributions to mathematics.

The following papers were communicated :—

*Perpetuants and Contra-perpetuants : Prof. E. B. Elliott.

On a Set of Intervals about the Rational Numbers : Mr. A. R. Richardson.

*Some Theorems connected with Abel's Theorem on the Continuity of Power Series : Mr. G. H. Hardy.

*A Question in the Theory of Aggregates : Prof. A. C. Dixon.

*The Canonical Forms of the Ternary Sextic and Quaternary Quartic : Prof. A. C. Dixon.

*On the Question of the Existence of Transfinite Numbers : Mr P. E. B. Jourdain.

*On the Accuracy of Interpolation by Finite Differences : Mr. W. F. Sheppard.

*On Two Cubics in Triangular Relation : Prof. F. Morley.

*On the Geometrical Interpretation of Apolar Binary Forms : Mr. C. F. Russell.

Thursday, May 10*th*, 1906.

Prof. A. R. FORSYTH, President, in the Chair.

Present seventeen members.

Mr. C. F. Russell was elected a member.

The following papers were communicated :—

The Substitutional Theory of Classes and Relations : Hon. B. A. W. Russell.

*The Expansion of Polynomials in Series of Functions : Dr. L. N. G. Filon.

*On the Motion of a Swarm of Particles whose Centre of Gravity describes an Elliptic Orbit of small Eccentricity round the Sun : Dr. E. J. Routh.

The Theory of Integral Equations : Mr. H. Bateman.

*On Linear Differential Equations of rank Unity : Mr. E. Cunningham.

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* Printed in this volume.

Thursday, June 14th, 1906.

Prof. A. R. FORSYTH, President, in the Chair.

Present fifteen members and a visitor.

Mr. W. H. Jackson was admitted into the Society.

Mr. Walter Bailey exhibited a collection of models of space-filling solids.

The following papers were communicated:—

The Algebra of Apolar Linear Complexes: Dr. H. F. Baker.

*Supplementary Note on the Representation of certain Asymptotic Series as Convergent Continued Fractions: Prof. L. J. Rogers.

On certain Special Types of Convertible Matrices: Mr. J. Brill.

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* Printed in this volume.

LIBRARY

[In the course of the Session the Library was transferred to a room on the first floor of the house, 22 Albemarle Street, where the meetings of the Society are held.]

Presents.

BETWEEN October, 1905, and December, 1906, the following presents were made to the Library:—

From the respective Authors or Publishers:—

Adams, J. L.—"The Infinity of the Starry Universe," 1906; and "The Milky Way," 1905.

Bachelier, M. L.—"Théorie Mathématique de Jeu" and "Théorie de la Spéculation."

Beckman, E. H. M.—"Geschiedenis der systematische Mineralogie" (from the Technische Hoogeschool te Delft).

Brioschi, Francesco.—"Opere matematiche," Tomo IV., 1906.

Bromwich, T. J. I'A.—"Quadratic Forms and their Classification by means of Invariant Factors," 1906.

Forsyth, A. R.—"Theory of Differential Equations," Part 4, vols. 5 and 6; Cambridge, 1906.

Geodetic Survey of South Africa, vol. III., "Report on the Geodetic Survey of Part of Southern Rhodesia," 1905.

Guccia, M. G. B.—"Un Théorème sur les Courbes algébriques planes d'ordre *n*."

Issaly.—"Théorie des Pseudo-Surfaces," 1902.

Myller, A.—"Gewöhnliche Differentialgleichungen höherer Ordnung," 1906.

Oettingen, A. von.—"Die perspektivischen Kreisbilder der Kegelschnitte," 1906.

Pittard-Bullock.—"The Power of the Continuum," Berlin, 1905.

Söhngen, N. L.—"Het ontstaan en verdwijnen van Waterstof," 1906.

Veronese, G.—"Il Vero nella Matematica," 1906.

Zeuthen, H. G.—"La Principe de Correspondance pour une Surface algébrique," 1906.

Coimbra: Academia Polyt. de Porto, Ann. Scientificos, vol. 1, nos. 2, 3, 1906.

Hamburg: Math. Gesellschaft, Mittheilungen, bd. 4, hefte 5, 6, 1906.

Indian Engineering, vol. 38, nos. 13–27, 1905; vol. 39, 1906; vol. 40, nos. 1–15, 1906.

Kansas: Univ. Science Bulletin, vol. 3, nos. 1–10, 1905–6.

London: Educational Times, vol. 58, nos. 531–536, 1905; vol. 59, nos. 537–548, 1906.

London: Educational Times Math. Questions and Solutions, New Series, vol. 8, 1905; vols. 9, 10, 1906.

London: Mathematical Gazette, vol. 3, nos. 53–60, 1905–6.

Nautical Almanac for 1906 (Appendix), and for 1909 (presented by the Admiralty).

Paris: L'Enseignement Math., ann. 7, no. 6, 1905; ann. 8, 1906.

Porto: Academia Polyt., Annaes, vol. 1, no. 1, 1905.

Tokyo: Physico-Math. Society, Proceedings, vol. 2, nos. 21–29, 1905–6; vol. 3, nos. 1–5, 1906.

Varsovie: Soc. des Naturalistes, Comptes Rendus, 1902–4, nos. 13–15, 1906.

Warsaw: Wiadomości Matem., tom 9, zeszyt 3–6, 1905; tom 10, zeszyt 1–3, 1906.

Exchanges.

Between October, 1905, and December, 1906, the following exchanges were received :—

American Journal of Mathematics, vol. 27, no. 4, 1905 ; vol. 28, 1906.
American Mathematical Society, Transactions, vol. 6, no. 4, 1905 ; vol. 7, 1906.
American Mathematical Society, Bulletin, vol. 12, nos. 2–10, 1906 ; vol. 13, nos. 1, 2, 1906.
American Philosophical Society, Proceedings, vols. 44, 45, and 46, 1906.
Amsterdam : Nieuw Archiev, deel 7, stuk 2, 3, 1906.
Amsterdam : Revue Semestrielle, tome 14, pts. 1, 2, 1906.
Amsterdam : Wiskundige Opgaven, deel 9, stuk 4, 5, 1906.
Belgique : Académie Royale des Sciences, Annuaire, 1906.
Belgique : Académie Royale des Sciences, Bulletin, 1905, nos. 6–12 ; 1906, nos. 1 8.
Berlin : Jahrbuch über die Fortschritte, bd. 34, hefte 2, 3, 1905 ; bd. 35, hefte 1, 2, 1906.
Berlin : Journal für die Mathematik, bd. 130, hefte 3, 4, 1905 ; bd. 131, 1906.
Berlin : Sitzungsberichte der K. Preuss. Akademie, 1905, nos. 39–53 ; 1906, nos. 1–38.
Bordeaux : Société des Sciences, Observations Pluviométriques et Thermométriques, 1905.
Bordeaux : Société des Sciences, Procès-Verbaux, 1905.
Bordeaux : Société des Sciences, Table Général des Matières, 1850 à 1900, 1905.
Cambridge Philosophical Society, Proceedings, vol. 13, pts. 3–5, 1905–6.
Cambridge Philosophical Society, Transactions, vol. 20, nos. 1–10, 1905–6.
Cambridge, Mass. : Annals of Mathematics, vol. 7, nos. 2–4, 1906 ; vol. 8, no. 1, 1906.
Canadian Institute, Transactions, no. 16, 1905.
Catania : Accademia Gioenia, Atti, Ser. 4, vol. 18, 1905.
Catania : Accademia Gioenia, Bollettino, fasc. 88–91, 1906.
Coimbra : Jornal de Sciencias Mathematicas, vol. 15, no. 6, 1905.
Edinburgh : Mathematical Society, Proceedings, vol. 22, 1904 ; Index to vols. 1–20.
Edinburgh : Royal Society, Proceedings, vol. 25, nos. 2–6, 1904 ; vol. 26, nos. 1–5, 1905.
Edinburgh : Royal Society, Transactions, vol. 41, pt. 1, 1904.
France : Société Mathématique, Bulletin, tome 34, fasc. 1–3, 1906.
Göttingen : Königl. Gesell. der Wissenschaften, Nachrichten, Math. Klasse, 1905, hefte 4, 5 ; 1906, hefte 1, 2.
Göttingen : Königl. Gesell. der Wissenschaften, Mittheilungen, 1905, heft 2 ; 1906, heft 1.
La Haye : Archives Néerlandaises, tome 10, liv. 5, 1905 ; tome 11, 1906.
Leipzig : Beiblätter zu den Annalen der Physik, bd. 29, hefte 21–34, 1905 ; bd. 30, hefte 1–23, 1906.
Leipzig : K. Sächsische Gesell., Math. Klasse, Berichte, 1905, nos. 3–6 ; 1906, nos. 1–5.
Leipzig : K. Sächsische Gesell., Math. Klasse, Abhandlungen, bd. 29, nos. 5–8, 1906.
Livorno : Periodico di Matematica, anno 21, fasc. 2–6, 1906 ; anno 22, fasc. 1, 2, 1906.
Livorno : Periodico di Matematica, Supplemento, anno 9, 1906 ; anno 10, fasc. 1, 1906.
London : Royal Society, Proceedings, Series A, vol. 76, no. 513, 1905 ; vol. 77, nos. 514–520, 1906 ; vol. 78, nos. 521–524, 1906 ; Series B, vol. 76, no. 513, 1905 ; vol. 77, nos. 514–521, 1906 ; vol. 78, nos. 522–527, 1906.
London : Royal Society, Transactions, Series A, vol. 205, 1906.
London : Physical Society, Proceedings, vol. 19, pts. 7, 8, 1905 ; vol. 20, pts. 1, 2, 1906.
London : Institution of Naval Architects, Transactions, vol. 46, 1904.
London : Institute of Actuaries, Journal, vol. 38, pts. 4–6, 1904 ; vol. 39, pt. 1, 1905.
London : Nature, vol. 72, nos. 1864–1878, 1905 ; vol. 73, 1879–1904, 1906 ; vol. 74, nos. 1905–1930, 1906 ; vol. 75, nos. 1931–1936, 1906.
Manchester Literary and Philosophical Society, Memoirs, vol. 48, pts. 2, 3, 1904 ; vol. 49, pt. 1, 1905.
Marseille : Annales de la Faculté des Sciences, tome 15, 1906.

Milano: Reale Istituto Lombardo, Rendiconti, vol. 38, fasc. 5–20, 1905; vol. 39, fasc. 1–16, 1906.
Milano: Reale Istituto Lombardo, Memorie, vol. 20, fasc. 5–8, 1905–6.
Modena: Regia Accademia, Memorie, vol. 5, 1905.
Napoli: Accademia delle Scienze, Rendiconti, vol. 11, fasc. 4–12, 1905; vol. 12, fasc. 1–8, 1906.
Napoli: Accademia delle Scienze, Atti, vol. 12, 1905.
Odessa: Société des Naturalistes, tome 18, 1905; tome 19, 1906.
Palermo: Rendiconti del Circolo Matematico, tomo 20, fasc. 2, 1905; tomo 21, 1906; tomo 22, fasc. 1, 2, 1906.
Paris: Bulletin des Sciences Mathématiques, tome 29, Sept.–Dec., 1905; tome 30, Jan.–Sept., 1906.
Paris: Journal de l'Ecole Polyt., cah. 10, 1905.
Roma: Reale Accademia dei Lincei, Rendiconti, vol. 14, sem. 2, fasc. 6–12, 1905; vol. 15, sem. 1, fasc. 1–12, and sem. 2, fasc. 1–9, 1906.
Roma: Reale Accademia dei Lincei, Rendiconti delle Sedute Solenne, vol. 2, 1906.
Stockholm: Acta Mathematica, bd. 30, pts. 1–3, 1906.
Torino: R. Accademia delle Scienze, Atti, vols. 38–40, and vol. 41, disp. 1–15, 1903–6.
Torino: R. Accademia delle Scienze, Osservazione Meteorologiche, 1906.
Toulouse: Faculté des Sciences, Annales, tome 7, fasc. 3, 4, 1905; tome 8, fasc. 1, 2, 1906.
Venezia: Atti del R. Istituto, tomo 63, 1904; tomo 64, 1905.
Warsaw: Prace Matematyczno-Fizyczne, tome 16, 1905; tome 17, 1906.
Washington: United States Naval Observatory, 2nd Series, vol. 4, pts. 1–3, and vol. 4, pt. 4, 1906.
Wien: Monatshefte für Mathematik, jahr. 17, 1906.
Zurich: Vierteljahrsschrift, hefte 3, 4, 1905; heft 1, 1906.

International Catalogue of Scientific Literature.

In the year April, 1905, to March, 1906 (inclusive), the following exchanges were sent in the first instance to Prof. Love to be indexed for the International Catalogue of Scientific Literature:—

"Proceedings of the Edinburgh Mathematical Society," Vol. XXIII., 1905.

"Proceedings of the Royal Society of Edinburgh," Vol. XXV., Nos. 7–12, 1905.

"Transactions of the Institution of Naval Architects," London, 1905.

"Journal of the Institute of Actuaries," Vol. XXXIX., Pts. 2–4, and Vol. XL., Pt. 1, 1905–6.

"Proceedings of the Manchester Literary and Philosophical Society," Vol. XLIX., Pts. 2, 3, and Vol. L., Pt. 1, 1905.

The following were also sent especially for the purposes of the Catalogue:—

"Mathematical Gazette," Nos. 51–55; London, 1905–6.

"Educational Times," Nos. 528–539; London, 1905–6.

"Journal of the Royal Statistical Society," Vol. XLVIII., Pts. 1–4; London, 1905.

"Proceedings of the Royal Irish Academy," Vol. XXV., Section A, No. 3; Dublin, 1905.

"Transactions of the Insurance and Actuarial Society of Glasgow," Series 5, Nos. 17, 18, 1905.

OBITUARY NOTICES

CHARLES JASPER JOLY

CHARLES JASPER JOLY, son of the late Rev. J. Swift Joly, of Athlone, was born in 1864, and spent most of his school days at the Galway Grammar School under Dr. Biggs. At the age of eighteen he entered Trinity College, Dublin, where he held a scholarship and subsequently a mathematical studentship. His subsidiary subject was experimental physics, and after graduating in 1886 he went to Berlin, where he studied under König. On the death of his father in 1887 he returned to Ireland, and read for a Fellowship, which he gained in 1894. By this time Joly's mathematical powers had fully developed, and he had become familiar with the calculus of quaternions, a subject which strongly attracted him, and to which most of his published work is related.

In 1897, with singular appropriateness, Joly was appointed Royal Astronomer of Ireland, thus occupying the post once held by the great inventor of quaternions. He now undertook a new edition of Hamilton's *Elements of Quaternions*, of which the first volume appeared in 1899 and the second in 1901. The only book of his own composition is *A Manual of Quaternions*, published in 1905. No one can read this without keenly regretting the author's early death. It is sufficiently elementary, at least in the earlier part, for the beginner to learn the principles of the subject; and it contains a great deal of original work, much of which is condensed into the form of examples. Physical applications receive considerable attention: there are sections dealing with kinematics, kinetics, wave theory, and electromagnetism. One of Joly's most interesting contributions to Hamilton's theory is his way of introducing the geometrical principle of duality. In some way or other this principle ought to come into closer contact with the formulæ of quaternions than it actually does: whether Joly's method is the best way towards this desirable end is a matter into which quaternionists might well inquire.

Joly's mathematical papers are not numerous (less than a score in all); but every one of them deserves to be read. With few exceptions, they are

contained in the *Proceedings* and *Transactions of the University of Dublin*. Among the most important, perhaps, are the following:—

"The Theory of Linear Vector Functions" (*Dubl. Trans.*, Vol. XXX., pp. 597–647).

"Properties of the General Congruency of Curves" (*ibid.*, Vol. XXXI., pp. 363–392).

"The Interpretation of a Quaternion as a Point Symbol" (*ibid.*, Vol. XXXII. A, pp. 1–16).

"The Quadratic Screw-system: a Study of a Family of Quadratic Complexes" (*ibid.*, Vol. XXXII. A, pp. 155–238).

"The Geometry of a Three-system of Screws" (*ibid.*, pp. 239–270).

"Quaternions and Projective Geometry" (*Phil. Trans.*, Vol. 201 A, pp. 223–327).

Joly was elected a member of the London Mathematical Society in April, 1902, but did not contribute to its *Proceedings*. He was elected Fellow of the Royal Society in 1904, and at the time of his death (January, 1906) was President of the International Association for the Promotion of the Study of Quaternions and Allied Subjects.

In private life Joly was loved as well as admired by those who knew him well. He was an expert mountain-climber, a devout student of Dante, and a lover of literature in general. How much he was regretted by his intimate friends is clearly shown by the notices which appeared, shortly after his decease, in the *Guardian* newspaper and elsewhere.

G. B. M.

ROBERT RAWSON

[For the statements of fact in this notice the Council is indebted to Rev. R. Harley.]

Robert Rawson was born at Brinsley, a little colliery village about nine miles from Nottingham, on the 22nd July, 1814. At the age of seven he began to earn his own living by working in the mines at the neighbouring village of Eastwood, and he continued to work as a collier for sixteen years. Having early learnt to read, and having come across a periodical containing mathematical questions which interested him, he was told on enquiring that he could learn how to solve them by reading a book on "arithmetic." So on a Saturday afternoon, after leaving the pit, he walked over to Nottingham, and bought a second-hand arithmetic for twopence. After a time he found that his arithmetic failed him beyond a limited range of questions, and was told that he wanted an "algebra." He procured one, and later an old "Euclid" and Simpson's *Fluxions*, He used to look through these books to discover the way in which to solve the questions that interested him. In after life he often lamented the deficiencies of his early training.

In 1837, when Robert Stephenson was beginning to build the Manchester and Leeds Railway, a controversy arose in a local newspaper as to how the level of the road ought to be altered at a curve. Rawson thought that none of the writers had solved the problem; so he wrote out his own solution, signed it, and sent it to the newspaper. This letter led to his being offered employment in the office of the railway engineer at Rochdale, where during the next five years he was chiefly occupied in calculations for the Engineer-in-Chief of the new line. At this time he became a constant contributor to the mathematical column of the York *Courant*, a newspaper which had a large circulation in the North of England. The column was edited by William Tomlinson, a very remarkable man, and entirely self-educated; and it seems to have been a source of inspiration to many—to none perhaps more than Rawson. When the railway was finished, in 1842, Rawson removed to Manchester to be near Eaton Hodgkinson, to whom Stephenson introduced him by letter as a "skilful calculator." He made a living as a private teacher of mathematics, and he added to his slender means by making calculations for engineers. He was employed by Stephenson to calculate the stresses in the girders of the Menai Bridge, and by Eaton Hodgkinson in calculations

relative to the strength of materials, and, in particular, to determine the strength of cast-iron pillars. He continued to write for the mathematical column of the York *Courant*, and he wrote also for the *Mathematician*, the *Lady's and Gentleman's Diary*, and other periodicals. He contributed to the *Memoirs of the Manchester Literary and Philosophical Society* papers on "The Summation of Series," "Definite Integration," "A New Mode of Representing Discontinuous Functions," and "Laplace's Theorem in the Theory of Attraction."

In 1847, on the recommendation of Hodgkinson, he was offered by the Admiralty the position of Head Master of the Dockyard School at Portsmouth; and he accepted it, though not without misgivings, for he had never been in a school before. He soon conquered the difficulties arising from his inexperience, and held the appointment for twenty-eight years with credit to himself and advantage to his pupils. Among these are numbered Sir Philip Watts, K.C.B., the present Director of Naval Construction; Sir John Durston, K.C.B., Engineer-in-Chief, R.N.; Sir James Williamson, C.B.; Prof. Francis Elgar, F.R.S.; and Mr. W. E. Smith, C.B., of the Admiralty. While at Portsmouth Rawson assisted in the experiments upon the stability of floating bodies which were carried out for the Admiralty at the instance of Moseley. The results of these experiments are described in Moseley's well known memoir "On the Dynamical Stability of Ships and on the Oscillations of Floating Bodies" (*Phil. Trans.*, 1851), in which a tribute is paid to Rawson's skill in designing experimental expedients. One of these, a mechanical arrangement "by which the position of the water-line was determined in the extreme position into which the vessel rolls," was described as "specially worthy of observation." Rawson also devised the screw compass—an instrument which determines at sight the pitch of the screw—and for this invention he received the thanks of the Admiralty and an expression of their appreciation of its practical importance.

Rawson published a treatise entitled *The Screw-Propeller: An Investigation of its Geometrical and Physical Properties, and its Application to the Propulsion of Vessels.* He wrote also some elementary books on arithmetic, mensuration, and trigonometry, which were supplied by the Admiralty to the boys of the Dockyard School. He contributed an account of the life and work of his friend Eaton Hodgkinson to the *Memoirs of the Manchester Literary and Philosophical Society* (1865), and many papers by him are published in these *Memoirs;* others will be found in the *Reports of the British Association*, the *Transactions of the Institution of Naval Architects*, and the *Messenger of Mathematics*. His single communication to the *Proceedings* of this Society was a paper,

published in Vol. IX. (1878), "A New Method of Finding Differential Resolvents of Algebraical Equations." His calculation of the complete cubic differential resolvent (*Brit. Assoc. Rep.*, 1886) was a very remarkable piece of work. He was a member of several scientific bodies, and he valued greatly the distinction of his election as an honorary member of the Manchester Literary and Philosophical Society. In 1894 he was placed on the Commission of the Peace for the County of Hampshire, and he discharged his duties as a magistrate with great zeal and assiduity up to within a month of his death, which took place at Havant on the 11th March, 1906.

NOTES AND CORRECTIONS.

Lt.-Col. A. Cunningham sends an abstract of his communication of date March 9th, 1906:—

(i.) The numbers of the form $N = q \, . \, 2^q + 1$ are remarkable for the simple rules for their arithmetical divisors, and also for the rarity of primes among them, there being no primes after $N = 3$ up to $q = 200$ (except possibly when $q = 141$). (ii.) The binomial form $N = (a^{4x} + 1)^3 + 1$ yields very high completely factorisable numbers: *e.g.*,

$$(2^{100} + 1)^3 + 1 = \frac{2^{75} - 1}{2^{25} - 1} \, \frac{2^{75} + 1}{2^{25} + 1} \, \frac{2^{150} + 1}{2^{50} + 1} \, 2 \, (2^{99} + 1),$$

which contains 91 figures. The complete factorisation of each of the large factors of this number has been given by Lucas.

Mr. H. Bateman sends the following correction of a paper by him published in this volume:—

On p. 97, equation (25) should be

$$P(s, t) = \Phi_1 (s, \lambda_0) \Psi_1 (t, \lambda_0) + \ldots + \Phi_\rho (s, \lambda_0) \Psi_\rho (t, \lambda_0),$$

where $\rho \leqslant p$.

Mr. Hardy sends the following corrections (due to Dr. Hobson) of his paper in this volume (pp. 247–265):—

On p. 251, Theorem I. *b* is incorrectly stated, as appears when we consider the particular case in which $f_n(x) = x^n$ $(0 \leqslant x < 1)$, $= 0$ $(x = 1)$ and $a_n = (-1)^n$. Dini's theorem, on which the proof depends, requires the continuity of $f_n(x)$. The following alterations should therefore be made:—

Theorem I. *b*. Omit "*uniformly*." Also omit the proof.

Add a new theorem (Theorem I. *b* 0)—"*If, in addition, the functions* $f_n(x)$ *are continuous, the series* $\Sigma a_n f_n$ *is uniformly convergent and continuous.*"

The proof given in the paper for I. *b* applies to this. The corollary may be omitted.

Theorem I. *b* 1. After "*subject*" add "*in* I. *b*."

In all the cases of interest the condition that each individual $f_n(x)$ is continuous is satisfied: and the rest of the paper is in no way affected.

PAPERS

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PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY

ON THE ARITHMETICAL NATURE OF THE COEFFICIENTS IN A GROUP OF LINEAR SUBSTITUTIONS OF FINITE ORDER (SECOND PAPER)

By W. BURNSIDE.

[Received October 12th, 1905.—Read November 9th, 1905.]

IN a former paper dealing with this question (*Proc. London Math. Soc.*, Series 2, Vol. 3, pp. 239–252) the subject was approached from the point of view of the reduction of the group when represented in the form of a group of permutations. One of the results arrived at was to determine a condition subject to which it is possible to exhibit an irreducible group of finite order in a form in which the coefficients are rational functions of the characteristics; but the question as to whether this condition was generally satisfied was not dealt with.

In the present paper the problem is approached from a different and, so to say, a more self-contained point of view.

An irreducible group of linear substitutions of finite order is supposed to be given in any one of its possible forms, without any assumption at all as to the nature of the coefficients. With regard to such a group the question arises as to whether it is possible to choose new variables, so that, when expressed in terms of them, the coefficients of the substitutions belong to an assigned domain of rationality. If the domain of rationality does not contain the characteristics of the group, it is certainly impossible to do this. The simplest domain of rationality to which the coefficients can belong is that defined by the characteristics.

It is shown here, by direct considerations, in which the theory of the reduction of the group, regarded as a permutation-group, is not introduced, that in general it is possible to exhibit the group so that the coefficients belong to the domain of rationality defined by the characteristics. There are exceptions to this general rule, and the whole of the exceptions are not here determined; but it is, I think, made clear that the cases which do not come under the rule are actually of the nature of exceptions.

1. I consider a discontinuous irreducible group of linear substitutions in m variables. The group itself is denoted by Γ, and any substitution of the group

$$x_i' = \sum_{j=1}^{j=m} s_{ij} x_j \quad (i = 1, 2, \ldots, m)$$

is denoted by S. The sum of the coefficients in the leading diagonal of S, or its characteristic, is denoted by χ_s. Such a group necessarily contains a set of m^2 substitutions $A^{(k)}$ $(k = 1, 2, \ldots, m^2)$ which are linearly independent* in the sense that the determinant

$$\begin{vmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{mm}^{(1)} \\ a_{11}^{(2)} & a_{12}^{(2)} & \cdots & a_{mm}^{(2)} \\ \cdots & \cdots & \cdots & \cdots \\ a_{11}^{(m^2)} & a_{12}^{(m^2)} & \cdots & a_{mm}^{(m^2)} \end{vmatrix}$$

is not zero.

This being the case, the equations

$$s_{ij} = \sum_{k=1}^{k=m^2} a_{ij}^{(k)} S_k \quad (i, j = 1, 2, \ldots, m) \tag{i}$$

are linearly independent, and determine the m^2 quantities S_k. The system of m^2 equations (i) may be replaced by

$$\sum_{i,j} s_{ij} a_{ji}^{(l)} = \sum_{k=1}^{k=m^2} S_k \sum_{ij} a_{ij}^{(k)} a_{ji}^{(l)} \quad (l = 1, 2, \ldots, m^2).$$

But $\sum_{ij} s_{ij} a_{ji}^{(l)}$ is the characteristic of $SA^{(l)}$. Hence the m^2 quantities S_k are determined by

$$\chi_{SA^{(l)}} = \sum_{1}^{m^2} S_k \chi_{A^{(k)}A^{(l)}},$$

and they are therefore rational functions of the characteristics of the group.

* *Proc. London Math. Soc.*, Series 2, Vol. 3, p. 433.

The m^2 equations (i) are equivalent to the single equation

$$S = \sum_1^{m^2} A^{(k)} S_k, \tag{ii}$$

expressing that the substitution (or matrix) S is a linear function of the substitutions (or matrices) $A^{(k)}$; and the coefficients which enter in this expression for S are rational functions of the characteristics of the group.

If S, T, U are three substitutions of the group, such that

$$ST = U,$$

then
$$\Sigma A^{(k)} U_k = \Sigma A^{(k)} S_k \, \Sigma A^{(l)} T_l. \tag{iii}$$

But, by (ii), $A^{(k)} A^{(l)}$ can be expressed linearly in terms of the m^2 independent substitutions. Let

$$A^{(k)} A^{(l)} = \Sigma A^{(p)} a_{klp}$$

be the equation so expressing it, the constants a_{klp} being rational functions of the characteristics. Then (iii) may be written

$$\sum_p A^{(p)} U_p = \sum_{k,l,p} a_{klp} S_k T_l A^{(p)}. \tag{iv}$$

The m^2 substitutions $A^{(p)}$ being linearly independent, this relation among the substitutions (or matrices) is equivalent to the system of m^2 equations

$$U_p = \sum_{k,l} a_{klp} S_k T_l \quad (p = 1, 2, \ldots, m^2). \tag{v}$$

Now the relation
$$ST = U$$

is also equivalent to the system of m^2 equations

$$u_{ij} = \sum_k t_{ik} s_{kj} \qquad (i, j = 1, 2, \ldots, m) \tag{vi}$$

between the coefficients of the three substitutions S, T, and U. Hence equations (v) is the form which equations (vi) take when the s's, t's, and u's are cogrediently replaced by new symbols, S's, T's, and U's, defined by the equations (i).

But, if the s's are regarded as the original and the u's as the transformed variables, while for the t's are taken in turn the coefficients of each substitution of Γ, then equations (vi) define a group of linear substitutions on m^2 variables which is simply isomorphic with Γ. Moreover in this group each set of m symbols with the same second suffix undergoes for each substitution of the group exactly the same transformation as the original x's; and there are m such sets. The group may then be denoted by $m\Gamma$, implying that m distinct sets of variables each undergo cogrediently the substitutions of the original group Γ.

If then, in (v), the S's are regarded as the original and the U's as the transformed variables, while the T's take in turn the values corresponding to each of the substitutions of Γ, the resulting system of equations is a form into which the group $m\Gamma$ can be thrown by a suitable choice of new variables. But the T's which correspond to the substitutions of Γ and the coefficients $a_{\kappa\lambda\mu}$ are rational functions of the characteristics of Γ. Hence $m\Gamma$ can be exhibited in a form in which all the coefficients are rational functions of the characteristics. Further, if in equations (vi) the t's are regarded as the original and the u's as the transformed variables, while for the s's the coefficients of each substitution of Γ are taken in turn, then the equations give another form of $m\Gamma$, every one of whose substitutions is permutable with every substitution of the previous form. Hence the same is also true of equations (v) when the T's are taken as the original and the U's as the transformed variables. The results of this paragraph may then be summed up as follows:—

Theorem.—If Γ is a discontinuous irreducible group of linear substitutions in m variables, and if $m\Gamma$ represents the group in m^2 variables that arises by carrying out the substitutions of Γ cogrediently on m sets of m variables each, then a group equivalent to $m\Gamma$ can be set up in two distinct forms on a set of m^2 variables, so that (i) the coefficients in each form are rational functions of the characteristics of Γ, and (ii) every substitution of the one form is permutable with every substitution of the other.

2. If Γ is a group of finite order, it will in general contain sub-groups for which one or more linear functions of the variables are invariants. In a paper recently published in the *Messenger of Mathematics* (Vol. xxxv., pp. 51–55) I have determined the nature of those exceptional cases in which the identical substitution of Γ is the only one for which there are linear invariants. They are very limited in number, and may be arranged in three classes, of which—

(i) the quaternion group in two variables;

(ii) the group generated by

$$x' = \omega x, \quad y' = \omega^2 y, \quad z' = \omega^4 z,$$
$$x' = y, \quad y' = z, \quad z' = ax,$$
$$\omega^7 = 1, \quad a^3 = 1;$$

(iii) the group of order 120 in two variables which is multiply isomorphic with the icosahedral group,

are typical.

Apart from these very special exceptions every group of linear substitutions of finite order has some sub-group, other than that consisting of the identical substitution only, for which one or more linear functions of the variables are invariant.

3. Let Γ now denote any given irreducible group of finite order on m variables. Further, let G and G' denote the two forms in which $m\Gamma$ can be set up on m^2 variables

$$x_1, x_2, \ldots, x_{m^2},$$

so that (i) the coefficients of the substitutions of G and G' are rational functions of the characteristics of Γ, and (ii) every substitution of G is permutable with every substitution of G'. Denote by ξ_1 a linear function

$$\Sigma A_i x_i$$

of the m^2 variables with arbitrary coefficients; and by

$$\xi_1, \xi_2, \ldots, \xi_n,$$

the functions into which ξ_1 is transformed by the n substitutions of a sub-group H', of order n, of G'. Then

$$\sum_1^n \xi_t$$

is the most general linear function of the variables which is invariant for H'. Unless Γ belongs to one of the exceptional cases above mentioned, G' must have some sub-group H' for which $\sum_1^n \xi_t$ is not identically zero. If in Γ the sub-group H' has i independent linear invariants, then in $m\Gamma$ there are mi linearly independent linear functions invariant for H'. Hence of the coefficients of $A_1, A_2, \ldots, A_{m^2}$ in $\sum_1^n \xi_t$, just mi are linearly independent. Now the coefficient of A_s in $\sum_1^n \xi_t$ is a linear function of the x's with coefficients which are rational in the characteristics of Γ. Hence mi linear functions of the x's with coefficients rational in the characteristics of Γ may be formed, each of which is invariant for H'; and no other linear function of the x's, which is linearly independent of these, is invariant for H'.

Now every substitution of G is permutable with every substitution of H'. Hence the substitutions of G must transform among themselves these mi linear functions of the x's; and the coefficients in the group of linear substitutions which so arises are rational functions of

the characteristics. This is equivalent to the statement that, unless Γ is one of the exceptions above mentioned, it is always possible to find a number i ($< m$) such that $i\Gamma$ can be exhibited in a form in which the coefficients are rational functions of the characteristics.

Denote by
$$y_1,\ y_2,\ \ldots,\ y_{mi},$$
the mi above linear functions. Any substitution S' of G' will change these into mi linearly independent functions

$$z_1,\ z_2,\ \ldots,\ z_{mi}.$$

These, and only these, are invariants for the sub-group $S'^{-1}H'S'$; and therefore these are transformed among themselves by every substitution of G. Moreover, since
$$S'^{-1}SS' = S,$$

if S is any substitution of G, the z's and the y's undergo, for each substitution of G, the same transformation. It may be the case that for certain substitutions S' of G', the y's are linear functions of the z's. This cannot be the case for every substitution of G', for then both G and G' would transform the y's among themselves.

It may, however, happen that when S' is suitably chosen some, but not all, linear functions of the y's can be expressed linearly in terms of the z's. If this is so, it is possible to determine coefficients A_s and B_t so that in
$$\Sigma A_s y_s + \Sigma B_t z_t$$

the coefficient of each x is identically zero. The equations expressing this are linear in the A's and B's with coefficients rational in the characteristics. Hence the linear functions, if any, of the y's which can be expressed in terms of the z's have coefficients which are rational in the characteristics. But, since the z's are transformed among themselves by G, those linear functions of the y's which can be expressed linearly in terms of the z's are transformed among themselves by G. If there are j of them, the j functions will be transformed among themselves by G, and the coefficients in the substitutions are still rational in the characteristics. Further, since Γ is irreducible, the number of symbols transformed among themselves by any component of G must be a multiple of m.

Hence, if the y's and z's are not linearly independent, it is possible to form mi' ($i' < i$) linear functions of the x's, with coefficients which are rational in the characteristics, that are transformed among themselves by G, and the coefficients of the resulting substitutions are rational in the characteristics.

This set of mi' functions may be represented by

$$y_1, y_2, \ldots, y_{mi'},$$

and may be dealt with as the original set of mi were.

In this way we finally arrive at a set of ma linear functions of the x's (a being as small as possible) with coefficients rational in the characteristics, which are transformed among themselves by G, the coefficients of the substitutions satisfying the same condition.

Represent them by $\quad y_1^{(1)}, y_2^{(1)}, \ldots, y_{ma}^{(1)}.$

A substitution S' of G' changes these into

$$y_1^{(2)}, y_2^{(2)}, \ldots, y_{ma}^{(2)},$$

and it is certainly possible to choose S' so that these are linearly independent of the preceding ones. If $m > 2a$, it must again be possible to take S'' in G' so that

$$y_1^{(3)}, y_2^{(3)}, \ldots, y_{ma}^{(3)},$$

into which the functions of the first line are changed by S'', are linearly independent of those in the first two lines. This process may be continued till a set of m^2 functions linearly equivalent to the original x's have been formed. Hence a must be a factor of m. Moreover, if $m = ab$, the functions in each of the b sets

$$y_1^{(t)}, y_2^{(t)}, \ldots, y_{ma}^{(t)}, \quad (t = 1, 2, \ldots, b)$$

undergo for each substitution of G the same transformation, the coefficients being rational functions of the characteristics.

Consider now any sub-group of Γ which has k independent linear invariants. In G' the sub-group has mk linear invariants; which may be denoted by

$$z_1, z_2, \ldots, z_{mk};$$

and by the process above used these may be expressed linearly in terms of the m^2 y's, with coefficients which are rational in the characteristics. The number denoted by a being, by supposition, as small as possible, k is not less than a. If k is greater than a, and if in the expression of the z's the $y^{(1)}$'s occur, the ma $y^{(1)}$'s may be eliminated from the equations which express the z's in terms of the y's. In the elimination other sets of y's besides the $y^{(1)}$'s may disappear; but there must remain $m(k-a)$ equations, giving $m(k-a)$ independent linear functions of the z's (with coefficients which are rational in the characteristics) in terms of certain remaining sets of y's. Since each set of

y's are transformed among themselves by G, these $m(k-a)$ functions of the z's are transformed among themselves by G, and the coefficients in the substitution are rational in the characteristics. If $k-a$ is greater than a, this process may be continued. Hence, unless k is a multiple of a, it will be possible to obtain a set of ma' $(a' < a)$ functions which are transformed among themselves by G with coefficients rational in the characteristics; and this is contrary to the supposition made. This result is equivalent to the following:—

Theorem.—An irreducible group of linear substitutions of finite order can certainly be exhibited in a form in which the coefficients in the substitutions are rational functions of the characteristics, unless there is an integer a (>1) such that the number of linear invariants for every subgroup is either zero or a multiple of a.

Corollary.—An irreducible group of linear substitutions in an odd prime number p of variables can be exhibited in a form in which the coefficients are rational functions of the characteristics, except in the case of the class of soluble groups $\{S, P\}$, defined by

$$S \sim x_1' = \beta x_1, \quad x_2' = \beta^s x_2, \quad \dots, \quad x_p' = \beta^{s^{p-1}} x_p;$$

$$T \sim x_1' = x_2, \quad x_2' = x_3, \quad \dots, \quad x_p' = \alpha x_1,$$

where α is a p^e-th root of unity, β an m-th (m prime to p) root of unity, and

$$s^p \equiv 1 \pmod{m}.$$

In fact, when the number of variables is prime a must be unity; and therefore the group can certainly be exhibited in the desired form, unless the identical substitution is the only one for which there are linear invariants. Such groups occur only among the second of the three classes mentioned above; and I have shown in a paper in the *Messenger of Mathematics*,* immediately preceding the one quoted above, that their form is that just stated.

4. I wish to emphasize the point that the cases in which an irreducible group of linear substitutions of finite order cannot be exhibited in a form in which the coefficients are rational functions of the characteristics are of the nature of exceptions.

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* Vol. xxxv., p. 49.

The general rule, subject to certain definite classes of exceptions, is that such a mode of exhibiting the group is possible.

The exceptions corresponding to the case in which a is zero I have actually determined. The familiar case of the quaternion group in two variables for which all the characteristics are rational, while any possible form of the group involves $\sqrt{(-1)}$ in the coefficients, suggested the possible occurrence of other similar exceptions and indeed led to this investigation. Whether there are also exceptions corresponding to values of a other than zero I cannot at present say. I have, however, examined a large number of groups for which the characteristics have been calculated, and in no case in which I have determined the corresponding multipliers have I come across an exception corresponding to a value of a other than zero.

THE CONTINUUM AND THE SECOND NUMBER CLASS

By G. H. HARDY.

[Received August 3rd, 1905.—Read November 9th, 1905.]

1. In a recent number of these *Proceedings** Dr. Hobson criticises (among other things) a construction which I gave in 1903† for a set of points of cardinal number $\aleph_1$ contained in the linear continuum (0, 1). This criticism is merely incidental to a much more comprehensive attack on the whole theory of Cantor's transfinite numbers, as it has been generally accepted by mathematicians, and in particular to the theory of cardinals elaborated by Mr. Whitehead and Mr. Russell, and expounded in the latter's *Principles of Mathematics*. It is, I believe, Mr. Russell's intention to reply to Dr. Hobson, and I should not wish to discuss the general question in the present communication, even if I felt competent to do so. My present object is a much more modest one. Besides the large question with which we are all concerned there is a smaller one which concerns only Dr. Hobson and myself. Each of us is of opinion that the other has made a mathematical mistake. It is with this smaller question that I propose to deal now, and I shall only refer to the larger issue in so far as is necessary if I am to make clear what the difference between us really is. If Dr. Hobson's views concerning cardinal numbers in general were correct, my construction would acquire a fundamental importance which I am not myself at all disposed to attach to it: I need therefore make no apology for considering in detail this particular part of Dr. Hobson's paper.

2. Before I proceed to discuss Dr. Hobson's objections to my construction, it will be convenient if I indicate a slight simplification‡ which can be made in it. Whether this alteration be made or not in no way affects the force of Dr. Hobson's arguments.

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* *Proc. London Math. Soc.*, New Series, Vol. 3, p. 170.

† *Quarterly Journal*, Vol. xxxv., p. 88.

‡ Suggested by a passage in Baire's *Leçons sur les Fonctions discontinues*, p. 25.

The principle of my construction was to associate with every number a of the second class an ascending sequence

$$(a) \qquad a_1,\ a_2,\ a_3,\ \dots$$

of positive integers by means of the three following rules:—

(i.) To the number 1 is to correspond the sequence

$$(1) \qquad 1,\ 2,\ 3,\ \dots.$$

(ii.) The sequence for $a+1$ is to be formed by omitting the first term of the sequence for a.

(iii.) If a is a number of the second kind (one with no immediate predecessor), we are to select a fundamental sequence (a_ν) of which a is the limit, and we are to traverse the array

$$\begin{array}{llll} (a_1) & a_{1,1}, & a_{1,2}, & a_{1,3},\ \dots, \\ (a_2) & a_{2,1}, & a_{2,2}, & a_{2,3},\ \dots, \\ & \dots & \dots & \dots \end{array}$$

diagonally, so that we obtain

$$(a) \qquad a_{1,1},\ a_{2,2},\ a_{3,3},\ \dots.$$

I found then that in order to assure ourselves that the sequences thus generated are all distinct it may be necessary to substitute for the fundamental sequence (a_ν) another sequence $(a_\nu+m_\nu)$, where the m_ν's are finite numbers formed successively according to a definite rule which I gave in my former paper.* I now wish to point out that this slight complication is quite unnecessary if, instead of defining the sequence for a by the simple equation

$$a_n = a_{n,n},$$

we take a_n to be *the greatest of the integers* $a_{1,n},\ a_{2,n},\ \dots,\ a_{n,n}$. It is then easy to prove, without the introduction of the numbers m_ν, that, if a, a' $(a < a')$ are any two numbers of the second class, there is a definite n from and after which $a'_n > a_n$; so that no two sequences can be the same. For, if this is true for all numbers $\leqslant a'$ (say), it is obviously true for all numbers $\leqslant a = a'+1$. We have therefore only to show that, if it is true for all numbers $\leqslant a_\nu$, where a_ν is an arbitrary member of a fundamental sequence (a_ν), it is true for all numbers $\leqslant a$, the number which immediately follows this fundamental sequence. According to the construction, a_n is the greatest of

$$a_{1,n},\ a_{2,n},\ \dots,\ a_{n,n}.$$

* *Loc. cit.*, p. 91.

Then, if $\beta < a$, we can find m so that $\beta < a_m$, and we can find n_0 so that $n_0 > m$ and $a_{m,n} > b_n$ for $n \geqslant n_0$. Thus

$$a_n \geqslant a_{m,n} > b_n,$$

and so the sequences for β and for a are distinct.

It follows that, if all the sequences which correspond to numbers $< a$ are distinct, all those which correspond to numbers $\leqslant a$ are distinct, whether a has an immediate predecessor or not; and therefore that *all* the sequences are distinct.

3. Dr. Hobson argues (p. 187) that this construction must be faulty for the following reason. Given any integer m, a least number a_m, he contends, can be found, such that for all numbers $a > a_m$ the second term of the corresponding sequence is greater than m. Taking a sequence of numbers

$$m_1 < m_2 < \ldots,$$

and forming the number a which is the limit of the fundamental sequence (a_{m_ν}), he deduces that the second term in the sequence for a is greater than m_ν, for all values of ν, *i.e.* is greater than any assignable integer, and therefore that no such sequence exists. And, in fact, he concludes that, if the sequences for the early numbers of the second class are formed as I formed them in my earlier paper, no sequence can be constructed to correspond to the number

$$\epsilon_\omega = \lim \epsilon_\nu,$$

where

$$\epsilon_1 = \lim \omega,\ \omega^\omega,\ \omega^{\omega^\omega},\ \ldots,$$

$$\epsilon_2 = \lim \epsilon_1,\ \epsilon_1^{\epsilon_1},\ \epsilon_1^{\epsilon_1^{\epsilon_1}},\ \ldots,$$

$$\ldots \quad \ldots \quad \ldots$$

$$\epsilon_{\nu+1} = \lim \epsilon_\nu,\ \epsilon_\nu^{\epsilon_\nu},\ \epsilon_\nu^{\epsilon_\nu^{\epsilon_\nu}},\ \ldots.*$$

I hardly think that Dr. Hobson can have realised how paradoxical his conclusion is. For it follows from the definition of the construction that, if sequences have been assigned by it for all numbers $< a$, a sequence is assigned by it for a. Therefore, if there is no sequence for a, we can find a number $< a$, say $a^{(1)}$, for which there is no sequence, a number $a^{(2)} < a^{(1)}$ for which there is no sequence, and so on. But, as it is impossible to find an infinite descending sequence $a, a^{(1)}, a^{(2)}, \ldots$, we shall find ultimately that there are no sequences at all; and even Dr. Hobson is not so sceptical as this.

* It is not difficult to prove that the numbers thus defined are the same as the first few of Cantor's ϵ-numbers.

My answer to Dr. Hobson's argument consists simply of a denial of his major premiss. It is not true that, given *any* integer m, a number a_m can be found such that, for *every* $a > a_m$, $a_2 > m$. The second terms of our sequences may go on increasing for a while, but sooner or later a sudden jump downwards will occur. This is most obvious, perhaps, if we consider a fundamental sequence of the type $(a+\nu)$.

If for a the second term is a_2, for $a+\nu$ the second term is $a_2+\nu$, and we can find a value of ν for which this number is greater than any assignable number. But it by no means follows that the second term in the sequence for $a+\omega$ is greater than any assignable number; on the contrary, it is a_3, the second term in the sequence for $a+1$, and is *less* than the second terms in the sequences for $a+2$, $a+3$, ... And, more generally, if b_n is the n-th term in the sequence for $a+\omega$, we can find a value of μ such that the n-th terms in the sequences for $a+\nu$ $(\nu \geqslant \mu)$ are all $> b_n$. It is equally true, as I showed in § 2, that, *given* ν, we can find n_0 so that for $n \geqslant n_0$ the figures in the sequence for $a+\omega$ are greater than those in the sequence for $a+\nu$. That these two propositions should simultaneously be true may seem paradoxical for a moment, but we have only to consider the sequences

$$\begin{array}{llllll}
(1) & 1, & 2, & 3, & 4, & \ldots, \\
(2) & 2, & 3, & 4, & 5, & \ldots, \\
 & \ldots & \ldots & \ldots & \ldots & \\
(\nu) & \nu, & \nu+1, & \nu+2, & \nu+3, & \ldots, \\
 & \ldots & \ldots & \ldots & \ldots & \\
(\omega) & 1, & 3, & 5, & 7, & \ldots,
\end{array}$$

to see that they are in reality perfectly consistent with one another, just as the propositions—

(i.) given m, we can find x_0 so that, for $x \geqslant x_0$, $e^x > x^m$;

(ii.) given $x_0\,(>1)$, we can find m_0 so that, for $m \geqslant m_0$, $x_0^m > e^{x_0}$

—are consistent with one another.

It may, no doubt, be the case that for a *particular* number m (*e.g.*, 1,000,000) all sequences later than an assignable sequence have second figures $> m$. Thus Dr. Hobson asserts that, for every $a \geqslant \omega^\omega$, $a_2 \geqslant 5$, and this may perfectly well be true. But, if it is true, it will depend on *two* facts:

(i.) that, for $a = \omega^\omega$, $a_2 \geqslant 5$;

(ii.) that *in constructing the sequences corresponding to higher*

numbers we never use a limit sequence $a_1, a_2, \ldots$ containing more than one term $< \omega^\omega$.

Whether (ii.) is true or not depends entirely on the particular "norm" chosen for forming the sequences. If I had chosen $1, \omega, \omega^\omega, \omega^{\omega^\omega}, \ldots$ instead of $\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots$ as the fundamental sequence for ϵ_1, the second term in the corresponding sequence would have been 3. And what Dr. Hobson's argument really proves is simply that, *however* the sequences are chosen, what may be true for some particular values of m cannot be true for *all* values of m, and that no such proposition as (ii.) above can be true for more than an enumerable sequence of values of a (such as ω^ω), which is otherwise obvious. And where his argument breaks down with regard to the particular sequence of numbers ($\epsilon_1, \epsilon_2, \ldots, \epsilon_\omega$) which he considers is in his not having noticed that the second term in the sequence for ϵ_ω is less than the second term in the sequences for $\epsilon_3, \epsilon_4, \ldots$ (if, as is natural, we take $\epsilon_1, \epsilon_2, \ldots$ as the fundamental sequence for ϵ_ω and grant that, as he asserts, the second term in the sequence for ϵ_ν continually increases with ν).

4. The arguments which Dr. Hobson urges specifically against me appear to me therefore to be invalid. These arguments are (as I think Dr. Hobson and I agree) quite independent of those used by him in his general attack on the theory of transfinite cardinals; and I fully admit that the latter cannot be answered in so summary a manner. These arguments will be discussed in detail by Mr. Russell. I shall only refer to them now in order to make clear the point that Dr. Hobson has made against a great deal of generally accepted mathematical reasoning, of which my construction is an average specimen. The point is this, that a great deal of such reasoning really depends on the acceptance of a certain logical postulate of which no proof has yet been given, namely, *the postulate of the existence of the multiplicative class.* If we have a class of mutually exclusive classes k, no one of which is null, the *multiplicative class of the k's* is defined as the class of classes each member of which contains one and only one member of each of the k's.* The class can always be defined, but it has never been proved that it is *never null*, that is to say, that it always contains at least one member.

Among mathematical proofs in which the existence of the multiplicative

* A. N. Whitehead, "On Cardinal Numbers" (*American Journal*, Vol. xxiv., p. 383).

class is assumed, if not universally, at any rate in cases more extensive than those for which its existence has been proved, I may instance Bernstein's and König's theorems concerning the exponentiation of cardinals, Bernstein's and my own proofs that the cardinal number of the continuum is greater than or equal to $\aleph_1$, and Borel's construction for a function of arbitrary class,* among many others.

The last instance is peculiarly instructive, as it shows how assumptions equivalent to that of the existence of the multiplicative class find their way into the writings even of mathematicians who can recognise the assumption elsewhere.

M. Borel wishes to show that functions exist which cannot be represented as double series of polynomials.† If

$$P_{\alpha,\beta}(x) = \sum_{\gamma=0}^{\alpha+\beta} c_{\alpha,\beta,\gamma}\, x^{\gamma}$$

(he says), every double series of polynomials can be written in the form

$$\sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} P_{\alpha,\beta}(x). \tag{1}$$

If this series converges for $0 \leqslant x \leqslant 1$, it represents for those values of x a function of class 0, 1, or 2; and every function of class 0, 1, or 2 can be defined in this way by a suitable choice of the constants $c_{\alpha,\beta,\gamma}$. "*Chaque fonction est même définie d'une infinité de manières, mais cela n'a pas d'inconvénient pour ce qui suit.*"

M. Borel's subsequent reasoning depends entirely on *one* representation of every function of classes 0, 1, and 2 having been selected from among the infinity of representations which correspond to each function, *i.e.*, on the existence of the multiplicative class of the classes formed by all the representations of any given function. Yet M. Borel, criticising Zermelo's article in the *Annalen,*‡ uses language which might have been used by Dr. Hobson: "il me semble que les objections que l'on peut y opposer valent contre tout raisonnement où l'on suppose un *choix arbitraire* fait une infinité non dénombrable de fois; de tels raisonnements sont en dehors du domaine des mathématiques."§

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* *Leçons sur les fonctions de variables réelles*, Note III., pp. 156–158.

† I am not implying any doubt of the correctness of the result.

‡ *Math. Ann.*, Bd. LX., p. 194.

§ The aggregate of functions considered above has the cardinal number of the continuum. The logical difficulty is the same whether the *choix arbitraire* has to be made an enumerable or a non-enumerable infinity of times; but, of course, it will generally be easier, in a particular case in which the infinity of times is enumerable, to get over the difficulty by substituting a "norm," *i.e.*, a set of rules for choosing, for the "arbitrary" acts of choice.

In the case of my construction the form in which the postulate is used is in the assumption of the existence of the multiplicative class of the class of classes formed by all the progressions whose limit is a given number a. To each a corresponds an infinity of such progressions: of these progressions we must select one for every a, and it certainly seems very paradoxical to suppose that the class whose members are defined to be all the various *aggregates of selected progressions* should be null, *i.e.*, possess no members. But, although we can define the class, we cannot (so far as can be seen at present) specify a single one among its members, and there seems to be no way of proving that there are members except by actually producing them.

The instance of the decimals referred to by Dr. Hobson is really not a parallel. For the class of decimals does exist: we can produce some at any rate of its members, for example ·0000..., ·0101.... Here the multiplicative class is that of an enumerable class of classes each of which contains the two members 0 and 1, and its existence can be proved.

The necessary axiom, if it is to be postulated, may be postulated in a variety of forms, that of the assumption of the universal existence of the multiplicative class, or its existence subject to restrictions,* or in either of Zermelo's two forms:

(*a*) That the product of any number of infinite cardinals cannot be zero.

(*b*) That a relation exists which correlates each class contained in a given class with *one* of its members.

It has been proved that (*b*) implies (*a*) and the universal existence of the multiplicative class, but whether or no the latter imply the former has not yet been decided. Mr. Russell has traced the consequences of the denial of the multiplicative axiom in the arithmetic of the transfinite numbers, and has shown that the question as to its truth or falsity has no bearing on the question of the Aleph-series and Burali-Forti's contradiction, which must be met in quite another way. There is therefore no reason for supposing that Zermelo's assumption is not valid except that it has not been proved. And that it has not been proved means simply that no general method has been given

* It is, of course, quite possible that the existence of the multiplicative class may not be universal, but may hold in extensive particular cases, *e.g.*, when the class of classes whose multiplicative class is required is well ordered. The multiplicative class certainly exists for any class of *well-ordered* classes, if a definite order is given with each class, but it has not been proved to exist for a *well-ordered* class of *any* classes.

for defining in finite terms, *i.e.*, by a finite number of repetitions of a finite number of symbols, *one* member of the multiplicative class whose existence is to be established. Even if we knew that it was *impossible* ever to define a single member of a class, it would not of course follow that members of the class did not exist, but there appears to be no way of proving the contrary, except by actually specifying a member or by showing that the hypothesis that there is no member leads to contradiction; and, awkward and paradoxical as the consequences of denying the multiplicative axiom are, it has yet to be shown that they are contradictory.

I am therefore, in default of proof, prepared to accept the multiplicative axiom* provisionally on the grounds

(i.) that to deny it appears to be paradoxical;

(ii.) that no *reason* has been given for denying it;

(iii.) that to deny it reduces to a state of chaos a great deal of very interesting mathematics.

* I do not imply that I accept Zermelo's proof that every aggregate is well ordered. I agree with Dr. Hobson in thinking it open to objection on other grounds.

ON "WELL-ORDERED" AGGREGATES

By A. C. DIXON.

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DR. HOBSON has lately pointed out the importance of the question whether any object which it is proposed to discuss can be defined in finite terms. It occurred to me, on reading his article, that the cardinal number of all objects which can be so defined could be found, and that from its determination important conclusions could be drawn.

By defining, describing, or specifying an object is meant stating such properties of it as distinguish it from all other objects of mental activity. For such a purpose we have at our disposal a great number of typographical devices, different alphabets, including Arabic figures, in different kinds of type and in different positions, as on, above, and below the line; also signs of operation and punctuation, including the blank space that separates words. The number of such signs may be reduced at the expense of circumlocution, but in any case it is finite, say p. If we take each sign to stand for a different digit in the scale whose radix is p, then each finite sequence of such signs will represent a different integer in that scale, and the cardinal number of such sequences is therefore $\aleph_0$. The specification of any object is such a sequence, and therefore the cardinal number of finite specifications, as also of the objects thereby described, is not more than $\aleph_0$.

Again, any finite integer can be so described; for it is only needful to begin with "one" and then repeat "and one" often enough. Hence the number of objects capable of finite description is not less than $\aleph_0$. More particularly, the number of numbers capable of specification in finite terms is not greater or less than $\aleph_0$.

Now the cardinal number of the real numbers in the continuum is greater than $\aleph_0$. Hence it follows that they cannot all be specified in finite terms. In fact, if one person were able to fix on a real number at random, he could not generally specify to any other person exactly what number he had chosen: he could specify to any desired degree of approximation, but not exactly, unless there were some rule, expressed in finite terms, by which the figure in any decimal place could be determined. Such a rule would distinguish the number in question from all others; but

numbers for which such a rule exists are the exception, just as commensurable and algebraic numbers are the exception.

The failure seems to be in human powers of apprehension; for a graphical method of specification is no more successful than a verbal one. If a point in a line is specified by means of a mark, two points are only distinguishable when their distance apart exceeds a certain limit, which may be reduced by practice and microscopes, but cannot conceivably vanish.

In any other aggregate whose cardinal number exceeds $\aleph_0$ it is similarly impossible to give a finite description of each individual member.

Secondly, it may be proved that, if any aggregate can be well ordered (*wohlgeordnet*) by any finite system of rules R, then each member of it can be specified in finite terms. For, if it is not true that every member can be so specified, there must be a first which cannot, and it may be described in finite terms as "the first in the series determined by the rules R which cannot be described in finite terms." This is absurd. The argument is so suspiciously simple that it will be worth while to discuss it further. By applying it to Cantor's ordinal numbers of the second class, we are led to a similar absurdity, involved in the phrase "the least number of the second class that cannot be specified in finite terms." If we ignore the absurdity for the moment, it is clear that this least number has no immediate predecessor, but must be the limit of a "fundamental series" in Cantor's sense. To construct such a series we could give different values to the finite number n in the phrase "the least number that cannot be specified without using more than n symbols." Take n to be a hundred, and we have the same absurdity as before, but relating now to the numbers of the first class. There can be no "least integer which cannot be specified by the use of at most a hundred letters." Hence even the aggregate of the natural numbers has not the property which characterizes well-ordered aggregates according to Cantor, that *every* aggregate contained in a well-ordered one has a first element. We are then justified in saying that no transfinite aggregate can be well ordered by a finite set of rules. The definition of the term "well ordered" needs to be so drawn up as to avoid the absurdity in question; or, if in its present form it meets this requirement, it would be well to emphasize the fact by a note on the point, calling attention to the way in which the word "every" is to be understood.

Numbers of the second class fall into two categories: those which can, and those which cannot, be specified in finite terms. In the second category there is no lowest member. The cardinal number of the members of the first category is neither less nor greater than $\aleph_0$. The proof

given by Cantor of Theorem D on p. 227 of Vol. XLIX., of the *Mathematische Annalen*, may be applied to the members of the first category, and shows that it is not possible by any *finite* set of rules to make them have a one-to-one correspondence with the numbers 1, 2, 3, ..., since when the law of a fundamental series is specified in finite terms the limit of that series is thereby also specified in finite terms.

Hence it is possible for an aggregate not to be enumerable by a finite set of rules even when its cardinal number is neither less nor greater than $\aleph_0$.

Cantor's two generating principles (*Math. Annalen*, Vol. XLIX., p. 226) do not seem to take us further than the first category of the second class. The well-ordered aggregate is an essential feature of the definitions of the third and higher classes, and until it is put on a proper footing we cannot be sure of the existence of ordinal numbers beyond the second class. Even in the second class the existence of numbers of the second category is open to doubt, since we have no means of forming them.

It is, perhaps, worthy of notice that in certain cases Cantor's way of taking the limit of a fundamental series is not that which at first sight seems most natural. For instance, take the series ω, $\omega^2+\omega$, $\omega^3+\omega^2+\omega$, ..., whose limit is ω^ω according to Cantor's convention. We may pass from the n-th member of the series to the $(n+1)$-th by a permutation which changes the type ω^n to $\omega^{n+1}+\omega^n$, the later terms $\omega^{n-1}+\omega^{n-2}+\ldots+\omega$ being unaffected. By this process we arrive at a limiting order type in which there is no first element; so that it is quite distinct from ω^ω.

In this kind of way it is easy to correlate an order type to each real number of the continuum. For, take the continuum between 0 and 1, and suppose each element of it expressed as a radical fraction in the scale of two. Let the places in which 1 occurs be in order, the a-th, b-th, c-th, ..., and take the correlated order type to be that which is the limit of the series ω^a, $\omega^b+\omega^a$, $\omega^c+\omega^b+\omega^a$, ... in the sense just explained.

These order types are not numbers of the second class, and it is clearly impossible to correlate the elements of the continuum to the ordinal numbers of the second class, without using some of those that are incapable of finite specification, if such exist.*

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* The idea on which this article is founded was mooted by me in a letter to Dr. Hobson in June of this year, and in a very short note sent to the Society in July. A referee reported unfavourably on this note, but revised his opinion after reading Herr König's paper on the question in the last number of the *Math. Annalen*. I have expanded the note by his advice, and added some developments of the idea.

ON THE ARITHMETIC CONTINUUM

By E. W. HOBSON.

[Received and Read November 9th, 1905.]

THE present communication is concerned with an important point relating to the definition of the irrational numbers of the arithmetic continuum which has been recently* raised by Prof. König. A distinction is introduced by König between those elements of the continuum which are capable of being "finitely defined" (*endlich definiert*) and those which are not capable of being defined in finite terms,† and he argues that the former elements form an enumerable aggregate E within the continuum, *i.e.*, an aggregate of cardinal number $\aleph_0$. The existence of those elements of the continuum which do not belong to the aggregate E being, in accordance with König's view, incapable of establishment by means of definitions in finite terms, such existence must be established by the method of postulation; each such element and the continuum itself being regarded as "possible conceptions" (*mögliche Begriffe*), that is, such that the postulation of them does not lead to contradiction. König refers for an analysis of this idea of the continuum to a treatment of the subject which has been given by Hilbert. A refusal to go beyond what can be defined by finite laws can only, in König's view, lead to the denial of the existence of the continuum and of the continuum problem.

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* "Ueber die Grundlagen der Mengenlehre und das Kontinuumproblem," *Math. Annalen*, Vol. LXI., September, 1905.

† König's explanation of what he means by finite definition is as follows:—"Ein Element des Kontinuums soll 'endlich definiert' heissen, wenn wir mit Hilfe einer zur Fixierung unseres wissenschaftlichen Denkens geeigneten Sprache in endlicher Zeit ein Verfahren (Gesetz) angeben können, das jenes Element des Kontinuums von jedem anderen begrifflich sondert, oder—anders ausgedrückt—für ein beliebig gewähltes k die Existenz einer und nur einer zugehörigen Zahl a_k ergibt. Dabei muss aber ausdrücklich betont werden, dass die hierin geforderte 'endliche' begriffliche Sonderung nicht mit der Forderung eines wohldefinierten oder gar endlichen Verfahrens zur Bestimmnng der a_k verwechselt werden darf." The last clause refers to the definition of the continuum employed by König, that, if $a_1, a_2, \ldots, a_k, \ldots$ is an enumerable sequence of positive integers (of type ω), the continuum is the aggregate of objects such as $(a_1, a_2, \ldots, a_k, \ldots)$. The idea that those numbers which are capable of finite definition form an enumerable set, with deductions similar to those made by König, occurred independently to Prof. A. C. Dixon, who communicated his views to me before the publication of König's paper.

König's theory, if well founded, is obviously of great importance in relation to our views as to the fundamental nature of the arithmetic continuum. I propose therefore to examine the distinction which König introduces between those elements of the continuum which belong to the aggregate E, and to the remainder, with a view to determine whether the distinction in relation to definition is well founded or not.

It will here be shewn that the distinction referred to is not a valid one, although it may be true that there exists in the continuum an enumerable set of irrational numbers each of which is capable of a formal definition of a character which is, in a certain aspect, simpler than definitions applicable to irrational numbers in general. The enumerable set referred to contains those irrational numbers, such as e, π, $\sqrt{2}$, ..., which are in common use in arithmetical analysis. It will, however, be shewn, by means of a discussion of the possible modes of formal definition of irrational numbers, that the distinction between the definitions applicable to the special class, and to other irrational numbers, is not of such a character as to justify our speaking of some irrational numbers as capable of finite definition, and of others as not so.

König argues that the irrational numbers capable of being defined in a form which in each case involves only a finite number of letters and symbols (*Buchstaben und Interpunktionszeichen*) must form an enumerable aggregate. Such irrational numbers, together with the rational numbers, he regards as finitely defined, and the other numbers of the continuum he characterizes as, in some special sense, ideal elements, these latter being incapable of finite definition. It must, however, in the first place be remarked that, if it be regarded as essential to a finite definition that it be expressible by means of a finite number of words and symbols, each of which has a definite and *unique* meaning, then no irrational number whatever is capable of such a definition. The simplest possible definition of any of the ordinary irrational numbers, such as π, e, ..., involves the use of a symbol n (or of some form of words equivalent to the use of such symbol), to which no unique meaning is attached, but which is capable of denoting the numbers of the integer sequence 1, 2, 3, ..., of type ω. Thus, for example, a definition of the number e may be given, in which the expression $1+\sum_1^n \frac{1}{n!}$ occurs, and no arithmetical definition of the number e can be given which dispenses with the use in some form of the "variable" n, of which the meaning is not unique.

König's argument may be applied to the aggregate of definitions of those irrational numbers, each of which can be defined in a form involving only a finite number of words and signs, and including the use

of this one "variable" n (1, 2, 3, ...). Assuming that it is possible to arrange such definitions in order, on the basis of the number and arrangement of the letters and signs, including n, which are employed in them, these numbers form an enumerable aggregate of type ω. However difficult it may be to imagine how this ordering of the definitions could be carried out, I shall assume, for the purposes of the present discussion, that this can be done, and therefore that the aggregate E_ι of all such numbers is enumerable. If to E_ι there be added the aggregate E_r of all the rational numbers, we have an aggregate E which I presume to be identical with König's aggregate E, and which I accordingly denote by the same letter.

An irrational number in general is an object which has a definite ordinal relation with the rational numbers in their so-called natural order. This conception of the nature of an irrational number is perhaps most immediately expressed in Dedekind's form of definition, in which an irrational number is regarded as being defined by a certain kind of section (*Schnitt*) of the aggregate of rational numbers; but it is also essential in Cantor's theory of irrational numbers. A particular irrational number is defined when we are supplied with the means of deciding, in respect of any arbitrarily assigned rational number, whether the irrational number ordinally precedes or ordinally succeeds the assigned rational number. Any definition of an irrational number, no matter how such definition be expressed, which satisfies this requisite, I shall speak of as an *adequate* definition of the irrational number in question. It is difficult to understand how any irrational number of which it is impossible to give in some form an *adequate* definition can be said to be defined at all, or to have a determinate ordinal relation with the rational numbers. To have recourse to the method of postulation, in order to provide elements in the continuum, would appear to involve the postulation of the existence of entities which are not clearly distinguishable from one another; for, if determinate ordinal relationship with the rational numbers is supposed to appertain to them, we are unable, in default of adequate definitions, to ascertain what that relationship in any particular case may be.

Confining our attention to the interval (0, 1), as we may do without real loss of generality, it is clear that a definition of a particular irrational number in this interval, which is adequate in the sense explained above, must supply us with directions for calculating, for any prescribed integer n, by an arithmetic process, the first n figures of the decimal representation of the irrational number. The number of steps in the process must be definite for any assigned value of n, but has no upper limit for all values of n. It is also clear that any form of definition which supplies us with

the directions mentioned above is an adequate definition, in that it enables us to assign the order of the number relatively to any arbitrarily assigned number. In particular, the numbers which belong to the enumerable set E are all capable of adequate definition.

Now let us assume that we have an enumerable set of numbers $x_1, x_2, \ldots, x_n, \ldots$ all in the interval (0, 1), and that each one of these numbers has an adequate definition. The set being enumerable, it can be placed in correspondence with the numbers 1, 2, 3, ... of the integer series, so that x_n is identifiable for each value of n.

Let the decimal representation of $x_1, x_2, \ldots$ be expressed as follows :—

$$
\begin{aligned}
x_1 &= \cdot p_{11} p_{12} p_{13} \cdots p_{1r} \cdots, \\
x_2 &= \cdot p_{21} p_{22} p_{23} \cdots p_{2r} \cdots, \\
\cdots & \quad \cdots \quad \cdots \quad \cdots \\
x_n &= \cdot p_{n1} p_{n2} p_{n3} \cdots p_{nr} \cdots, \\
\cdots & \quad \cdots \quad \cdots \quad \cdots .
\end{aligned}
$$

In case any of the numbers are rational numbers not expressed by recurring decimals, we may suppose all the figures to be 0 after some fixed one. On the above hypothesis that x_n is, for any particular value of n, an identifiable number possessing an adequate definition, we are in possession of the means of calculating the digit p_{nr}, for any assigned values of n and r, by a finite process dependent upon those assigned values. In particular, we have the means of calculating p_{nn}, for any assigned value of n. It can now be shewn that an adequate definition can be given of a number which is not contained in the set $x_1, x_2, \ldots$. For example, let us consider the number N represented by $\cdot a_1 a_2 \ldots a_n \ldots$, where $a_n = p_{nn} + (-1)^{p_{nn}}$, and is thus essentially different from p_{nn}: if $p_{nn} = 9$, then $a_n = 8$; and, if $p_{nn} = 0$, then $a_n = 1$; and so on. This number N is adequately defined; for, in accordance with the hypothesis, we have the means of calculating p_{nn}, and thence a_n, and thus $\cdot a_1 a_2 \ldots a_n$, can be calculated in a finite number of steps. Moreover, N cannot be identical with any of the numbers $\cdot x_1, x_2, \ldots$; for it differs from each of them in at least one figure.

Let us now assume that the set of numbers $x_1, x_2, x_3, \ldots$ contains all the numbers of the set E which are in the interval (0, 1).

We have seen that the new number N is also capable of adequate definition; it thus appears that there exist numbers capable of adequate definition which do not belong to the set E. It does not, however, follow that the number N is capable of being defined in a form of words and signs which involves only the use of the one "variable" n, and

which would be such that the definitions of $x_1, x_2, \ldots,$ together with their law of order, were all collected together and merged in one definition. In fact, this cannot be the case; otherwise N would itself belong to E, contrary to hypothesis. It thus appears that any formal definition of the number N must involve a reference to the numbers $x_1, x_2, \ldots$ explicitly, or to some other such sequence of numbers not identical with the integer sequence $1, 2, 3, \ldots$.

We are thus led to the consideration of a type of definition of irrational numbers. A definition of this type contains, besides the "variable" n, a reference to an aggregate $a_1, a_2, a_3, \ldots$ of numbers* which must be regarded as having been already defined and arranged in the order type ω. If we use one general symbol a_n to denote any of the already defined numbers $a_1, a_2, a_3, \ldots,$ in the same sense in which n denotes any of the numbers $1, 2, 3, \ldots,$ then the particular definition is expressible by a finite number of words and unique signs together with the symbols n and a_n; and such a definition is an adequate definition of a particular irrational number. We may, for convenience, speak of the numbers $a_1, a_2, a_3, \ldots$ as the parameters of the definition. All the numbers capable of being defined in forms which involve the use of one and the same set of parameters would, by a repetition of König's argument, form an enumerable set. It does not, however, follow that all numbers capable of a definition of this kind, when various sets of parameters are taken into account, form an enumerable set. In fact, reasoning similar to that employed above may be applied to shew that this cannot be the case. For, let us suppose, if possible, that all the numbers so definable form an enumerable set which can be denoted by $x_1, x_2, x_3, \ldots x_n, \ldots$; then the same reasoning as was applicable when $\{x_n\}$ was taken to be the set E suffices to shew that a number $\overline{N}$, not belonging to the $\{x_n\}$, admits of a definition of the type considered; and thus that there is a contradiction in the assumption that all the numbers definable in this manner are contained in the enumerable set $\{x_n\}$. The proof is, in fact, a modification of one of Cantor's proofs that the arithmetic continuum is not enumerable; and it completes that proof, by shewing that a number can be adequately *defined* which does not belong to the assumed enumerable set.

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* It is easy to construct directly definitions of irrational numbers in which the parameters cannot be taken to be the integers $1, 2, 3, \ldots$. For example, an irrational number M may be adequately defined as follows:—Let the decimal representation of M be such that the n-th figure is identical with the n-th figure in the decimal representation of $2^{1/P_n}$, where P_n denotes the n-th of the prime numbers $2, 3, 5, 7, 11, \ldots$. Since P_n is not expressible in finite terms as a function of n, the parameters of the definition may be taken to be the set of prime numbers.

Any particular number which admits of a definition of the type considered is capable of definition in a variety of ways, involving the use of various sets of parameters. For example, we may, in defining the number N, make use of the rational parameters $\cdot p_{11}, \cdot p_{21}p_{22}, \cdot p_{31}p_{32}p_{33}, \ldots,$ instead of the parameters represented by endless decimals.

An irrational number belonging to the set E has the peculiarity that it is capable of definition in a form which involves the use of the numbers 1, 2, 3, ..., denoted by n, without the employment of any other sequence of the same type ω. As has been shewn, other definitions of irrational numbers can be given which involve the use of sets of parameters $a_1, a_2, \ldots$, which can be denoted by a_n; these parameters being numbers which have already been defined, and must be taken as data in the definitions in question. The definitions of the numbers of E are only that particular case of the more general type of definition which arises when $a_1, a_2, \ldots$ can be identified with 1, 2, 3, ..., which are therefore the parameters used in a definition of one of the numbers of E. The possession of this peculiarity does not, however, justify the use of the term "finite definition" as in any peculiar sense applicable to the numbers of E. These numbers, like the others, are only capable of a definition involving the use of an ordered infinity of numbers (parameters) regarded as data in virtue of previous definition. The term "finitely defined" is, in fact, an expression not free from ambiguity. In one sense every irrational number capable of a definition involving the use of a set of parameters, and which is therefore adequately defined, is finitely defined, since a single variable may be used to denote the parameters. But, as each such definition, whether the irrational number belong to E or not, implies the existence of an infinity of separate entities taken as data, and contains directions for carrying on a process which is essentially endless, it follows that such definition cannot, in a more fundamental sense of the term, be said to be finite. Nevetheless, the process of making an ordinal comparison of the defined number with any assigned rational number is a finite one in which only a finite number of parameters is employed.

It appears therefore that, on the assumption of the possibility of ordering those formal definitions in which a finite number of words, signs, and symbols are employed, in the order type ω, the distinction drawn by König between finitely defined numbers and others not finitely defined is not a valid one, and that the numbers which are capable of formal definition involving only a finite number of words, signs, and symbols do not form an enumerable set. König asserts that it is necessary to admit the existence of elements in the continuum which go beyond "finite laws" in

his sense of the term, and that there exist elements "die wir nicht 'zu Ende' denken können," and which are yet free from contradiction. Of no irrational number can the expression "zu Ende denken" be rightly used if we regard the number as in process of formation, from the point of view of the process itself. The warrant of the uniqueness of the object defined is contained in the definition itself, the determinancy of the process being the test of the adequacy of the definition to supply us with the conception of a distinct object. It is unnecessary that any part of the process have been actually carried out, and all questions as regards the mere practicability of the process are irrelevant. Any definition which is adequate in the sense defined above, no matter how the definition may be expressed, or what implications are contained in it, is sufficient to supply us with the conception of an object which has a definite ordinal relation with the rational numbers.

If we regard the continuum as containing every number capable of adequate definition, in whatever form, and as containing no elements which are to be regarded as in any special sense ideal, it appears that the continuum so conceived has the properties which are essential to its fitness to be regarded as the domain of the real variable. For it is connex, *i.e.*, having given two numbers in it, other numbers ordinally between the two can be defined: this follows from the connexity of the aggregate of rational numbers. Again it is perfect; for, if $x_1, x_2, x_3, \ldots$ be any defined convergent sequence of numbers, each of which is adequately defined, the limit x of the sequence is definable adequately in a form involving the use of $x_1, x_2, x_3, \ldots$ as parameters; and, conversely, any adequately defined number can be exhibited as the limit of a convergent sequence of other numbers, in particular of rational numbers. The continuum so conceived thus possesses the two properties of being connex, and of being perfect, and these are sufficient for the purposes of arithmetic analysis.

Some mathematicians appear to have the impression that there must in some sense exist in the continuum numbers incapable of adequate definition, and only capable of representation as endless decimals in which each figure is to be regarded as quite arbitrarily assigned. In the first place, as has just been shewn, the arithmetic continuum is complete for the purposes for which it exists, without taking into account such nebulous entities as lawless decimals, even if any precise meaning can be assigned to the assertion of their existence. Moreover, in respect of such a lawless infinite decimal, it cannot be rightly asserted that the object exists as a single whole; only so much of it exists at any one time as is represented by the figures which have been actually chosen at that time, and the

number of such figures must be finite. In the case of an adequately defined irrational number, on the other hand, its existence is quite independent of the number of figures in the decimal representation of it which may at any one time have been calculated, or indeed of whether any of them are ever calculated. The process of arbitrarily choosing figures one after the other, without cessation, involves the idea of endlessness only, and this is quite distinct from the truly infinite process which can be regarded as defining a definite object. In the latter case the process regarded from outside is a completed one embodied in the law which dominates it; in the former case it is impossible to regard the process from the outside.

For reasons which I have elsewhere* explained, there appears to be no adequate reason for thinking that any unenumerable aggregate is capable of being normally ordered (*wohlgeordnet*), and this of course includes the case of the continuum. The proof which König has given that the continuum cannot be normally ordered depends, however, on the distinction which he has drawn between those elements which are finitely defined and those which are only ideal, and stands or falls with the validity of this distinction.

* "On the General Theory of Transfinite Numbers and Order Types," *Proc. London Math. Soc.*, Ser. 2, Vol. 3.

ON SOME DIFFICULTIES IN THE THEORY OF TRANSFINITE NUMBERS AND ORDER TYPES

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DR. HOBSON's most interesting paper in the *Proceedings of the London Mathematical Society** raises a number of questions which must be answered before the principles of mathematics can be considered to be at all adequately understood. I do not profess to know the complete answers to these questions, and most of the present paper will consist only of tentative suggestions, made as possibly a step towards the true solutions, not as themselves constituting solutions. With the greater part of Dr. Hobson's paper I find myself in agreement; my purpose, therefore, will not be in the main polemical, but rather to carry the discussion a stage further by introducing certain distinctions which I believe to be relevant and important, and by generalizing as far as possible the difficulties and contradictions hitherto discovered in the theory of the transfinite.

There are two wholly distinct difficulties to be considered in the theory of transfinite cardinal numbers, namely:

(1) The difficulty as to *inconsistent* aggregates (as they are called by Jourdain);

(2) The difficulty as to what we may call Zermelo's axiom.†

These two difficulties do not seem to be clearly distinguished by Dr. Hobson; yet they are, so far as appears, largely independent and of very different degrees of importance. The first leads to definite contradictions, and renders all reasoning about classes and relations, *prima facie*, suspect; while the second merely raises a doubt as to whether a certain much used axiom is true, without showing that any *fundamental* difficulties arise either from supposing it true or from supposing it false. I shall consider these difficulties separately, beginning with the first, because it is more fundamental.

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* Series 2, Vol. 3, pp. 170–188.

† See his "Beweis, dass jede Menge wohlgeordnet werden kann," *Math. Annalen*, Vol. LIX., pp. 514–516. For statements of various forms of this axiom see the third part of this paper.

I.

When Dr. Hobson speaks of the necessity of a *norm* for constituting an aggregate, he appears sometimes to suppose that the *norm* is absent or ill-defined in the case of inconsistent aggregates, at other times to suppose it absent where Zermelo's axiom requires it. But the two cases are, in reality, quite distinct. The doubt as to the truth of Zermelo's axiom arises from the impossibility of discovering a *norm* by which to select one term out of each of a set of classes, while the difficulty of inconsistent aggregates arises from the presence of a perfectly definite *norm* combined with the demonstrable absence of a corresponding aggregate. This suggests that a *norm* is a necessary but not a sufficient condition for the existence of an aggregate; if so, the complete solution of our first set of difficulties would consist in the discovery of the precise conditions which a norm must fulfil in order to define an aggregate. Logical determinateness, it seems, is not sufficient, as Dr. Hobson supposes (p. 173), and the meaning which he attaches to the term *aggregate* (*ib.*) appears to be too wide. This is proved by a perfectly strict argument, which I shall try to state after explaining some ways of generating inconsistent aggregates.

In the first place, since the discussion belongs to symbolic logic, which already possesses technical names for the ideas we require, it is desirable to compare Dr. Hobson's terms with those in current use. What he calls a *norm* is what I call a *propositional function.* A *propositional function of* x is any expression $\phi ! x$ whose value, for every value of x, is a proposition; such is "x is a man" or "$\sin x = 1$." Similarly we write $\phi ! (x, y)$ for a propositional function of two variables; and so on.

In this paper I shall use the words *norm*, *property*, and *propositional function* as synonyms.

The word *aggregate* is used sometimes with an implication of order, sometimes without; I shall use *class* where there is no implication of order, and where there is order I shall consider the *relation* of *before* and *after* which generates the order. This last is necessary because every class which can be ordered at all can be ordered in many ways; so that only the ordering relation, not the class, determines what the order is to be. A *relation* will be used in an extensional sense, *i.e.*, so that two relations are identical provided each holds whenever the other holds. We shall find that a propositional function $\phi ! x$ may be perfectly definite, in the sense that, for every value of x, $\phi ! x$ is determinably true or determinably false, while yet the values of x for which $\phi ! x$ is true do not form a class. And, similarly, we shall find that a propositional function $\phi ! (x, y)$

may be in the same sense definite, without there being any relation R which holds between x and y when and only when $\phi!\,(x, y)$ is true.

In order to eliminate at the outset a number of considerable but irrelevant difficulties, I may point out that the argument we are about to consider does not depend upon this or that view as to the nature of classes and relations. The refutable assumption as to the nature of classes and relations is only this: that a class is always uniquely determined by a *norm* or property containing one variable, and that two norms which are not *equivalent* (*i.e.*, such that, for any value of the variable, both are true or both false) do not determine the same class, with a similar assumption as regards relations. It is in no way essential to the argument to suppose that classes and relations are taken in *extension*, *i.e.*, that two equivalent norms determine the same class or relation. Thus the argument proves that a norm itself is in general not an entity; that is, if we make statements of the form $\phi!\,x$ about a number of different values of x, we cannot pick out an entity ϕ which is the common *form* of all these statements, or is the property assigned to x when we state $\phi!\,x$. In other words, a statement about x cannot in general be analyzed into two parts, x and what is said about x. There is no harm in talking of norms or properties so long as we remember this fact; but, if we forget it, we become involved in contradictions.

The two contradictions first discovered concerned respectively the greatest ordinal and the greatest cardinal.* Of these the cardinal contradiction is the simpler, and lends itself more readily to the removal from arithmetic to logic which I wish to effect for both. I shall therefore consider it first.

The cardinal contradiction is simply this: Cantor has a proof† that there is no greatest cardinal, and yet there are properties (such as "$x = x$") which belong to *all* entities. Hence the cardinal number of entities having such a property must be the greatest of cardinal numbers. Hence a contradiction.

If every logically determinate norm defines a class, there is no escape from the conclusion that there is a cardinal number of all entities. For, in that case, the norm "$x = x$" defines a class, which contains all entities: call this class V. Then the norm "u is similar to V" defines a set of classes which may be taken as being the cardinal number of V,

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* The contradiction concerning the greatest ordinal was first set forth by Burali Forti, "Una questione sui numeri transfiniti," *Rendiconti del Circolo Matematico di Palermo*, 1897. The contradiction concerning the greatest cardinal is discussed in my *Principles of Mathematics*, § 344 *ff*.

† *Jahresbericht der Deutschen Mathematiker-Vereinigung*, Vol. I., 1892, p. 77.

i.e., the greatest cardinal number.* Thus, if every logically determinate norm defines a class, it is impossible to escape the conclusion that there is a greatest cardinal.

The other horn of the dilemma yields more interesting results. Cantor's proof that there is no greatest cardinal may be simplified into the following:—Let u be any class, and R a one-one correlation of all the members of u to some (or all) of the classes contained in u. There are such correlations, since one of them is obtained by correlating each member of u with the class whose only term is that member. Consider now the following norm:—"x is a member of u, but is not a member of the class with which R correlates it." Suppose this norm defines a class w. Then w is omitted from the correlation; for, if w were correlated with x, then, if x is any member of w, it follows from the definition of w that x is not a member of its correlate, *i.e.*, is not a member of w; while, conversely, if x is not a member of w, it is a member of its correlate, *i.e.*, of w. Hence the supposition that w is the correlate of x leads to a contradiction. Hence, in any one-one correlation of all the terms of u with classes contained in u, at least one class contained in u is omitted. Therefore, whatever class u may be, there are more classes contained in u than there are members of u.

We may test this conclusion, in the case of the class of all entities, by constructing, according to the method of the proof of the Schröder-Bernstein theorem, an actual one-one correlation of all terms with all classes, and then considering the class which Cantor shows to be omitted. This process leads us to the consideration of the norm: "x is not a class which is a member of itself." If this norm defines a class w, then the class w is omitted from our correlation. But it is easy to see that this norm does not define a class at all. For, if it defined a class w, we should find that, if w is a member of itself, then it is not a member of itself, and *vice versa*. Hence there is no such class as w. Essentially the same argument may be stated as follows:—If u be any class, then, when $x = u$, the statement "x is not an x" is equivalent to "x is not a u." Hence, whatever class u may be, there is one value of x—namely, u—for which "x is not an x" is equivalent to "x is not a u"; thus there is no class w such that "x is not an x" is always equivalent to "x is a w." Hence, again, this norm does not define a class.

We thus find that, quite apart from any view as to the nature of cardinals, and without any considerations belonging to arithmetic, we can prove that at least one perfectly determinate norm does not define a class.

* I shall consider later Dr. Hobson's objection to this definition.

By the same method, we can easily construct other such norms. Take any class u for which we can correlate all entities to some u's by a one-one correlation. By the method of the proof of the Schröder-Bernstein theorem, construct an actual one-one correlation of all the members of u to all classes contained in u, and then consider the norm: "x is a member of u which is not a member of its correlate according to the correlation in question." This norm does not define a class. Thus from the class of all relations we obtain the norm: "R is not a relation which is a member of its own domain." From the class of all couples we obtain the norm: "v is such that the couple whose members are (1) the class of all entities, (2) v, is not a member of v." Thus it appears that the contradiction dealt with in chapter x. of my *Principles of Mathematics* is a special case of a general type of contradictions which result from supposing that certain propositional functions determine classes, when, in fact, they do not do so. The above method of discovering such propositional functions is not required for proving, when they are discovered, that they are of the sort that do not define classes. In each case it is easy to discover a definite simple contradiction, analogous to that discussed in the above mentioned chapter, which results from supposing that the propositional functions in question do determine classes.

In like manner propositional functions of two variables do not always determine relations. For example, "R does not have the relation R to S" does not determine a relation T between R and S, *i.e.*, it is not equivalent, for all values of R and S, to "R has the relation T to S." For, if it were, substituting T for R and for S, we should have "T does not have the relation T to T," equivalent to "T has the relation T to T," which is a contradiction.

The following contradiction, of an analogous type to those discussed above, shows that a norm or property is not always an entity which can be detached from the argument of which it is asserted. Consider the norm "x does not have any property which it is." If this assigns to x the property θ, then "x has the property θ" is equivalent to "x does not have any property which it is." Hence, substituting θ for x, "θ has the property θ" is equivalent to "θ does not have any property which it is," which is equivalent to "θ does not have the property θ"; whence a contradiction. The solution, in this case, is that properties are not always (if ever) separable entities which can be put as arguments either to other properties or to themselves. Thus, when we speak of properties we are sometimes (if not always) employing an abbreviated form of statement, which leads to errors if we suppose that the properties we are speaking of are genuine entities.

We have thus reached the conclusion that some norms (if not all) are not entities which can be considered independently of their arguments, and that some norms (if not all) do not define classes. Norms (containing one variable) which do not define classes I propose to call *non-predicative;* those which do define classes I shall call *predicative.* Similarly, by extension, a norm containing two variables will be called predicative if it defines a relation; in the contrary case it will be called non-predicative. Thus we need rules for deciding what norms are predicative and what are not, unless we adopt the view (which, as we shall see, has much to recommend it) that *no* norms are predicative.

I come now to Burali-Forti's contradiction concerning the greatest ordinal, and I shall show how this too reduces to a simple logical contradiction resulting from supposing that a certain non-predicative function is predicative.

Burali-Forti's contradiction may be stated, after some modification, as follows:—If u is any segment of the series of ordinals in order of magnitude, the ordinal number of u is greater than any member of u, and is, in fact, the immediate successor of u (*i.e.*, the limit if u has no last term, or the immediate successor of the last term if u has a last term). The ordinal number of u is always an ordinal number, and is never a member of u. But now consider the whole series of ordinal numbers. This is well ordered, and therefore should have an ordinal number. This must be an ordinal number, and yet must be greater than any ordinal number. Hence it both is, and is not, an ordinal number, which is a contradiction.

To generalize this contradiction, put $\phi ! x$ in place of "x is an ordinal," and $f`u$* in place of "the ordinal number of $u$." Then in the case of the ordinals $\phi$ and $f$ are such that, if all the members of $u$ satisfy $\phi$, then $f`u$ satisfies ϕ and is not a member of u. Whenever these two conditions are satisfied for all values of u, one or other of two conclusions follows: namely, either (1) $\phi ! x$ is not a predicative property; or (2), if $\phi ! x$ is predicative and defines the class w, then there must be no such entity as $f`w$. This is proved very simply as follows:—If there is such a class as $w$, and such an entity as $f`w$, then, since every member of w satisfies ϕ, it follows that $f`w$ satisfies $\phi$; but, conversely, $f`w$ must be not a member of w, and must therefore not have the property ϕ, since w consists of all terms having the property ϕ. In the special case of the ordinals, our two alternatives are: (1) the ordinals do not form a class; (2) although they form a class, they have no ordinal number. The second alternative is

* The inverted comma may be read "of." The notation $f`u$ means the same as $f(u)$, but is for several reasons more convenient.

equivalent to the assumption that either the whole series of ordinals is not well ordered, or, if it is well ordered, the dual property "α and β are ordinal numbers, and α is less than β" is non-predicative; so that the series as a whole has no definite type, *i.e.*, no ordinal number. The supposition that the whole series of ordinals is not well ordered can be disproved*; hence we are left with the alternatives that either (1) the property "x is an ordinal number" is non-predicative, or (2), though "x is an ordinal number" is predicative, "x and y are ordinal numbers and x is less than y" is non-predicative.

We have seen that Burali-Forti's contradiction is a particular case of the following:—

"Given a property ϕ and a function f, such that, if ϕ belongs to all the members of u, $f‘u$ always exists, has the property ϕ, and is not a member of u; then the supposition that there is a class w of all terms having the property ϕ and that $f‘w$ exists leads to the conclusion that $f‘w$ both has and has not the property ϕ."

This generalization is important, because it covers all the contradictions that have hitherto emerged in this subject. In the case of the class of terms which are not members of themselves, we put "x is not a member of x" for $\phi ! x$, and u itself for $f‘u$. In this case, owing to the fact that $f‘u$ is u itself, we have only one possibility: namely, that "x is not a member of x" is non-predicative. In other cases, we have two possibilities, and it may often be difficult to decide which of them to choose.

When we have a pair such as ϕ and f above, we can define, in terms of f alone, without introducing ϕ, a series ordinally similar to that of all ordinals, and obtain, as regards this series, a contradiction analogous to Burali-Forti's, provided f satisfies certain conditions. We do this as follows:—Taking any class x, for which $f‘x$ exists, take $f‘x$ as the first term of our series, take the f of the class got by adding $f‘x$ to x as the second term, and so on. Generally, the successor of any term is the f of the class consisting of that term together with all its predecessors and x, and the successor of a class u of terms having no maximum is the f of the class consisting of the whole segment defined by the class u. This gives Cantor's two principles of formation, and we can define the property of occurring in this series by

* This supposition can be disproved (by the generalized form of induction which applies throughout any well-ordered series) by means of the theorem that every segment of the series of ordinals is well ordered. It is not disproved by Jourdain's theorem, that every series which is not well ordered must contain a part of type $*\omega$; for this theorem depends upon Zermelo's axiom, of which the truth is doubtful.

the generalized form of induction.* We may then, subject to certain conditions as to f, substitute for ϕ the property of occurring in the f-series starting from x. If f has the property that, if u is composed of terms of the above series, then $f‘u$ exists and is not a member of u, it will follow that the whole series does not form a class; for, if it did, its f would both be and not be a member of the series. In the particular case of the ordinals, if u is a class of ordinals, $f‘u$ is their immediate successor; the whole series of ordinals can be generated by the above method, starting from 0. In the case of "x is not an x," $f‘x$ is x itself: if we start from any class which is not a member of itself, and proceed by the above method, we obtain a series, like the series of all ordinals,† consisting entirely of classes which are not members of themselves, and the series as a whole does not form a class.

The above considerations point to the conclusion that the contradictions result from the fact that, according to current logical assumptions, there are what we may call *self-reproductive* processes and classes. That is, there are some properties such that, given any class of terms all having such a property, we can always define a new term also having the property in question. Hence we can never collect *all* the terms having the said property into a whole; because, whenever we hope we have them all, the collection which we have immediately proceeds to generate a new term also having the said property. It is probable, in view of the above general form for all known contradictions, that, if ϕ is any demonstrably non-predicative property, we can actually construct a series, ordinally similar to the series of all ordinals, composed entirely of terms having the property ϕ. Hence, if the terms satisfying ϕ can be arranged in a series ordinally similar to a segment of the series of ordinals, it follows that no contradiction results from assuming that ϕ is a predicative property. But this proposition is of very little use, until we know how far the series of ordinals goes; and at present it is not easy to see where this series begins to be non-existent, if such a bull may be permitted.

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* This is done as follows:—A property is *inductive* in the f-series if whenever it belongs to a class u it belongs to the class got by adding $f‘u$ to u, and whenever it belongs to each of a set of classes it belongs to their logical sum, *i.e.*, to the class of members of members of the set. A term belongs to the "f-series starting from x" if it possesses every property which is possessed by x and is inductive in the f-series.

† I owe the proof that this series is well ordered and ordinally similar to the series of all ordinals to Mr. G. G. Berry, of the Bodleian Library.

II.

We have now seen the nature of the contradictions which beset the theory of the transfinite: we have seen that they are not an isolated few, but can be manufactured in any required number by a recipe; we have seen that all of them belong to a certain definite type, and we have seen that none of them are essentially arithmetical, but all belong to logic, and are to be solved, therefore, by some change in current logical assumptions. I propose, in this section, to consider three different directions in which such a change may be attempted. I shall endeavour to set forth the advantages and disadvantages of each of the three, without *deciding* in favour of any one of them.

What is demonstrated by the contradictions we have considered is broadly this: "A propositional function of one variable does not always determine a class."*

In view of this fact, it is open to us, *prima facie*, to adopt one or other of two theories. We may decide that all ordinary straightforward propositional functions of one variable determine classes, and that what is needed is some principle by which we can exclude the complicated cases in which there is no class. In this view, the state of things is like that in the differential calculus, where every commonplace continuous function has a derivative, and only rather complicated and recondite functions have to be excluded. The other theory which suggests itself is that there are no such things as classes and relations and functions as entities, and that the habit of talking of them is merely a convenient abbreviation.

The first of these two theories itself divides into two, according as we hold that what classes have to avoid is excessive size, or a certain characteristic which we may call zigzagginess. Of these, the second is the more conservative, *i.e.*, it preserves more of the theory of the transfinite than the first. Both preserve more of it than does the theory that there are no such things as classes. I shall consider these three theories in the following order, and by the following names:—

A. The zigzag theory.

B. The theory of limitation of size.

C. The no classes theory.

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* Here it is to be understood that the arguments which show that there is not always a class also show that there is not always a separable entity which is the propositional function (as opposed to its value); also that some propositional functions of two variables do not determine a relation either in intension or in extension, if we mean by a relation a separable entity which can be considered apart from related terms.

A. *The Zigzag Theory.*

Each of the three theories can be recommended as plausible by the help of certain *a priori* logical considerations. In the zigzag theory, we start from the suggestion that propositional functions determine classes when they are fairly simple, and only fail to do so when they are complicated and recondite. If this is the case, it cannot be bigness that makes a class go wrong; for such propositional functions as "x is not a man" have an exemplary simplicity, and are yet satisfied by all but a finite number of entities. In this theory, as well as in the theory of limitation of size, we define a *predicative* propositional function as one which determines a class (or a relation, if it contains two variables); thus in the zigzag theory the negation of a predicative function is always predicative. In other words, given any class u, all the terms which are not members of u form a class which may be called the class not-u.

If now $\phi!x$ is a non-predicative function, it follows that, given any class u, there must either be members of u for which $\phi!x$ is false, or members of not-u for which $\phi!x$ is true. (For, if not, $\phi!x$ would be true when, and only when, x is a member of u; so that $\phi!x$ would be predicative.) It thus appears that $\phi!x$ fails to be predicative just as much by the terms it does not include as by the terms it does. Again, given any class u, the property $\phi!x$ belongs either to some, but not all, of the members of u, or to some, but not all, of the members of not-u. This is the zigzag property which gives its name to the theory we are considering. This theory is specially suggested by the argument of Cantor's proof that there is no greatest cardinal. This proof, as we have already seen, constructs a would-be class w by the norm "x is not a member of the class with which it is correlated by the relation R," where R is a relation which correlates individuals with classes. Such would-be classes, as we saw, are very apt to be not classes, and they all have a certain zigzag quality, in the fact that x is a w when x is not a member of its correlate, and is not a w when x is a member of its correlate.

The full development of this theory requires axioms as to the kinds of functions that are predicative. It has the great advantage that it admits as predicative all functions which can be stated simply, and only excludes such complicated cases as might well be supposed to have strange properties.*

The principal objection to this theory, so far as it is at present de-

* For suggestions of a solution more or less on the above lines, see my *Principles of Mathematics*, §§ 103, 104.

veloped, is that the axioms as to what functions are predicative have to be exceedingly complicated, and cannot be recommended by any intrinsic plausibility. This is a defect which might be remedied by greater ingenuity, or by the help of some hitherto unnoticed distinction. But hitherto, in attempting to set up axioms for this theory, I have found no guiding principle except the avoidance of contradictions; and this, by itself, is a very insufficient principle, since it leaves us always exposed to the risk that further deductions will elicit contradictions. The general postulate, that predicative propositional functions must have a certain simplicity, does not lend itself readily to the decision whether this or that propositional function has the requisite degree of simplicity. Nevertheless, since these difficulties are all such as further research might conceivably remove, the theory is not to be rejected wholly, but is rather to be retained as one of those that are possible. Speaking broadly, one may say that it applies better to cardinal than to ordinal contradictions: it deals more readily with such difficulties as that of the class of classes which are not members of themselves than with such difficulties as that of Burali-Forti.

The zigzag theory, in some form or other, is that assumed in the definitions of cardinal and ordinal numbers as classes of classes (if numbers are supposed to be entities). For all these classes of classes, if they are legitimate, must contain as many members as there are entities altogether; hence, if bigness makes classes go wrong, as we suppose in the "limitation of size" theory, cardinals and ordinals so defined will be illegitimate classes. Dr. Hobson has various criticisms on these definitions of cardinals and ordinals; but on the zigzag theory his criticisms can, I think, be all satisfactorily met.

Dr. Hobson says*: "It has been seen that the assumptions that an ordered aggregate possesses a definite order type and a definite cardinal number, which can be treated as objects, lead to the contradiction pointed out by Burali-Forti." This statement seems to me somewhat too sweeping. It is quite open to us to hold every ordered aggregate possesses a definite cardinal number, and that every ordered aggregate which is ordinally similar to a segment of the ordinals in order of magnitude possesses a definite ordinal number. All that Burali-Forti's contradiction forces us to admit is that there is no *maximum* ordinal, *i.e.*, that the function "α and β are ordinal numbers, and α is less than β" and all other functions ordinally similar to this one are non-predicative. In the same way the difficulty of the greatest cardinal is met by denying that the

* P. 176, beginning of No. 5.

defining functions of Cantor's omitted classes are predicative in certain cases. Thus we conclude that in this theory there is a greatest cardinal, but there is no greatest ordinal; in each case contradictions are avoided by regarding certain functions as not predicative.

Dr. Hobson distinguishes two methods of establishing the existence of a class of mathematical entities: the genetic method and the method by postulation. He rejects the former, as regards cardinals and ordinals; but he seems not to perceive that this can only be done by recognizing that there may be no class even where there is a perfectly definite *norm*. From his No. 2 one would suppose that he regards the *norm* as a sufficient condition for the class; yet, later on, he refuses to admit classes which are defined by unimpeachable *norms*. It seems hardly correct to say, as he does: "In the genetic method, as applied to the construction of the whole series of ordinal members, this notion of correspondence plays no part." (No. 6, p. 177.) It is the notion of correspondence which defines the class of relations constituting an ordinal number; this class consists of all the relations which are *like** a certain given relation. "The existence of a number," he truly says, "is constantly inferred from that of a single unique ordered aggregate." (*Ib.*) But there can be no objection to this procedure, unless on the ground that, when P is given, "Q is like P" is not predicative in respect of Q.

It is, of course, very easy to prove, when we have one series of a certain type, that there are an infinite number of series of the same type. To do this we need only substitute other terms for the terms of our series. Suppose, *e.g.*, our series is composed of numbers. We may substitute Socrates for any terms of our series; this will give as many new series of the same type as there are terms in the given series. If our series is infinite, we can obtain $\aleph_0$ series of the same type by merely knocking off terms at the beginning; and so on. Thus, if multiplicity of series of a given type is desired, there is no difficulty in obtaining as many series of the given type as there are entities altogether, *i.e.*, the maximum cardinal number of series of the given type. (For, instead of Socrates, we may substitute any other term not occurring in our series.) Thus it is not the case that the genetic method involves "the setting up of a scale of standards, to which standards no aggregates not consisting of the preceding numbers conform" (No. 6, p. 179), though I do not see what harm there would be if this were the case.

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* I use *like* to mean *ordinally similar to*. For the precise definition cf. *Principles of Mathematics*, § 253.

The same remark applies to the criticism (No. 7, p. 179) of the definition of a cardinal as a class of similar classes.* It is very easy to show that the number of classes similar to the class of numbers from 0 to n is as great as the total number of entities; and, even if no other class were similar to this class, that would not, so far as I can see, constitute any objection. The number $n+1$ would, in that case, be the class whose only member is the class of numbers from 0 to n.

Dr. Hobson explains that his opinion and mine are at variance as to the definition of cardinals because I, unlike him, "regard the activities of the mind as irrelevant in questions of existence of entities" (No. 7, p. 180). This is a philosophical difference, and, like all philosophical differences, it ought not to be allowed to affect the detail of mathematics, but only the interpretation. Mathematics would be in a bad way if it could not proceed until the dispute between idealism and realism has been settled. When a new entity is introduced, Dr. Hobson regards the entity as *created* by the activity of the mind, while I regard it as merely *discerned;* but this difference of interpretation can hardly affect the question whether the introduction of the entity is legitimate or not, which is the only question with which mathematics, as opposed to philosophy, is concerned.

There is another passage in No 7 (p. 179) which calls for explanations, namely, the following:—"Russell objects to the conception of a number as the common characteristic of a family of equivalent aggregates on the ground that there is no reason to think that such a single entity exists with which the aggregates have a special relation, but that there may be many such entities. The mind does, however, in point of fact, in the case of finite aggregates at least, recognize the existence of such single entity, the number of the aggregates; and this is a valid result of our mental activity, subject to the law of contradiction."

In the first place, it is not merely the case that "there *may* be many such entities," but that there demonstrably are as many as there are entities altogether. Given any many-one relation having the property that when, and only when, u and v are similar classes, there is an entity a to which both u and v have the relation S, the converse domain of S (*i.e.*, the terms to which classes have the relation S) will have all the formal properties of cardinal numbers.† Now, if there is one such

* This definition is due to Frege. See his *Grundlagen der Arithmetik*, Breslau, 1884, pp. 79, 85.

† For a development of this point of view see § 2 of "La Logique des Relations," *Revue de Mathématiques*, Vol. VII.

relation as S, it is very easy to prove that there are as many as there are entities altogether; and, if there is no such relation as S, then there are no such entities as cardinal numbers. (There might be cardinal numbers for some classes and not for others, if there was a relation such as S which had some classes in its domain, but no relation such as S which had *all* classes in its domain.)

The supposition that there is no such relation as S is disproved by the fact that the relation of a class to the class of all classes similar to it has the properties we wish S to have. This disproof is rejected by Dr. Hobson, since he considers that it involves improper classes. His position seems to be that, at least in the case of finite aggregates, "the mind" immediately recognizes a certain relation of the sort required. The simplest formal statement of this point of view is, roughly, as follows:—

In beginning cardinal arithmetic we introduce a new indefinable S, concerning which we lay down the indemonstrable properties: *

(1) S is a many-one relation;

(2) Every finite class (and, presumably, some infinite classes) have the relation S to some term;

(3) When, and only when, two finite classes (and, presumably, some pairs of infinite classes) are similar they both have the relation S to the same term;

(4) Things which are not classes do not have the relation S to anything.

The reason that S has to be indefinable and the above propositions indemonstrable is that, if we regard the above propositions as giving a definition of S "by postulates," they do not determine S, since an infinite number of relations (if any) fulfil the above conditions, and every entity will, for a suitable S, be the cardinal number (in respect of that S) of some class which has a cardinal number. Moreover, the recognition by "the mind," which Dr. Hobson speaks of, is precisely the process of introducing an indefinable. It is a process of which, in certain cases, I fully recognize the validity and the necessity; but indefinables and indemonstrables are to be diminished in number as much as possible.† Moreover, in the case supposed, where Dr. Hobson says

* It is probably possible to simplify the statement of these indemonstrables.

† This is merely the truism with which Dedekind begins "Was sind und was sollen die Zahlen," namely: "Was beweisbar ist, soll in der Wissenschaft nicht ohne Beweis geglaubt werden."

that "the mind" recognizes such entities, I am unable to agree: if he said "my mind," I should have taken his word for it; but, personally, I do not perceive such entities as cardinal numbers, unless as classes of similar classes.

B. *The Theory of Limitation of Size.*

This theory is naturally suggested by the consideration of Burali-Forti's contradiction, as well as by certain general arguments tending to show that there is not (as in the zigzag theory) such a thing as the class of all entities. This theory naturally becomes particularized into the theory that a proper class must always be capable of being arranged in a well-ordered series ordinally similar to a segment of the series of ordinals in order of magnitude; this particular limitation being chosen so as to advoid Burali-Forti's contradiction.* We still have the distinction of predicative and non-predicative functions; but the test of predicativeness is no longer simplicity of form, but is a certain limitation of size. In this theory, if u is a class, "x is not a member of u" is always non-predicative; thus there is no such class as not u.

The reasons recommending this view are, roughly, the following:—We saw, in the first part of this paper, that there are a number of processes, of which the generation of ordinals is one, which seem essentially incapable of terminating, although each process is such that the class of all terms generated by it (or a function of this class) ought to be the last term generated by that process. Thus it is natural to suppose that the terms generated by such a process do not form a class. And, if so, it seems also natural to suppose that any aggregate embracing all the terms generated by one of these processes cannot form a class. Consequently there will be (so to speak) a certain limit of size which no class can reach; and any supposed class which reaches or surpasses this limit is an improper class, *i.e.*, is a non-entity. The existence of self-reproductive processes of this kind seems to make the notion of a totality of all entities an impossible one; and this tends to discredit the zigzag theory, which admits the class of all entities as a valid class.

This theory has, at first sight, a great plausibility and simplicity, and I am not prepared to deny that it is the true solution. But the plausibility and simplicity tend rather to disappear on examination.

Let us first recall the generalizations of Burali-Forti's contra-

* This view has been advocated by Jourdain, "On the Transfinite Cardinal Numbers of Well-ordered Aggregates," No. 4, *Phil. Mag.*, January, 1904; also "On Transfinite Cardinal Numbers of the Exponential Form," *Phil. Mag.*, January, 1905.

diction which we obtained in the first part of this paper. The fundamental proposition is: "Given a property ϕ and a function f such that, if ϕ belongs to all the members of any class u, then $f'u$ always exists and has the property ϕ, but is not a member of u, it follows that either ϕ is non-predicative or, if ϕ is predicative and determines the class w, then $f'w$ does not exist."

The theory of limitation of size neglects the second alternative (that $f'w$ may not exist), and decides for the first (that ϕ is not predicative). Thus, in the case of the series of ordinals, the second alternative is that the whole series of ordinals has no ordinal number, which is equivalent to denying the predicativeness of "α and β are ordinal numbers, and α is less than β." The adoption of this alternative would enable us to hold that all ordinals do form a class, and yet there is no greatest ordinal. But the theory in question rejects this alternative, and decides that the ordinals do not form a class. The only case in which this is the only alternative is when $f'u$ is u itself; otherwise we always have a choice.

A great difficulty of this theory is that it does not tell us how far up the series of ordinals it is legitimate to go. It might happen that ω was already illegitimate: in that case all proper classes would be finite. For, in that case, a series ordinally similar to a segment of the series of ordinals would necessarily be a finite series. Or it might happen that ω^2 was illegitimate, or ω^ω or ω_1 or any other ordinal having no immediate predecessor. We need further axioms before we can tell where the series begins to be illegitimate. For, in order that an ordinal α may be legitimate, it is necessary that the propositional function "β and γ are ordinal numbers less than α, and β less than γ" should be predicative. (Here, of course, "less than α" must be replaced by some property not involving α, but such that, if α is legitimate, the property is equivalent to being less than α.) But our general principle does not tell us under what circumstances such a function is predicative.

It is no doubt intended by those who advocate this theory that all ordinals should be admitted which can be defined, so to speak, from below, *i.e.*, without introducing the notion of the whole series of ordinals. Thus they would admit all Cantor's ordinals, and they would only avoid admitting the maximum ordinal. But it is not easy to see how to state such a limitation precisely: at least, I have not succeeded in doing so. The merits of this theory, therefore, would seem to be less than they at first appear to be.

C. *The No Classes Theory.*

In this theory classes and relations are banished altogether.* It is not necessary to the theory to assume that no functions determine classes and relations; all that is essential to the theory is to abstain from assuming the opposite. This is the strong point of the theory we are now to consider: the theory is constituted merely by abstinence from a doubtful assumption, and thus whatever of mathematics it permits us to obtain is indubitable in a way which anything involving classes or relations cannot be. The objections to the theory are (1) that it seems obvious to common sense that there are classes; (2) that a great part of Cantor's theory of the transfinite, including much that it is hard to doubt, is, so far as can be seen, invalid if there are no classes or relations; (3) that the working out of the theory is very complicated, and is on this account likely to contain errors, the removal of which would, for aught we know, render the theory inadequate to yield the results even of elementary arithmetic.

To explain fully how this theory is to be developed would take too much space. Some of its main points may, however, be briefly set forth.

Instead of a function $\phi ! x$, where the notation inevitably suggests the existence of something denoted by "ϕ," we proceed as follows:—Let p be any proposition, and a a constituent of p. (We may say broadly that a is a constituent of p if a is mentioned in stating p.) Then let "$p\,\dfrac{x}{a}$" denote what p becomes when x is substituted for a in the place or places where a occurs in p. For different values of x this will give us what we have been accustomed to call different values of a propositional function. In place of ϕ we have now two variables, p and a: in respect to the different values of $p\dfrac{x}{a}$, we may call p the prototype and a the *origin* or *initial subject*. (For a may be taken as being, in a generalized sense, the subject of p.) Consider now such a statement as "$p\dfrac{x}{a}$ is true for all values of x." Let b be an entity which is not a constituent of p, and put $q = p\,\dfrac{b}{a}$; then "$q\,\dfrac{x}{b}$ is true for all values of x" is

* It must be understood that the postulate of the existence of classes and relations is exposed to the same arguments, *pro* and *con.*, as the existence of propositional functions as separable entities distinct from all their values. Thus, in the theory we are considering, anything said about a propositional function is to be regarded as a mere abbreviation for a statement about some or all of its values.

equivalent to "$p\frac{x}{a}$ is true for all values of x." Thus, subject to a certain reservation, the statement "$p\frac{x}{a}$ is true for all values of x" is independent of the initial subject a, and thus may be said to depend only upon the *form* of p.* Statements of this sort replace what would otherwise be statements having propositional functions for their arguments. For example, instead of "ϕ is a unit function" (*i.e.*, "There is one, and only one, x for which $\phi ! x$ is true"), we shall have "There is an entity b such that $p\frac{x}{a}$ is true when, and only when, x is identical with b." There will not now be any such entity as the number 1 in isolation; but we shall be able to define what we mean by "One, and only one, proposition of the type $p\frac{x}{a}$ (for a given p and a) is true." Instead of saying "The class u is a class which has only one member," we shall say (as above) "There is an entity b such that $p\frac{x}{a}$ is true when, and only when, x is identical with b." Here the values of x for which $p\frac{x}{a}$ is true replace the class u; but we do not assume that these values collectively form a single entity which is the class composed of them.

There is not much difficulty in re-wording most definitions so as to fit in with the new point of view. But now the existence theorems become hard to prove. We can manufacture enough different propositions to show what is now equivalent to the existence of ω and $\aleph_0$, though the process is cumbrous and artificial. We shall be able, by continuing a similar process, to prove the existence of various transfinite ordinal types. But we shall not be able to prove the existence of *all* the usual ordinal types. I do not know at what point the series begins to be non-existent; but I cannot at present, in this method, prove the existence of ω_1 or $\aleph_1$, which must therefore be considered for the moment as undemonstrated.

I hope in future to work out this theory to the point where it will appear exactly how much of mathematics it preserves, and how much it forces us to abandon. It seems fairly clear that ordinary arithmetic, analysis, and geometry, and, indeed, whatever does not involve the later

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* The reservation is merely that the initial subject must not occur in the prototype except in the places which we wish to be variable. For example, if our prototype is "$3 > 2$," and our initial subject is 3, the substitution of x for 3 gives "$x > 2$." But, if we now take 2 as our initial subject, so that our prototype is "$2 > 2$," the substitution of x gives "$x > x$," which is not the propositional function we want.

transfinite numbers, can be stated, though in a roundabout and difficult way, without the use of classes and relations as independent entities. A certain amount, also, of transfinite arithmetic can be preserved; but it is not easy to discover how much. The theory is safe, but drastic; and, if, in fact, there are classes and relations, it is unnecessarily difficult and complicated. For the present, therefore, it may be accepted as one way of avoiding contradictions, though not necessarily *the* way.

III.

I come now to the second of our difficulties, namely, the doubt as to the truth of Zermelo's axiom. This is dealt with by Dr. Hobson in his Nos. 10 and 11, with which I find myself in complete agreement.*

All that I wish to do is to state the question in various forms, and to point out some of its bearings. I shall assume the existence of classes and relations, for the sake of simplicity of statement. The difficulty is of a different kind, and is more easily apprehended by this form of statement.

Zermelo's axiom asserts the possibility of picking out one from each of the classes contained in a given class (excepting the null class). It has hitherto been commonly assumed by mathematicians, and Zermelo has the merit of explicitly mentioning the assumption. The axiom may be stated as follows:—" Given any class w, there is a function $f‘u$ such that, if u is an existent† class contained in w, then $f‘u$ is a member of u." That is, the axiom asserts that we can find some rule by which to pick out one term from each existent class contained in w. The axiom may also be stated: " Given a set k of all existent classes contained in a certain class w, there is a many-one relation R, whose domain is k, which is such that, if u is a member of k, the term to which u has the relation R is a member of u." The axiom can be stated in a form which does not involve classes, functions, or relations, but I shall not give this form of statement, as its complication makes it almost unintelligible.

A simple illustration may serve to show the nature of the difficulty as regards this axiom, and to introduce the analogous " multiplicative axiom." Given $\aleph_0$ pairs of boots, let it be required to prove that the number of boots is even. This will be the case if all the boots can be divided into

* Though I do not agree with his special criticism of Mr. G. H. Hardy in No. 12, according to which the second figure in Mr. Hardy's sequences " would have indefinitely great values for all numbers β of the second class, and thus that for sufficiently great ordinal numbers of the second class the corresponding sequences can have no existence."

† An *existent* class is a class having at least one member.

two classes which are mutually similar. If now each pair has the right and left boots different, we need only put all the right boots in one class, and all the left boots in another: the class of right boots is similar to the class of left boots, and our problem is solved. But, if the right and left boots in each pair are indistinguishable, we cannot discover any property belonging to exactly half the boots. Hence we cannot divide the boots into two equal parts, and we cannot prove that the number of them is even. If the number of pairs were finite, we could simply choose one out of each pair; but we cannot choose one out of each of an infinite number of pairs unless we have a *rule* of choice, and in the present case no rule can be found.

The problem involved in the above illustration raises grave difficulties in regard to many elementary theorems about multiplication of cardinals. Multiplication has been defined as follows by Mr. A. N. Whitehead:—*

Let k be a set of classes no two of which have any common terms. Then we define the "multiplicative class of k" (denoted by $\times$'k) as the class formed by picking one and only one term out of each class belonging to k, and doing this in all possible ways. That is, one member of $\times$'k is a class consisting of one member of each class belonging to k. Then the number of terms in $\times$'k is defined to be the product of the numbers of the various classes belonging to k. This definition is perfectly satisfactory when the number of classes which are members of k is finite, and also when each class which is a member of k has some peculiar term (for example, if each is given as a well-ordered series, and we can pick out the first term). But in other cases it is not obvious that there is any rule by which we can pick out just one term of each member of k, and therefore it is not obvious that $\times$'k has any members at all. Hence, as far as the definition shows, the product of an infinite number of factors none of which is zero might be zero. Thus, in the case of the boots, we wished to pick out one boot from each pair, but we could find no rule by which this was to be done.

What is required is not that we should actually be able to pick out one term from each class which is a member of k, but that there should be (whether we can specify it or not) at least one class composed of one term from each member of k. It there is one, there must be many, unless all the members of k are unit classes; for, given one such class, if u is a member of k, and x is the member of u which is picked out, we can substitute for x any other member of u—say y—and we still have a member of $\times$'k. Thus the axiom we need may be stated: "Given a

* *American Journal of Mathematics*, October, 1902.

mutually exclusive set of classes k, no one of which is null, there is at least one class composed of one term out of each member of k."

This axiom is more special than Zermelo's axiom. It can be deduced from Zermelo's axiom; but the converse deduction, though it may turn out to be possible, has not yet, so far as I know, been effected. I shall call this the *multiplicative* axiom.

The multiplicative axiom has been employed constantly in proofs of theorems concerning transfinite numbers. It is open to everybody, as yet, to accept it as a self-evident truth, but it remains possible that it may turn out to be capable of disproof by *reductio ad absurdum.* It may also, of course, be capable of proof, but that is far less probable. A class of classes of which this axiom holds may conveniently be called a *multipliable* class of classes.

The above axiom is required for identifying the two definitions of the finite. We may define a finite cardinal number

(*a*) As a cardinal number which obeys mathematical induction starting from 0;

(*b*) As a cardinal number such that any class which has that number contains no part similar to itself.

We will for the present call any number of the kind (*a*) an *inductive* number, and any number of the kind (*b*) a *finite* number. Then it is easy to prove that all inductive numbers are finite; that every class whose number is infinite contains a part whose number is $\aleph_0$ (where $\aleph_0$ is defined as the number of inductive numbers), and *vice versa;* and that, if the number of classes contained in a finite class is always finite, then all finite numbers are inductive numbers. But, so far as I know, we cannot prove that the number of classes contained in a finite class is always finite, or that every finite number is an inductive number.*

The multiplicative axiom is also required for proving that the number of terms in α sets of β terms is $\alpha \times \beta$, *i.e.*, for connecting addition and multiplication. We cannot even prove, without this axiom, that the number of terms in α sets of β terms is always the same. Similarly, we cannot prove that the product of α factors each equal to β is β^α (taking

* Burali-Forti has shown that the two definitions of the finite can be identified if we assume the following axiom:—"If u is any class of existent classes, the number of members of u is less than or equal to the number of members of members of u." ("Le Classi finite," *Proceedings of the Accademia Reale delle Scienze di Torino*, 1896–7.) This axiom leads at once to the result that the number of classes contained in a finite class must be finite, whence the conclusion follows, as above indicated. The axiom as it stands is untrue: it is necessary to assume that the classes are mutually exclusive, or something analogous. Whether it will then give the desired result I do not know.

Cantor's definition of exponentiation),* or even that it is always the same number.

And in the case of the $\aleph_0$ pairs of boots we cannot prove that the number of boots is $\aleph_0$ (*i.e.*, $\aleph_0+\aleph_0$), except in the case where we can distinguish right and left boots.

The existence of $\times$ 'k can be proved whenever any method exists of picking out one term from each member of k. If, for example, all terms which are members of members of k belong to some one well-ordered series, we get a member of $\times$ 'k by picking out the first terms of the various members of k (k being assumed to be a set of mutually exclusive existent classes). It does not follow that $\times$ 'k exists when every member of k can be well ordered: for there will always be many ways of well ordering each member of k, and we need some rule for picking out one, in each case, of the various possible ways of well ordering each member. That is, we need a term of the multiplicative class of the class of which a single member is the class of relations by which a single member of k is well ordered.

If k is any set of mutually exclusive existent classes, and if we form another class k' by substituting for every member u of k the class (u') of all existent classes contained in u, then k' is a set of mutually exclusive existent classes, and $\times$ 'k' exists, since k is a member of $\times$ 'k' (because each u is a member of its u').

Assuming that k is a set of mutually exclusive existent classes, there are certain cases in which the existence of $\times$ 'k can be proved, because there is some structure which enables us to pick out particular terms from members of k. Such, for example, is the following case:— Suppose there is some one-many relation P, such that each term of k consists of all the terms to which some term of the domain of P has the relation P, and suppose further that every term of the domain of P has the relation P to itself: then the domain of P is a member of the multiplicative class of k.

If k is a set of mutually exclusive existent classes, $\times$ 'k exists when, and only when, there is a one-one relation S whose domain is k, and which relates each class u, belonging to k, to a member of u; for, when this condition is satisfied, the converse domain of S is a member of $\times$ 'k; and, given a term of $\times$ 'k, the relation of members of k to the corresponding members of the given term of $\times$ 'k is an S fulfilling the above conditions. Another way of stating the same thing is that $\times$ 'k exists when, and only when, there is some function f'u such that,

* *Math. Annalen*, Vol. XLVI., § 4.

if u is a member of k, $f'u$ is a member of u; for the f's of the various members of k make up a term of k. A *sufficient* condition for the existence of $\times' k$ whenever k is a set of mutually exclusive existent classes is that there should exist a function $f'u$ such that, if u is any existent class, $f'u$ is a member of u. This is equivalent to Zermelo's axiom,* and it is not, so far as I know, a *necessary* condition for the existence of $\times' k$ in all such cases.

Zermelo's axiom is a generalized form of the multiplicative axiom, and is interesting because he has shown † that, if it is true, then every class can be well ordered. Since it is doubtful whether all classes obey Zermelo's axiom, we may define a Zermelo class as one which does obey the axiom; that is, w is a Zermelo class if there is at least one many-one relation R such that the domain of R consists of all existent classes contained in w, and if u has the relation R to x, then x is a member of u. That is, a class w is a Zermelo class if there is a method of correlating each existent class contained in w with one of its members. Zermelo proves that any class w for which this holds can be well ordered. The converse is obvious; for, if w is well ordered, we correlate each existent class u, contained in w, with the first term of u, which gives a relation R satisfying the above conditions. Hence Zermelo's axiom holds of those classes which can be well ordered, and of no others.

By applying his axiom to the class of all entities, we find that, if it holds universally, there must be a function $f'u$ such that, if u is any existent class, then $f'u$ is a member of u. Conversely, if there is such a function, Zermelo's axiom is obviously always satisfied. Hence, if there is a class of all entities, his axiom is equivalent to: "There is a function $f'u$ such that, if u is any existent class, $f'u$ is a member of u."

I think that Zermelo's axiom, applied in its functional form, and without the assumption that there are classes or relations, leads to the result that any propositional function only satisfied by terms of one type is such that all the terms satisfying it can be well ordered. If it should appear, on other grounds, that this is not always true, it would follow that Zermelo's axiom, in its functional form, is false. Whether or not it is true in the form in which it applies only to classes is a question which requires for its answer a previous decision as to what propositional functions are predicative: the more we restrict the notion of *class* the more likely this form of Zermelo's axiom is to be true, and the less

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* Assuming that there is a class of all entities. But, if there is no such class, we only have to adopt the statement of Zermelo's axiom which does not assume that there are classes.

† *Math. Ann.*, Vol. LIX., pp. 514–516.

information it gives us. To discover the conditions subject to which Zermelo's axiom and the multiplicative axiom hold would be a very important contribution to mathematics and logic, and ought not to be beyond the powers of mathematicians.

It is easy to see that, if w is any Zermelo class, and k is a set of mutually exclusive existent classes which between them contain all the terms w and no more, then k is a multipliable class of classes. For every member of k is an existent class contained in w; hence, if we pick out one term from each existent class contained in w, we incidentally pick out one term from each member of k. Thus the universal truth of Zermelo's axiom involves the universal truth of the multiplicative axiom. The converse, so far as I know, has not been proved, and may or may not be true.

It should be observed that, both in the case of Zermelo's axiom and in that of the multiplicative axiom, what we are primarily in doubt about is the existence of a norm or property such as will pick out one term from each of our aggregates; the doubt as to the existence of a *class* which will make this selection is derivative from the doubt as to the existence of a norm.

The problem concerned in such cases is like that of the "lawless" decimal, which reduces to the problem of the "lawless" class of finite integers. If we consider all the classes that can be formed of finite integers, it seems at first sight obvious that many will consist of a perfectly haphazard collection, not definable by any formula. But this is open to doubt. It would seem that, as Dr. Hobson urges, an infinite aggregate requires a norm, and that such haphazard collections as seem conceivable are really non-entities. In the case of Cantor's "proof" that there are more classes of finite numbers than there are finite numbers, it is shown that no one denumerable set of formulæ will cover all classes of finite numbers; but the class shown to be left over in each case is defined by a formula in the process of showing that it is left over. Thus this process gives no ground for thinking that there are classes of finite numbers which are not definable by a formula.

To sum up: there are two analogous axioms—Zermelo's and the multiplicative axiom—which have been habitually employed by mathematicians in reasonings about the transfinite, but which, most likely, are not true without some restriction. Without them, we cannot, so far as at present appears, identify the two definitions of the finite, or establish the usual relations of addition, multiplication, and exponentiation. If Zermelo's axiom were true, every class would be well ordered, and also, I think, every aggregate of terms possessing some property. But in this

respect the problem considered in our second part is dependent upon that considered in our first part.

The general position advocated in the foregoing paper may be briefly stated as follows:—

When we say that a number of objects all have a certain property, we naturally suppose that the property is a definite object, which can be considered apart from any or all of the objects which have, or may be supposed to have, the property in question. We also naturally suppose that the objects which have the property form a *class*, and that the class is in some sense a new single entity, distinct, in general, from each member of the class. Both these natural suppositions can be proved, by arguments so short and simple that they scarcely admit a possibility of error, to be at any rate not *universally* true. We may, in view of this fact, adopt one of two courses: we may either decide that the assumptions in question are *always* false, or endeavour to find conditions subject to which they are true, these conditions being such as to exclude the cases where the falsehood of the assumptions can be proved. The latter course has the advantage of being more consistent with common sense, and of preserving more of Cantor's work; but it has, as yet, the disadvantage of great uncertainty and artificiality in detail, owing to the absence of any broad principle by which to decide as to which functions are predicative. The former course, in practice, merely involves abstaining from the doubtful assumptions, and does not commit us to the view that they are false; it is therefore, so long as any doubt subsists, the prudent plan to pursue the former course as far as possible. It appears on examination that, without supposing either of the suspected assumptions to be *ever* true, we can construct ordinary mathematics and most of the theory of the transfinite; and in this development we meet with no contradictions, so far as is known at present. Whether it is possible to rescue more of Cantor's work must probably remain doubtful until the fundamental logical notions employed are more thoroughly understood. And whether, in particular, Zermelo's axiom is true or false is a question which, while more fundamental matters are in doubt, is very likely to remain unanswered. The complete solution of our difficulties, we may surmise, is more likely to come from clearer notions in logic than from the technical advance of mathematics; but until the solution is found we cannot be sure how much of mathematics it will leave intact.

[*Note added February 5th*, 1906.—From further investigation I now feel hardly any doubt that the no-classes theory affords the complete solution of all the difficulties stated in the first section of this paper.]

ON THE HESSIAN CONFIGURATION AND ITS CONNECTION WITH THE GROUP OF 360 PLANE COLLINEATIONS

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THE Hessian configuration is the name given to a set of nine points in a plane which lie three by three on twelve straight lines. Its most familiar form is that given by the nine inflexions of a real cubic curve. The object of the first part of this memoir is to establish the existence of the configuration and to deduce its principal properties, especially the nature of the group of collineations for which the configuration is invariant, from a purely geometrical point of view. This group in its abstract form and in its analytical form as a group of linear substitutions in three variables has formed the subject of several investigations. The earliest is due to M. Jordan (*Traité des Substitutions*, pp. 302–305); while one of the most recent is given by Herr Weber (*Lehrbuch der Algebra*, Vol. II., pp. 400–410). None of these investigations with which I am acquainted, however, approaches the problem from the point of view which most naturally presents itself, namely, as a question of pure projective geometry. This is the point of view here taken, and it is contended that both the properties of the configuration and the nature of the group thereby appear in a clearer light.

In the second part of the memoir it is shewn that, starting from the Hessian configuration, there may be constructed a very remarkable configuration of 45 points of which the following are some of the properties:—

The line joining any two of the points passes through either one, two, or three others. The points lie 5 by 5 on 36 lines, 4 by 4 on 45 lines, and 3 by 3 on 120 lines. From the 45 points just 10 Hessian configurations can be formed, each two of which have just one of the points in common.

Finally, it is shown that such a configuration is invariant for a group of 360 collineations, which is simply isomorphic with the alternating group on six symbols.

The existence of such a group of collineations, which was established by H. Valentiner (*Die endelige Transformations-Gruppen Theorie*, 1889) on analytical grounds, is here shewn to follow from purely geometrical considerations.

As regards notation, some mode of representing collineations in a plane is necessary. Any such collineation is completely determined by the positions of the points A', B', C', D' into which it changes four given points A, B, C, D, no three of which are collinear. The symbol

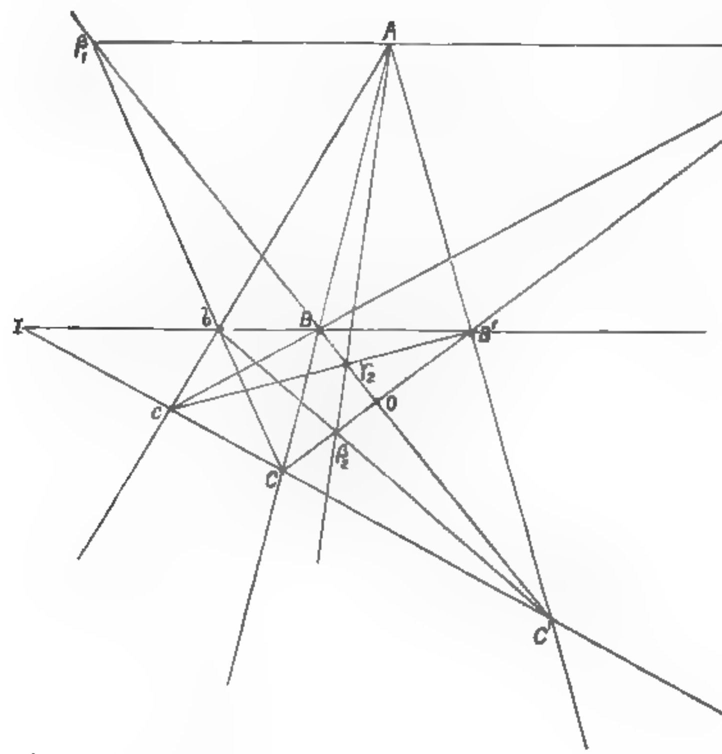
$$\begin{pmatrix} A, & B, & C, & D \\ A', & B', & C', & D' \end{pmatrix}$$

therefore completely specifies such a collineation. If the collineation is one of finite order, it will permute sets of points cyclically. Its effect on such sets of points may be represented in the usual way by the symbol

$$(PQR \ldots T).$$

I.

1. Take four points B, C, B', C', no three of which lie in a line. Let BC and $B'C'$ meet in A. Through A draw Abc, meeting BB' and CC' in b and c. Denote by β_1 and γ_1 the points in which bC and cB meet BC'



and CB' respectively; and by β_2 and γ_2 the points in which bC' and cB' meet $B'C$ and $C'B$.

To the pencil of lines through A, of which Abc is one, correspond the two projective ranges described by b and c on BB' and CC'. At the same time β_1, γ_1, β_2, γ_2 describe projective ranges on BC', CB', $B'C$, $C'B$ respectively.

Particular positions of the four points β_1, γ_1, β_2, γ_2 are determined by the table :—

Abc	β_1	γ_1	β_2	γ_2
ABC	B	C	O	O
$AB'C'$	O	O	B'	C'
AI	C'	B'	C	B

where O is the intersection of BC' with $B'C$, and I that of BB' with CC'. The projective ranges β_1, γ_1 on the lines BC' and CB', having a self-corresponding point, viz., O, are in perspective. Also in two particular positions, viz., those corresponding to the positions ABC and AI of Abc, $\beta_1\gamma_1$ passes through A. Hence the projective ranges β_1, γ_1 are in perspective with respect to A. Similarly, the projective ranges β_2, γ_2 are in perspective with respect to A.

Again β_2 and γ_1 give projective ranges on the line CB'. To the three positions

$$O,\ B',\ C$$

of β_2 there correspond the three positions

$$C,\ O,\ B'$$

of γ_1. Hence the projective transformation of CB' which changes the first of these projective ranges into the second is

$$\begin{pmatrix} OB'C \\ COB' \end{pmatrix}.$$

If this projective transformation is repeated three times, it leads to

$$\begin{pmatrix} OB'C \\ OB'C \end{pmatrix}$$

which, leaving three distinct points of the line unchanged, leaves every point unchanged. Hence the projective transformation is a projective transformation of order 3.

It has therefore two distinct (imaginary if B, C, B', C' are real) fixed points. The two projective ranges β_2, γ_1 on CB' have therefore two distinct self-corresponding points (which are imaginary when the four original points are real). Denote them by B'' and B''_0. Similarly the two projective ranges γ_2, β_1 on $C'B$ have two self-corresponding points C'' and C''_0. Also, since $A\beta_1\gamma_1$, $A\beta_2\gamma_2$ are straight lines, to each of B'' and B''_0 there must correspond one of C'' and C''_0, such that the lines joining the corresponding pairs pass through A; say $AB''C''$ and $AB''_0C''_0$.

To the positions B'', C'' of β_2, γ_1 and γ_2, β_1 corresponds a definite position $AB'''C'''$ of Abc: and to the positions B''_0, C''_0 there corresponds another definite position $AB'''_0C'''_0$ of Abc.

Consider now the nine points

$$A,\ B,\ C,\ B',\ C',\ B'',\ C'',\ B''',\ C'''.$$

It follows immediately from the figure that

$$\begin{gathered} ABC,\quad AB'C',\quad AB''C'',\quad AB'''C''', \\ BB'B''',\quad BB''C''',\quad BC'C'', \\ CC'C''',\quad CC''B''',\quad CB'B'', \\ B'C''C''',\quad C'B''B''' \end{gathered}$$

are straight lines. In other words, the straight line joining any two of these nine points passes through a third; or the nine points lie three by three on 12 straight lines. The same is also true of the nine points

$$A,\ B,\ C,\ B',\ C',\ B''_0,\ C''_0,\ B'''_0,\ C'''_0,$$

where the last four points are distinct from the last four of the previous set.

2. The existence of a Hessian configuration is thus proved, and it is shown that the given construction leads to one or the other of two distinct configurations. The next point to consider is how these two configurations are related. With this object, consider the effect of the projective transformation of order 2 defined by

$$\begin{pmatrix} B\ C\ B'C' \\ C'B'C\ B \end{pmatrix}$$

on either of them. This transformation permutes the five points A, B, C, B', C' among themselves. It leaves the line CB' unchanged, and effects on it the projective transformation of order 2

$$\begin{pmatrix} OB'C \\ OCB' \end{pmatrix}.$$

The previously considered transformation of order 3 on CB', viz.,

$$\begin{pmatrix} OB'C \\ COB' \end{pmatrix},$$

of which B'' and B''_0 are the fixed points, is changed into its inverse by the transformation of order 2; for obviously

$$\begin{pmatrix} OB'C \\ OCB' \end{pmatrix}\begin{pmatrix} OB'C \\ COB' \end{pmatrix}\begin{pmatrix} OB'C \\ OCB' \end{pmatrix} = \begin{pmatrix} OB'C \\ B'CO \end{pmatrix}.$$

Hence B'' and B''_0 are permuted by the projective transformation of the plane of order 2. Similarly, C'' and C''_0 are permuted by it. But when seven of the points of a Hessian configuration are known the remaining two can be determined by drawing straight lines. Hence the projective transformation which changes

$$A,\ B,\ C,\ B',\ C',\ B'',\ C''$$

into

$$A,\ B,\ C,\ B',\ C',\ B''_0,\ C''_0$$

necessarily changes the pair B''', C''' into the pair B'''_0, C'''_0.

The two Hessian configurations are therefore transformed each into the other by the projective transformation

$$\begin{pmatrix} B\ C\ B'C' \\ C'B'C\ B \end{pmatrix}.$$

Returning now again to the construction, it involves not only four points, but also a particular pair of lines through them. Three such pairs may be drawn, viz., BC, $B'C'$, meeting in A; BB', CC', meeting in I; $B'C$, BC', meeting in O. If B, C, B', C' belong to a Hessian configuration of nine points, either A, I, or O must also belong to the configuration. Suppose, in fact, that neither A nor I belongs to it. Then, besides B, C, B', C', the configuration has one distinct point on each of the lines BC, $B'C'$, BB', CC'; so that there is only one remaining point. Hence O, in which $B'C$, BC' intersect, must be a point of the configuration. Now I and O do not belong to the two distinct configurations already determined which contain A. Hence in all just six distinct Hessian configurations can be constructed to contain any given four points, no three of which lie in a line. Moreover the set of 24 plane collineations which permute among themselves the four points B, C, B', C' also obviously permute A, I, O; and it has been seen that one of these collineations which leaves A unchanged permutes the two corresponding Hessian configurations. Therefore the six distinct Hessian configurations which can be constructed to contain four given points (no three of which lie on a straight line) are transitively permuted among themselves by the group of 24 collineations which permutes the four points.

But any four points of a plane, no three of which are collinear, can be projected into any other four. Hence any one Hessian configuration can be projected into any other. It follows from this that the distribution of the nine points on twelve lines given on p. 57 is quite general.

[*February*, 1906.—The six Hessian configurations each of which contains the four points B, C, B', C', while each also contains either A, I, or O, contain in all just 12 other points, each of which occurs in two of the configurations. These 12 points lie three by three on 8 lines, two of which pass through each of the four points B, C, B', C'. The two lines through B, containing 6 of the 12 points, are the two (imaginary) fixed lines of the collineation which, leaving B unchanged, permutes B', C, C' cyclically; and the 6 points are the points in which these two fixed lines of the collineation meet $B'C$, CC', and $C'B'$. That a set of 12, and not 24, points arises in this way follows from the fact that the (imaginary) fixed lines of the collineations which leave B (or B') unchanged and permute cyclically B', C, C' (or B, C, C') meet CC' in the same pair of points.]

3. From the twelve lines just four sets of three may be formed such that each set contains all nine points. These sets are :—

$$\begin{array}{lll} ABC, & B'C''C''', & C'B''B''' ; \\ AB'C', & BB''C''', & CB'''C'' ; \\ AB''C'', & BB'B''', & CC'C''' ; \\ AB'''C''', & BC'C'', & CB'B''. \end{array}$$

The lines of any one set intersect those of any other in a point belonging to the configuration. The three lines of any one set intersect in three points which do not belong to the configuration. There thus arises a set of twelve points, whose relations to the configurations will be determined.

The collineation of order 2 defined by

$$\begin{pmatrix} BCB'C' \\ CBC'B' \end{pmatrix}$$

is a perspective with A for its vertex (or fixed point), and IO in the figure for its axis or fixed line. Since it leaves A unchanged and permutes B, C, B', C', it must either leave unchanged or permute the two configurations which have these five points in common. Now neither B_0'', C_0'', B_0''' nor C_0''' lies on AB''. Hence the collineation leaves unchanged each of the configurations. It therefore permutes B'' with C'' and B''' with C'''. Similarly, there is a collineation of order 2 with any other one of the nine points for its vertex which permutes the remaining eight points of the configuration in pairs. The axes of these perspectives corresponding to

$$A,\ B,\ C,\ B',\ C',\ B'',\ C'',\ B''',\ C''',$$

as vertices will be denoted by

$$a,\ b,\ c,\ b',\ c',\ b'',\ c'',\ b''',\ c'''.$$

The perspective of order 2 with A for its vertex and a for its axis—say Aa —leaves four of the twelve lines, viz., ABC, $AB'C'$, $AB''C''$, $AB'''C'''$, unchanged, and permutes the remaining pairs which belong to the four sets, viz.,

$$B'C''C''',\ C'B''B''';\quad BB''C''',\ CB'''C'';$$

$$BB'B''',\ CC'C''';\quad BC'C'',\ CB'B''.$$

So each other perspective such as Bb leaves unchanged one of the twelve lines in each of the four sets of three and permutes the remaining two. But the only points which are unchanged for a perspective other than its vertex are the points on its axis. Hence the set of twelve points which arise from the intersections of the twelve lines lie four by four on the set of nine lines

$$a,\ b,\ c,\ b',\ c',\ b'',\ c'',\ b''',\ c'''.$$

These nine lines then conversely pass three by three through the twelve points. In fact, each of the three perspectives Aa, Bb, Cc permutes $B'C''C'''$ and $C'B''B'''$; so that a, b, c pass through that one of the twelve points which is determined by the intersection of $B'C''C'''$ and $C'B''B'''$. This point may be conveniently denoted by abc; and then to each of the twelve lines such as ABC will correspond uniquely one of the twelve points, viz., abc; just as to each of the nine points such as A there corresponds uniquely one of the nine lines, viz., a.

4. The configuration has hitherto been regarded as consisting of the original nine points. The phrase may now be used in a more extended sense as including:—

(α) a set of nine points, (γ) a set of twelve points,

(β) a set of twelve lines, (δ) a set of nine lines.

The points of (α) lie three by three on the lines of (β). The intersections of the lines of (β) other than the points of (α) are the points of (γ), and these lie four by four on the lines of (δ). Moreover there is a unique one-to-one correspondence between the elements of (α) and (δ), and also between the elements of (β) and (γ).

5. It has been seen that the configuration is unchanged by the nine perspectives of order 2 of which Aa is typical. It will now be shown that there is also a system of perspectives of order 3, of which the points of (γ) and the corresponding lines of (β) are the vertices and axes, for which also the configuration is invariant.

Consider the collineation defined by

$$\begin{pmatrix} B'C'B''C'' \\ C''B''B'''C''' \end{pmatrix}.$$

This collineation changes $B'C'$ and $B''C''$ into $B''C''$ and $B'''C'''$; and therefore leaves A unchanged. It also leaves $B'C''C'''$ and $C'B''B'''$, two lines intersecting in abc, unchanged. Hence it leaves every line through abc unchanged; *i.e.*, it is a perspective of which abc is the vertex. Now b is a line through abc, and $AB'''C'''$, $CB'B''$; $AB'C'$, $CC''B'''$; $AB''C''$, $CC'C'''$; are the three pairs of the twelve lines which intersect on b. But the collineation in question changes $B'B''$ into $C''B'''$, $B'C'$ into $B''C''$, $B''C''$ into $B'''C'''$. Hence it permutes cyclically the three points in which the above three pairs of lines meet b. The collineation, when repeated three times, leaves therefore every line through abc and every point on b unaltered; and therefore it leaves every point in the plane unaltered. It is therefore a collineation of order 3, and changes C''' and B''' into B' and C' respectively. Since it leaves A unchanged and permutes B', C', B'', C'', B''', C''' among themselves, it must leave the configuration unchanged; and therefore it must leave the two remaining points B and C unchanged. The collineation is therefore a perspective of order 3 of which abc is the vertex and ABC is the axis. Similarly, the configuration is invariant for each of the perspectives of order 3 for which one of the twelve points is the vertex, and the corresponding one of the twelve lines is the axis.

It has been seen that, given 5 of the 9 points belonging to the configuration, such as A, B, C, B', C', they determine one of the 9 lines, viz., a; and then from these (which may be all real) just two configurations may be formed.

Suppose now that three collinear points A, B, C of the 9 are given, and the corresponding three concurrent lines a, b, c. These again may be all real. Containing these an infinite number of configurations may be formed, any two of which are in perspective, with abc for the vertex and ABC for the axis of the perspective.

The three points A, B, C and the three lines a, b, c are permuted among themselves by the collineations Aa, Bb, Cc, as also must be every Hessian configuration containing them. Now Aa followed by Bb is a collineation of order 3 of which abc is a fixed point and ABC a fixed line, A, B, C being permuted cyclically on it.

This collineation has two (imaginary) fixed lines i, j through abc. Since $B'C''C'''$ and $C'B''B'''$ are invariant for it, they must coincide with i, j. Hence, to construct the configuration, B' may be taken to be *any* point on i, and the remaining 5 points are then determinate. Any two

such Hessian configurations which have A, B, C, a, b, c in common must clearly have either all the rest of the 9 points and 9 lines in common (*i.e.*, be identical) or have none of them in common. In the latter case the remaining 6 of the 9 points of each configuration lie in threes on the lines i and j.

6. From the perspectives of order 3 arise the whole of the collineations for which the configuration is invariant. There are just four, with their inverses, which leave the point A unchanged. These are

$$abcABC,\quad ab'c'AB'C',\quad ab''c''AB''C'',\quad ab'''c'''AB'''C'''.$$

The permutations which they give of the eight points, other than A, are respectively

$$(B'C''C''')\,(C'B''B'''),\quad (BB''C''')\,(CC''B'''),$$
$$(BB'B''')\,(CC'C'''),\quad (BC'C'')\,(CB'B''),$$

and their inverses, the unchanged points being in each case unwritten, those written being permuted cyclically.

From the collineations written it is clear that collineations arise giving all possible even permutations of the four lines ABC, $AB'C'$, $AB''C''$, $AB'''C'''$. It remains to determine whether any collineation for which the configuration is invariant can give an odd permutation of the lines; and, secondly, what collineations leave the configuration and each of the four lines invariant.

Since all possible even permutations of the lines occur, it is sufficient to consider a collineation which interchanges ABC, $AB'C'$. Such a collineation must be either

$$\begin{pmatrix} BCB'C' \\ C'B'CB \end{pmatrix},\quad \begin{pmatrix} BCB'C' \\ B'C'BC \end{pmatrix},\quad \begin{pmatrix} BCB'C' \\ B'C'CB \end{pmatrix},\quad \text{or}\quad \begin{pmatrix} BCB'C' \\ C'B'BC \end{pmatrix}.$$

Of these it is shewn in § 2 that the first does not leave the configuration invariant. The second arises by combining the first with

$$\begin{pmatrix} BCB'C' \\ CBC'B' \end{pmatrix},$$

for which the configuration is invariant. The configuration therefore is not invariant for the second. The third and fourth are inverses of each other, and it is sufficient to consider one of them. But the third arises on combining the two perspectives of order 3, $(BB''C''')(CC''B''')$ and $(BC''C')(CB''B')$, of which $ab'c'$, $ab'''c'''$ are the vertices and $AB'C'$, $AB'''C'''$ the axes. It therefore gives an even permutation of the four lines. There are therefore no collineations for which the configuration is invariant that give an odd permutation of the four lines.

A collineation which leaves each of the four lines unchanged must permute or leave unchanged the members of each B, C pair. Consider then the collineation

$$\begin{pmatrix} BCB'C' \\ BCC'B' \end{pmatrix}$$

which permutes one pair and leaves unchanged the members of another. If this left $AB''C''$ and $AB'''C'''$ unchanged, it would leave every line through A unchanged, and would be a perspective with A for its vertex, which it is not. Hence the only collineation, other than identity, which leaves each of the four lines and the configuration unchanged is the perspective of order 2, Aa.

The number of even permutations of four symbols is twelve. Hence the order of the greatest group of collineations for which the configuration and the point A are invariant is 24. This sub-group contains the perspective of order 2, Aa, as a self-conjugate operation, and in respect of it is simply isomorphic with a tetrahedral group. Moreover, it contains no collineations of order 2 except the perspective Aa. Now A may be changed into any one of the other eight points by collineations for which the configuration is invariant. Hence the order of the greatest group for which the configuration is invariant is 216. It may be noticed, as following obviously from the present point of view, that the only collineations of order 2 in the group are the nine perspectives of the set Aa.

7. For the sequel two sub-groups of the G_{216}, which leaves the configuration invariant, are of special importance. The first is the sub-group generated by the 9 perspectives of order 2. Since these are the only perspectives of order 2, this sub-group is an invariant sub-group of the G_{216}. Its order is 18, and besides the perspectives of order 2 it contains 8 collineations (not perspectives) of order 3 and identity. This is at once verified by the permutations of the 9 points that the perspectives of order 2 give rise to. The fixed points of the 8 collineations of order 3 (each occurring with its inverse) are the 12 γ-points of the configuration. This sub-group will be called the G_{18}. In respect of it the G_{216} has been shewn to be simply isomorphic with a tetrahedral group. The latter has three sub-groups of order 2, forming a conjugate set. The G_{216} has therefore three sub-groups of order 36 (each containing the G_{18}) which form a conjugate set. Any one of these will be denoted by G_{36}. They arise by combining the G_{18} with any one of the collineations of order 4 belonging to the G_{216}. Such a collineation of order 4, arising by combining the two perspectives of order 3, $ab'c'AB'C'$ and $ab''c''AB''C''$, gives

the permutation

$$(BB''CC'')(B'B'''C'C''')$$

of the 8 points other than A; and the particular G_{36}, which is made use of in the sequel, is the group that is generated by the G_{18} and this collineation of order 4.

II.

8. I consider now sets of points which are permuted by the G_{18} that arises from the nine collineations of order 2. In general, such a set of points will consist of 18 members; but, if one of the points is on one of the nine lines of the Hessian configuration, the set will have only 9 members. In this case a uniform notation will be used,

$$A_i,\ B_i,\ C_i,\ B'_i,\ C'_i,\ B''_i,\ C''_i,\ B'''_i,\ C'''_i,$$

denoting the set of 9 permuted points lying respectively on

$$a,\ b,\ c,\ b',\ c',\ b'',\ c'',\ b''',\ c'''.$$

When the Hessian configuration and one point of such a set is given, the others are determined by drawing straight lines: *e.g.*, B'_i is the point of intersection of b' with the line joining A_i to C'.

Four such sets, with suffixes 1, 2, 3, 4, are formed as follows:—

Through A draw $AB_1B'_2$, meeting b, b' in B_1 and B'_2; and construct the sets 1 and 2. Join A to B''_1, and let it meet b' in B'_3. Form the set 3. Join A to B'''_2, and let it meet b in B_4. Form the set 4. As $AB_1B'_2$ turns round A, AB'''_3 and AB''_4 describe superposed projective pencils, which must have two self-corresponding rays. Let $AB_1B'_2$ and $A\bar{B}_1\bar{B}'_2$ be the positions of the original line which lead to the self-corresponding rays in these two projective pencils. Then the four sets of 9 points each which arise from A_1, A_2, A_3, A_4 are such that

$$AB_1C_1B'_2C'_2,\quad AB''_1C''_1B'_3C'_3,\quad AB_4C_4B'''_2C'''_2,\quad AB''_4C''_4B'''_3C'''_3,$$

and the other 32 symbols that arise from them by the collineations of the G_{18} represent straight lines.

The same statement is true for the four sets of 9 points each which arise from $\bar{A}_1$, $\bar{A}_2$, $\bar{A}_3$, $\bar{A}_4$.

Moreover, these are the only two sets of 36 points (consisting each of 4 sets of 9 points conjugate in respect of the G_{18}) for which the statement is true.

The G_{18} is contained self-conjugately in a G_{72} formed by combining

it with the collineations of order 4 denoted by the permutations

$$(BB''CC'')(B'B'''C'C'''),$$
$$(BB'CC')(B''C'''C''B'''),$$
$$(BB'''CC''')(B'C''C'B'').$$

Each of these collineations then must either leave unchanged or permute the two sets of 36 points. It is easily verified that the second and third collineations permute the sets; and therefore the first must change each set with itself. If *io* be used to denote the set into which the collineation $(BB''CC'')(B'B'''C'C''')$ changes the set i, then it changes the four lines

$$AB_1C_1B'_2C'_2, \quad AB''_1C''_1B'_3C'_3, \quad AB_4C_4B'''_2C'''_2, \quad AB''_4C''_4B'''_3C'''_3$$

into the lines

$$AB''_{10}C''_{10}B'''_{20}C'''_{20}, \quad AC_{10}B_{10}B'''_{30}C'''_{30}, \quad AB''_{40}C''_{40}C'_{20}B'_{20}, \quad AC_{40}B_{40}C'_{30}B'_{30},$$

and, apart from sequence, the one set of lines is identical with the other. Hence, on comparison, the set 10 is 4, 40 is 1, 20 is 3, and 30 is 2; in other words, the collineation $(BB''CC'')(B'B'''C'C''')$ permutes the two sets 1 and 4 together and the two sets 2 and 3.

Considering then one set of 36 points, in respect of the G_{36} that arises on combining the G_{18} with the collineation $(BB''CC'')(B'B'''C'C''')$, it consists of two sets of 18 points each, each set being transitively permuted among themselves by the G_{36}. Moreover, the 36 points lie four by four on the set of 36 lines given in the following table, which themselves are permuted transitively in two sets of 18 each by the G_{36}. Through each of the nine original points of the Hessian configuration just 4 of the 36 lines pass. Hence the original 9 points and the 36 constructed from them form a set of 45, which lie five by five on a set of 36 lines that pass four by four through each of them.

TABLE I.

$AB_1C_1B'_2C'_2$,	$AB_4C_4B'''_2C'''_2$,	$AB''_1C''_1B'_3C'_3$,	$AB''_4C''_4B'''_3C'''_3$,
$BA_1C_1B''_2C'''_2$,	$BA_4C_4C''_2C'_2$,	$BB'_1B'''_1B''_3C'''_3$,	$BB'_4B'''_4C''_3C'_3$,
$CB_1A_1B'''_2C''_2$,	$CB_4A_4B'_2B''_2$,	$CC'''_1C'_1B'''_3C''_3$,	$CC'''_4C'_4B'_3B''_3$,
$B'C''_1C'''_1A_2C'_2$,	$B'C''_4C'''_4B''_2C_2$,	$B'B'''_1B_1A_3C'_3$,	$B'B'''_4B_4B''_3C_3$,
$C'B'''_1B''_1B'_2A_2$,	$C'B'''_4B''_4B_2C''_2$,	$C'C_1C'''_1B'_3A_3$,	$C'C_4C'''_4B_3C''_3$,
$B''C'_1B'''_1C'''_2B_2$,	$B''C'_4B'''_4C_2B'_2$,	$B''A_1C''_1C'''_3B_3$,	$B''A_4C''_4C_3B'_3$,
$C''C'''_1B'_1C_2B'''_2$,	$C''C'''_4B'_4C'_2B_2$,	$C''B''_1A_1C_3B'''_3$,	$C''B''_4A_4C'_3B_3$,
$B'''B''_1C'_1C''_2C_2$,	$B'''B''_4C'_4A_2C'''_2$,	$B'''B_1B'_1C''_3C_3$,	$B'''B_4B'_4A_3C'''_3$,
$C'''B'_1C''_1B_2B''_2$,	$C'''B'_4C''_4B'''_2A_2$,	$C'''C'_1C_1B_3B''_3$,	$C'''C'_4C_4B'''_3A_3$.

9. Although the foregoing geometrical construction is formally sufficient to determine the system of points and lines, it cannot be actually carried out. Indeed, of the nine points and nine lines of the Hessian configuration not more than either (i) three points and three lines, (ii) five points and one line, or (iii) one point and five lines, can be real. An actual specification of the points and lines is necessarily analytical. The formulæ are as follows. Taking ABC, $B'C'''C''$, $C'B''B'''$ as the sides of the triangle of reference, the nine collineations of order 2 of the G_{18} with their fixed points and fixed lines are given by the table:—

$(\omega^3 = 1.)$

	Substitution.	Fixed Point.	Fixed Line.
Aa	(x, z, y)	$x = 0, \quad y+z = 0$	$y-z = 0$
Bb	$(x, \omega z, \omega^2 y)$	$x = 0, \quad y+\omega z = 0$	$y-\omega z = 0$
Cc	$(x, \omega^2 z, \omega y)$	$x = 0, \quad y+\omega^2 z = 0$	$y-\omega^2 z = 0$
$B'b'$	(z, y, x)	$y = 0, \quad z+x = 0$	$z-x = 0$
$C'''c'''$	$(\omega^2 z, y, \omega x)$	$y = 0, \quad z+\omega x = 0$	$z-\omega x = 0$
$C''c''$	$(\omega z, y, \omega^2 z)$	$y = 0, \quad z+\omega^2 x = 0$	$z-\omega^2 x = 0$
$C'c'$	(y, x, z)	$z = 0, \quad x+y = 0$	$x-y = 0$
$B''b''$	$(\omega y, \omega^2 x, z)$	$z = 0, \quad x+\omega y = 0$	$x-\omega y = 0$
$B'''b'''$	$(\omega^2 y, \omega x, z)$	$z = 0, \quad x+\omega^2 y = 0$	$x-\omega^2 y = 0$

With this table there is no difficulty in carrying out the calculation, which presents no point of interest. The result is that the four points A_1, A_4, A_2, A_3 are

$$\lambda, 1, 1; \quad \frac{\lambda}{(\omega-\omega^2)\lambda-\omega^2}, 1, 1; \quad -\frac{1+\lambda}{\lambda}, 1, 1; \quad \omega^2-1+\omega\lambda, 1, 1,$$

where λ is an assigned root of

$$\lambda^2+(3+4\omega)\lambda-2\omega^2 = 0,$$

the other root giving the four points $\bar{A}_1$, $\bar{A}_4$, $\bar{A}_2$, $\bar{A}_3$. From the coordinates of the A's those of the B's, &c., can be calculated by the previous table.

This analytical specification of the points may be used to verify the existence of further collinearities among them in a very simple manner. The fact that an adequate figure cannot be drawn, owing to most of the points being imaginary, renders a geometrical treatment of this point very difficult to carry out.

The coordinates of B_4''', which is changed into A_4 by the collineation $C'''c'''$, are

$$\omega^2, 1, \frac{\lambda\omega}{\lambda(\omega-\omega^2)-\omega^2},$$

and, since

$$\begin{vmatrix} \omega^2, & 1, & \dfrac{\lambda\omega}{\lambda(\omega-\omega^2)-\omega^2} \\ 1, & 1, & -1 \\ \lambda, & 1, & 1 \end{vmatrix} = 0,$$

B_4''' lies in a line with A and A_1. Now $AB_4'''C_4'''$ is a line; and hence $AA_1B_4'''C_4'''$ is a line. The collineation

$$(BB''CC'')(B'B'''C'C''')$$

changes A, A_1, B_4''', C_4''' into A, A_4, C_1', B_1' respectively. Hence $AA_4C_1'B_1'$ is also a line. It may

be similarly verified that $AA_2B_3''C_3''$ and $AA_3B_2C_2$ are lines. Since B_4''' is changed into C_4''' by Aa, which leaves A and A_1 unchanged, $B_4'''C_4'''$ divide AA_1 harmonically; and similarly $B_3''C_3''$ divide AA_2 harmonically. It also follows directly from the coordinates of A_1, A_4, A_2, A_3 that A_2A_3 divide A_1A_4 harmonically.

10. The table of the 36 lines containing the 45 points five by five may now be supplemented by one of 45 lines which contain the same 45 points four by four.

TABLE II.

$AA_1B_4'''C_4'''$,	$AA_4C_1'B_1'$,	$AA_2B_3''C_3''$,	$AA_3B_2C_2$,	$A_1A_4A_2A_3$,
$BB_1C_4''C_4'$,	$BB_4C_1'''B_1''$,	$BB_2B_3'B_3'''$,	$BB_3C_2A_2$,	$B_1B_4B_2B_3$,
$CC_1B_4'B_4''$,	$CC_4C_1''B_1'''$,	$CC_2C_3'''C_3'$,	$CC_3A_2B_2$,	$C_1C_4C_2C_3$,
$B'B_1'B_4''C_4$,	$B'B_4'C_1'A_1$,	$B'B_2'B_3'''B_3$,	$B'B_3'C_2''C_2'''$,	$B_1'B_4'B_2'B_3'$,
$C'C_1'B_4C_4''$,	$C'C_4'A_1B_1'$,	$C'C_2'C_3C_3'''$,	$C'C_3'B_2'''B_2''$,	$C_1'C_4'C_2'C_3'$,
$B''B_1''C_4B_4'$,	$B''B_4''B_1C_1'''$,	$B''B_2''A_3C_3''$,	$B''B_3''C_2'B_2'''$,	$B_1''B_4''B_2''B_3''$,
$C''C_1''C_4'B_4$,	$C''C_4''B_1'''C_1$,	$C''C_2''B_3''A_3$,	$C''C_3''C_2'''B_2'$,	$C_1''C_4''C_2''C_3''$,
$B'''B_1'''A_4C_4'''$,	$B'''B_4'''C_1C_1''$,	$B'''B_2'''B_3B_3'$,	$B'''B_3'''B_2''C_2'$,	$B_1'''B_4'''B_2'''B_3'''$,
$C'''C_1'''B_4'''A_4$,	$C'''C_4'''B_1''B_1$,	$C'''C_2'''C_3'C_3$,	$C'''C_3'''B_2'C_2''$,	$C_1'''C_4'''C_2'''C_3'''$.

Each line of this table, the formation of which from the previous data is obvious, contains 4 of the 45 points, and in each the first pair divide the second pair harmonically. Further, there are just 4 of the set of lines passing through any one of the 45 points. This and the previous table contain implicitly all the properties of the configuration of 45 points which has been constructed.

11. An inspection of the two tables shews that the set of 45 points is invariant for collineations which do not belong to the G_{36} in connection with which the set arises.

Consider in particular the perspective of order 2, defined by

$$\begin{pmatrix} BCB_2''C_2'' \\ B_2''C_2''BC \end{pmatrix}.$$

Since ABC, $AB_2''C_2''$, $BA_1C_1B_2''C_2'''$, $CB_1A_1B_2'''C_2''$, $BA_4C_4C_2''C_2'$, $CB_4A_4B_2'B_2''$ are straight lines, A, A_1, A_4 are unchanged by the perspective; and three lines through A_1, being unchanged, every line through A_1 must be unchanged. Hence A_1 is the fixed point and AA_4 the fixed line of the perspective; and B_1', C_1', being points on AA_4, are unchanged. Further,

since $A_1A_4A_2A_3$, $AA_1B_4'''C_4'''$, $B'B_4'C_1'A_1$, $C'C_4'A_1B_1'$ are lines in each of which the first pair of points divide the second pair harmonically, A_2, A_3 are permuted by the perspective, as also are B_4''', C_4'''; B', B_4'; and C', C_4'. The perspective therefore leaves A, A_1, A_4, B_1', C_1' unchanged, and gives the permutations

$$(A_2A_3)(BB_2'')(CC_2'')(B'B_4')(C'C_4')(B_4'''C_4''').$$

Further permutations are obtained from the tables by taking a pair of lines intersecting in a common point (belonging to the 45) and determining the lines into which they are changed by the perspective. Thus $BB_3A_2C_2$ and $C'C_4C_4'''B_3C_3''$ become $B''B_2''A_3C_3''$ and $B''C_4'B_4'''C_2B_2'$; so that B_3 and B'' are permuted by the perspective. Continuing in this way, it may be very easily verified that the whole of the 45 points are permuted by the perspective of which A_1 is the fixed point and AA_4 the axis, the actual permutations being:—

$$A,\ A_1,\ A_4,\ B_1',\ C_1' \text{ unchanged,}$$

$$(BB_2'')(CC_2'')(B'B_4')(C'C_4')(B''B_3)(C''C_3)(B'''C_1''')(C'''B_1'''),$$

$$(A_2A_3)(B_1B_2''')(C_1C_2''')(B_1''B_3''')(C_1''C_3''')(B_2B_3'')(C_2C_3''),$$

$$(B_2'C_4)(C_2'B_4)(B_3'B_4'')(C_3'C_4'')(B_4'''C_4''').$$

Similarly, it may be shewn that the perspective of order 2 of which A_2 is the fixed point and AA_3 the fixed line permutes the 45 points among themselves.

Now, for the G_{36}, A_1 and A_2 are each one of a set of 18 conjugate points; and AA_4, AA_3 each belong to a set of 18 conjugate lines. From the two perspectives of which A_1 and A_2 are the fixed points, and AA_4, AA_3 the fixed lines, there thus arises a set of 36 perspectives of order 2, for every one of which the configuration of 45 points is invariant. These, with the original nine perspectives of order 2, belonging to the G_{36}, give a set of 45, each with one of the 45 points for fixed point and one of the 45 lines (of Table II.) for fixed line. A set of five is

$$A(A_1A_4A_2A_3),\quad A_1(AA_4),\quad A_4(AA_1),\quad A_2(AA_3),\quad A_3(AA_2),$$

and the remainder are formed by replacing A by B, C, B', C', B'', C'', B''', or C'''.

An inspection of the permutations of the 45 points given by $A_1(AA_4)$ shews that they form a single conjugate set with respect to the group G of collineations generated by the G_{36} and the perspective of order 2, $A_1(AA_4)$. Hence the set of 45 perspectives of order 2 forms a single conjugate set of collineations for G. For a group of plane collineations of finite order cannot contain two distinct perspectives of order 2 with a common vertex.

12. The four lines

$$ABC, \quad AB'C', \quad AB''C'', \quad AB'''C'''$$

are changed by $A_1(AA_4)$ into

$$AB_2''C_2'', \quad AB_4'C_4', \quad AB_3C_3, \quad AC_1'''B_1'''.$$

Thus the 16 points of the configuration which do not lie on either of the 4 lines of Table I., or of the 4 lines of Table II., that pass through A, lie by pairs on 8 other lines through A. Of these lines which contain the 45 points three by three there are 120, and through each point of the 45 eight of these lines pass. From this it follows that the straight line joining any two points of the configuration passes through either one, two, or three others.

13. The Hessian configuration

$$A, B, C, B', C', B'', C'', B''', C''',$$

formed of 9 out of the 45 points, is changed by $A_1(AA_4)$ into the Hessian configuration $A, B_2'', C_2'', B_4', C_4', B_3, C_3, C_1''', B_1'''$,

containing only one point in common with the previous one. Moreover, no one of the set of 12 (β) lines of the first coincides with one of the set of 12 of the second; and each of these sets of 12 belongs to the set of 120 of the last paragraph.

Any Hessian configuration into which the first is changed by a collineation belonging to G must have for its (β) lines 12 from the set of 120 just mentioned. If two configurations have one of these lines, say ABC, in common, the three points A, B, C on it belong to each, and the three lines a, b, c are (δ) lines for each. But then, by § 5, the remaining 12 points making up the configurations would lie on two lines, 6 on each. Now no 6 of the 45 points lie on a line. Hence no two Hessian configurations, formed from the 45 points by the collineations of G, can have a (β) line in common. There are then at most 10 such Hessian configurations. On the other hand,

$$A, B, C, B', C', B'', C'', B''', C'''$$

is changed into another Hessian configuration having any single one of the 9 points written in common with it, by a suitably chosen one of the 45 perspectives of order 2. For instance, $B_1''(B''B_4'')$ changes it into one having B'' in common with it. There are then at least 10 such Hessian configurations. Combining the two results, it follows that from the original Hessian configuration just 10 Hessian configurations can be formed by the collineations of G, having for their (β) lines the lines of the set of 120. Each two of these Hessian configurations have just one point in common; and,

conversely, each point of the 45 enters in just two Hessian configurations. Moreover, the 10 Hessian configurations are transitively permuted by the collineations of G; and any collineation which changes each Hessian configuration into itself changes each of the 45 points into itself, and is therefore the identical collineation. But it has already been seen that the greatest sub-group of the G_{216} for which a Hessian configuration is invariant, which leaves invariant the set of 45 points, is a G_{36}. Hence the order of G is 360. It follows immediately that G is the greatest group of collineations for which the 45 points are invariant.

14. The six lines in the upper left-hand corner of Table I., viz.,

$$\begin{array}{lll} AB_1C_1B_2'C_2', & AB_4C_4B_2'''C_2''', & \text{say } l_1,\ l_4, \\ BA_1C_1B_2''C_2''', & BA_4C_4C_2''C_2', & \text{say } l_2,\ l_5, \\ CB_1A_1B_2'''C_2'', & CB_4A_4B_2'B_2'', & \text{say } l_3,\ l_6, \end{array}$$

contain just 15 of the 45 points, which constitute their complete intersection. The collineations Aa, Bb, Cc, and $A_1(AA_4)$ permute these lines among themselves, giving in fact the permutations

$$(l_2l_3)(l_5l_6),\quad (l_3l_1)(l_6l_4),\quad (l_1l_2)(l_4l_5),\quad (l_1l_4)(l_5l_6).$$

The collineation of order 3, $abcABC$, gives the permutation

$$(l_1l_2l_3)(l_4l_5l_6);$$

and therefore this collineation followed by the perspective $A_1(AA_4)$ leaves l_5 unchanged, and gives the permutation

$$(l_1l_2l_3l_4l_6)$$

of the other five. The six lines are therefore permuted among themselves by an icosahedral group of 60 collineations, and they hence form one of not more than six such sets of six lines which are permuted transitively by G. Now the perspective $A_2(AA_3)$ leaves A unchanged, and changes the lines l_1, l_2, l_3, l_4, l_5, l_6 into another set of six. Every point of the 45 then occurs in at least two such conjugate sets of 15, which constitute the complete intersection of a set of 6 lines. Hence there are not less than six sets of six lines transitively permuted by the group. The icosahedral group of 60 collineations is therefore the greatest sub-group of G for which the set of lines l_1, l_2, l_3, l_4, l_5, l_6 is invariant; and by the collineations of G just six such sets arise which are permuted transitively. Each of the 45 points occurring in just two sets, any collineation which leaves each set unchanged must leave each point of the 45 unchanged, and is the identical collineation. The group of collineations G is therefore

simply isomorphic with a group of permutations of six symbols. Hence, the order of G being 360, it is simply isomorphic with the alternating group of six symbols.

Of the eight lines which pass through A and contain just three of the 45 points, two, viz., ABC, $AB''_2C''_2$, occur in connection with the 15 points which form the complete intersection of l_1, l_2, l_3, l_4, l_5, l_6. Two more must occur in connection with the complete intersection of the six lines into which l_1, l_2, l_3, l_4, l_5, l_6 are changed by the perspective $A_2(AA_3)$. There cannot therefore be more than one other set of six lines, containing l_1, and having 15 of the 45 points for their complete intersection. An inspection of Table I. shews that there is just one other such set, viz.,

$$\begin{array}{ll} AB_1C_1B'_2C'_2, & AB''_1C''_1B'_3C'_3, \\ B'''_1B''_1C'A_2B'_2, & B'''_1B'B_1C'_3A_3, \\ C'''_1B'C''_1C'_2A_2, & C'''_1C'C_1A_3B'_3. \end{array}$$

The 36 lines can therefore be divided in just two distinct ways into six sets of six each, such that the complete intersection of any set of six is 15 of the 45 points; and each of these two sets of six are permuted transitively among themselves by the collineations of the group.

ON THE REPRESENTATION OF CERTAIN ASYMPTOTIC SERIES AS CONVERGENT CONTINUED FRACTIONS

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1. It has been proved by Prof. T. Muir (*Edinburgh Trans.*, Vol. XXVII., 667–671) that, in general, a power series in x, $a_0+a_1x+a_2x^2+\ldots$, can be represented as a continued fraction of the form

$$\frac{e_0}{1-}\,\frac{e_1x}{1-}\,\frac{e_2x}{1-}\ldots,$$

and, for convenience of reference, it will be useful to indicate the proof of the determinant expressions which give the e's in terms of the a's.

If P_n and Q_n are respectively numerator and denominator of the n-th convergent left unreduced, so that

$$P_1=e_0,\quad P_2=e_0,\quad P_n=P_{n-1}-e_{n-1}xP_{n-2};$$

and $$Q_1=1,\quad Q_2=1-e_1x,\quad Q_n=Q_{n-1}-e_{n-1}xQ_{n-2},$$

it will be readily seen that P_{2n-1}, P_{2n}, Q_{2n-1} are of degree $n-1$ in x, and that Q_n is of degree n.

Moveover, if E_n denote $\dfrac{e_n}{1-}\,\dfrac{e_{n+1}x}{1-}\ldots$, then

$$E_0Q_n-P_n=E_0E_1\ldots E_nx^n. \tag{1}$$

If $n=2m$, we get, by equating coefficients of $x^m, x^{m+1}, \ldots, x^{2m}$, the relation

$$e_0e_1\ldots e_{2m}=\alpha_{2m}/\alpha_{2m-2}, \tag{2}$$

where $$\alpha_{2m}=\begin{vmatrix} a_{2m}, & a_{2m-1}, & \ldots, & a_m \\ a_{2m-1}, & a_{2m-2}, & \ldots, & a_{m-1} \\ \vdots & \vdots & \vdots & \vdots \\ a_m, & a_{m-1}, & \ldots, & a_0 \end{vmatrix},$$

which is a determinant of a type called persymmetric.

Also, by considering the coefficients of x^{2m+1}, we get

$$e_0e_1\ldots e_{2m}(e_1+e_2+\ldots+e_{2m+1})=\alpha_{2m,1}/\alpha_{2m-2}, \tag{3}$$

where $\alpha_{2m,1}$ denotes what α_{2m} becomes when the suffixes of the first row are all increased by unity.

Similarly, by making $n = 2m+1$, we get

$$e_0 e_1 \dots e_{2m+1} = a_{2m+1}/a_{2m-1} \tag{4}$$

and
$$e_0 e_1 \dots e_{2m+1}(e_1+e_2+\dots e_{2m+2}) = a_{2m+1,\,1}/a_{2m-1}, \tag{5}$$

where
$$a_{2m+1} = \begin{vmatrix} a_{2m+1}, & a_{2m}, & \dots, & a_{m+1} \\ a_{2m}, & a_{2m-1}, & \dots, & a_m \\ \vdots & \vdots & \vdots & \vdots \\ a_{m+1}, & a_m, & \dots, & a_1 \end{vmatrix},$$

and $a_{2m+1,\,1}$ is what a_{2m+1} becomes when the suffixes of the first row are all increased by unity.

The converse relations, viz., the expressions giving $a_0, a_1, \dots$ in terms of the e's, can be determined successively, but apparently not generally. For, if δ denote an operation which changes a_n into a_{n+1}, we get $\delta a_n = a_{n,\,1}$, both when n is even and when n is odd.

Hence, if $n > 1$,

$$\delta e_0 e_1 \dots e_n = \frac{a_n}{a_{n-2}}\left(\frac{a_{n,\,1}}{a_n} - \frac{a_{n-2,\,1}}{a_{n-2}}\right), \quad \text{by (2) or (4),}$$

$$= e_0 e_1 \dots e_n (e_n+e_{n+1}), \quad \text{by (3) or (5),}$$

therefore
$$\delta e_n = e_n (e_{n+1} - e_{n-1})$$

while
$$\delta e_0 = e_0 e_1, \quad \text{and} \quad \delta e_1 = e_1 e_2.$$

Thus
$$\begin{aligned} a_0 &= e_0, \\ a_1 &= e_0 e_1, \\ a_2 &= \delta e_0 e_1 = e_0 e_1 (e_1+e_2), \\ a_3 &= \delta e_0 e_1 (e_1+e_2) = e_0 e_1 (e_1+e_2)^2 + e_0 e_1 e_2 e_3, \\ &\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots . \end{aligned}$$

The $2n$-th convergent denotes that rational algebraic fraction in x, of degree in numerator $n-1$ and of degree in denominator n, which is identical with the given series up to $2n$ terms, *i.e.*, as far as x^{2n-1}; while the $(2n+1)$-th convergent is identical with the given series up to $2n+1$ terms, and has numerator and denominator both of degree n.

The alternate convergents thus form a series of fractions which may in certain cases approach a finite limit, even when the given series does not absolutely converge.

Such will be found to be the case when the series is of the asymptotic type derived from certain definite integrals by continued process of integration by parts.

It will be necessary first to establish certain lemmata which will hereafter be found important.

LEMMA I.—The $2m$-th convergent of the fraction

$$\frac{e_0}{1-}\,\frac{e_1 x}{1-}\,\frac{e_2 x}{1-}\,\cdots$$

is identical with the m-th convergent of

$$\frac{e_0}{1-e_1x-}\,\frac{e_1e_2x^2}{1-(e_2+e_3)x-}\,\frac{e_3e_4x^2}{1-(e_4+e_5)x-}\,\cdots.$$

For, from the relation

$$P_{2m}=P_{2m-1}-e_{2m-1}xP_{2m-2},\quad P_{2m-1}=P_{2m-2}-e_{2m-2}xP_{2m-3},\quad \ldots,$$

we may, by eliminating P_{2m-1}, P_{2m-3}, obtain

$$P_{2m}=\{1-(e_{2m-2}+e_{2m-1})x\}P_{2m-2}-e_{2m-3}e_{2m-2}x^2P_{2m-4},$$

together with a similar relation connecting alternate Q's; which two equations, since the necessary initial conditions are satisfied, give the required result.

LEMMA II.—If $\quad \dfrac{e_0}{1-}\,\dfrac{e_1x}{1-}\,\cdots = e_0+\dfrac{f_1x}{1-}\,\dfrac{f_2x}{1-}\,\cdots,$

then $\quad f_1=e_0e_1,\quad f_2=e_1+e_2,\quad f_2f_3=e_2e_3,\quad f_3+f_4=e_3+e_4,\quad \ldots.$

For $$E_0=\frac{e_0}{1-xE_1}=e_0+\frac{e_0e_1x}{\dfrac{e_1}{E_1}-e_1x},$$

while $$E_1=\frac{e_1}{1-e_2x-}\,\frac{e_2e_3x^2}{1-(e_3+e_4)x-}\,\cdots\quad\text{(by Lemma I.)};$$

therefore $$E_0=e_0+\frac{e_0e_1x}{1-(e_1+e_2)x-}\,\frac{e_2e_3x^2}{1-(e_3+e_4)x-}\,\cdots.$$

But $$E_0=e_0+\frac{f_1x}{1-f_2x-}\,\frac{f_2f_3x^2}{1-(f_3+f_4)x-}\,\cdots\quad\text{(by Lemma I.)},$$

whence, by comparing the two expressions for E_0, we obtain the required relations between the f's and the e's.

LEMMA III.—If in a continued fraction of the second class

$$\frac{e_0}{1-}\,\frac{e_1}{1-}\,\frac{e_2}{1-}\,\cdots,$$

where the e's are all positive, the convergent denominators are all positive and the fraction is known to be convergent, then also will the fraction

$$\frac{e_0}{1-}\,\frac{f_1}{1-}\,\frac{f_2}{1-}\,\cdots$$

be convergent, provided $f_n < e_n$.

Let p_n/q_n be the n-th convergent of the first fraction, and P_n/Q_n that of the second.

Then, since $p_{n+1}=p_n-e_n p_{n-1}, \ldots,$ we have

$$\frac{p_{n+1}}{q_{n+1}}-\frac{p_n}{q_n}=\frac{e_n q_{n-1}}{q_{n+1}}\left(\frac{p_n}{q_n}-\frac{p_{n-1}}{q_{n-1}}\right).$$

Now it is evident that, if $\frac{f_n Q_{n-1}}{Q_{n+1}}<\frac{e_n q_{n-1}}{q_{n+1}}$ for all values of n above some fixed integer, the second continued fraction converges more rapidly than the first.

Now
$$\frac{e_n q_{n-1}}{q_{n+1}}=\frac{q_n}{q_{n+1}}-1 \quad\text{and}\quad \frac{q_n}{q_{n+1}}=\frac{1}{1-}\,\frac{e_n}{1-}\,\frac{e_{n-1}}{1-}\cdots\frac{e_2}{1-e_1}.$$

Since, however, $f_1<e_1$, $f_2<e_2$,

$$\frac{f_2}{1-f_1}<\frac{e_2}{1-e_1},$$

and, since $f_3<e_3$,
$$\frac{f_3}{1-}\,\frac{f_2}{1-f_1}<\frac{e_3}{1-}\,\frac{e_2}{1-e_1},$$

and generally
$$\frac{Q_n}{Q_{n+1}}<\frac{q_n}{q_{n+1}},$$

which is the condition required in order that the second continued fraction should converge more rapidly than the first.

COROLLARY.—If $m_0, m_1, m_2, \ldots>1$, then $\frac{e_0}{m_0-}\,\frac{e_1}{m_1-}\,\frac{e_2}{m_2-}\ldots$ converges more rapidly than $\frac{e_0}{1-}\,\frac{e_1}{1-}\ldots.$

As Lemma IV., I introduce for reference the well known condition for fractions of the first class

$$\frac{e_0}{1+}\,\frac{e_1}{1+}\,\frac{e_2}{1+}\ldots,$$

where all the e's are positive; viz., that, if the fraction be reduced to the form

$$\frac{1}{d_0+}\,\frac{1}{d_1+}\,\frac{1}{d_2+}\ldots,$$

it will converge, provided at least one of the series $d_0+d_2+\ldots$ or $d_1+d_3+\ldots$ is divergent.

It may be remarked that in this case the fraction

$$\frac{e_0}{1+}\,\frac{e_1 x}{1+}\,\frac{e_2 x}{1+}\ldots$$

is also convergent for all positive values of x.

2. Let u_n denote $\int_0^\infty \mathrm{sn}^n t\, e^{-t/x}\, dt$. Then, if $n > 2$,

$$u_n = \Big[-xe^{-t/x}\,\mathrm{sn}^n t\Big]_0^\infty + x\int_0^\infty \frac{d}{dt}\,\mathrm{sn}^n t\, e^{-t/x}\, dt = x^2\int_0^\infty \frac{d^2}{dt^2}\,\mathrm{sn}^n t\, e^{-t/x}\, dt$$

$$= x^2 n(n-1)\, u_{n-2} - n^2 x^2 (1+k^2)\, u_n + n^2(n+1)\, x^2 u_{n+2}.$$

If $n = 1$, we have

$$u_1 = x\int_0^\infty \mathrm{cn}\, t\, \mathrm{dn}\, t\, e^{-t/x}\, dt$$

$$= \Big[-x^2\,\mathrm{cn}\, t\, \mathrm{dn}\, t\, e^{-t/x}\Big]_0^\infty + x^2\int_0^\infty e^{-t/x}\{-(1+k^2)\,\mathrm{sn}\, t - 2k^2\,\mathrm{sn}^3 t\}\, dt$$

$$= x^2 - x^2(1+k^2)\, u_1 - 2k^2 x^2 u_3,$$

whence $$u_1 = \frac{x^2}{1+(1+k^2)x^2 - 2k^2x^2u_3/u_1}$$

$$= \frac{x^2}{1+(1+k^2)x^2-}\;\frac{1.2^2.3k^2x^4}{1+3^2(1+k^2)x^2-}\;\frac{3.4^2.5k^2x^4}{1+5^2(1+k^2)x^2-}. \qquad (1)$$

Let $x = 1/u$; then

$$\int_0^\infty \mathrm{sn}\, t\, e^{-ut}\, dt = \frac{1}{u^2+1+k^2-}\;\frac{1.2^2.3k^2}{u^2+3^2(1+k^2)-}\;\frac{3.4^2.5k^2}{u^2+5^2(1+k^2)-}\cdots.$$

To prove that this fraction is convergent for all real values of u and k, where $k < 1$, let us write

$$u^2 = (1+k^2)\, v^2 \quad\text{and}\quad \frac{2k}{1+k^2} = m.$$

$$(1+k^2)\int_0^\infty \mathrm{sn}\, t\, e^{-ut}\, dt = \frac{1+k^2}{(1+k^2)(v^2+1)-}\;\frac{2^2.1.3.k^2}{(1+k^2)(v^2+3^2)-}\;\cdots$$

$$= \frac{1}{v^2+1-}\;\frac{m^2.1.3}{v^2+3^2-}\;\frac{2^2m^2.3.5}{v^2+5^2-}$$

$$= \frac{1}{v^2+1-}\;\frac{m^2}{\frac{1}{3}(v^2+3^2)-}\;\frac{2^2m^2}{\frac{1}{5}(v^2+5^2)-}\;\frac{3^2m^2}{\frac{1}{7}(v^2+7^2)-}. \qquad (2)$$

If $v = 0$, this becomes

$$\frac{1}{1-}\;\frac{m^2}{3-}\;\frac{2^2m^2}{5-}\;\frac{3^2m^2}{7-} \qquad (3)$$

where $m < 1$.

This fraction is known to be convergent and equal to $m^{-1}\tanh^{-1} m$; so that, by Lemma III., the right-hand side of (2) is also convergent.

If $k = 1$, the identity (1) becomes

$$\int_0^\infty \tanh t\, e^{-t/x}\, dt = \frac{x^2}{1+2x^2-}\, \frac{1.2^2.3x^4}{1+2.3^2x^2-}\, \frac{3.4^2.5x^4}{1+2.5^2x^2-},$$

which, by Lemma I.,

$$= \frac{x^2}{1+}\, \frac{1.2x^2}{1+}\, \frac{2.3x^2}{1+}\, \frac{3.4x^2}{1+} \dots. \tag{4}$$

The convergence of this fraction may be established also from Lemma IV.

From considerations similar to the foregoing, we have also

$$\int_0^\infty \mathrm{sn}^2 t\, e^{-t/x}\, dt = \frac{2x^3}{1+2^2(1+k^2)x^2-}\, \frac{2.3^2.4k^2x^4}{1+4^2(1+k^2)x^2-}, \tag{5}$$

of which the convergency is, by Lemma III., more rapid than that of $\frac{m^2}{2-}\, \frac{3^2m^2}{4-}\, \frac{5^2m^2}{6-} \dots$, which, again, is more rapid than that of (3).

3. If v_n denote $\int_0^\infty \mathrm{cn}^n t\, e^{-t/x}\, dt$, we shall, as in the last section, obtain a general linear relation connecting v_{n-2}, v_n, and v_{n+2}.

By Lemma I., it will be found that the integral may be expressed in the form

$$\frac{x}{1+}\, \frac{x^2}{1+}\, \frac{2^2k^2x^2}{1+}\, \frac{3^2x^2}{1+}\, \frac{4^2k^2x^2}{1+} \dots, \tag{1}$$

the convergency of which can be established by Lemma IV.

Similarly,

$$\int_0^\infty \mathrm{dn}\, t\, e^{-t/x}\, dt = \frac{x}{1+}\, \frac{k^2x^2}{1+}\, \frac{2^2x^2}{1+}\, \frac{3^2k^2x^2}{1+} \dots,$$

and, by putting $k = 1$,

$$\int_0^\infty \mathrm{sech}\, t\, e^{-t/x}\, dt = \frac{x}{1+}\, \frac{x^2}{1+}\, \frac{2^2x^2}{1+}\, \frac{3^2x^2}{1+} \dots.$$

4. If Landen's theorem be applied to the results of § 2, (1) and (5), we get from (1), after some reductions,

$$\int_0^\infty \frac{\mathrm{sn}\, t\, \mathrm{cn}\, t}{\mathrm{dn}\, t} e^{-t/x}\, dt$$

$$= \frac{x^2}{1+2(1+k'^2)x^2-}\, \frac{1.2^2.3k^4x^4}{1+2.3^2(1+k'^2)x^2-}\, \frac{3.4^2.5k^4x^4}{1+2.5^2(1+k'^2)x^2-\dots}$$

$$= \frac{x^2}{1+2^2x^2-2k^2x^2-}\, \frac{1.2^2.3k^4x^4}{1+2^2.3^2x^2-2.3^2k^2x^2-}\, \frac{3.4^2.5k^4x^4}{1+2^2.5^2x^2-2.5^2k^2x^2-\dots},$$

which also

$$= \frac{x^2}{1+2^2x^2-}\, \frac{1.2k^2x^2}{1-}\, \frac{2.3k^2x^2}{1+6^2x^2-}\, \frac{3.4k^2x^2}{1-}\, \frac{4.5k^2x^2}{1+10^2x^2-\dots}$$

by Lemma I.; the x of the lemma being the k^2 of this section. So, too,

$$\int_0^\infty \frac{k^2 \operatorname{sn}^2 t \operatorname{cn}^2 t}{\operatorname{dn}^2 t} e^{-t/x} dt = \frac{k^2x^3}{1+2.2^2(1+k'^2)x^2-} \; \frac{2.3^2.4k^4x^4}{1+2.4^2(1+k'^2)x^2-\dots},$$

whence
$$\int_0^\infty \frac{1-k^2 \operatorname{sn}^4 t}{\operatorname{dn}^2 t} e^{-t/x} dt = x+\frac{k^2x^3}{1+2.2^2(1+k'^2)x^2-\dots}.$$

The right-hand side of this equation may be treated by Lemma II., where k^2 is the x of the lemma; and, transforming the elliptic function at the same time, we have

$$\int_0^\infty \frac{2e^{-t/x}}{1+\operatorname{dn} 2t} dt = \frac{x}{1-} \; \frac{2k^2x^2}{1+4^2x^2-} \; \frac{2.3k^2x^2}{1-} \; \frac{3.4k^2x^2}{1+8^2x^2-\dots}.$$

5. A very general theorem relating to the conversion of a definite integral of the form $\int_0^\infty f(t)\, e^{-t/x} dt$ may be obtained from the following considerations.

Suppose $f(x)$ to be represented by a power series in x with positive integral indices, and that the number of terms is indefinite. Then it is not difficult to see that we have sufficient arbitrary constants at our disposal to assume that

$$f(x+y) = A_0 f(x) f(y) + A_1 f_1(x) f_1(y) + A_2 f_2(x) f_2(y) + \dots \qquad (1)$$

where A_0, A_1, A_2 are independent of x and y. Also the leading power of x in $f_n(x)$ may be assumed to be x^n, the leading coefficient being at present arbitrary. If $f(x)$ consists of a finite number of terms, the functions f_1, f_2, ... are, of course, finite in number; otherwise the number of these functions is indefinite, but, as we are only concerned with the manner of deducing successively the constants A_0, A_1, A_2, ... and the coefficients in $f_1, f_2, \dots$ from those in f, we need not consider the question of convergency of any of the series.

We shall, moreover, assume that the constants A_0, A_1, A_2, ... do not vanish identically, which assumption includes the supposition that $f(0)$ is not equal to zero.

Differentiating (1) n times for y, and putting $y = 0$, we have

$$\frac{d^n}{dx^n} f(x) = \text{linear function of } f, f_1, \dots, f_n.$$

Using all such deductions for values of n from 1 to n, we see that f_n is a linear function of f and its first n derivatives, and that the condition that the leading power of x shall be x^n makes this representation of f_n unique.

Hence $Lf_{n-1}+M\frac{d}{dx}f_{n-1}+Nf_{n-2}$, where L, M, N are constants, is a linear function of f and its first n derivatives, and in general its leading power of x is x^{n-2}.

But we may so choose L, M, and N that the coefficients of x^{n-2} and x^{n-1} are zero, and in this case the function can be none other than a constant multiple of f_n. We may write then

$$f_n = Lf_{n-1}+M\frac{d}{dx}f_{n-1}+Nf_{n-2}. \tag{2}$$

Now, let
$$\int_0^\infty f_n(t)\,e^{-t/x}\,dt = x^{n+1}\phi_n(x).$$

Then
$$\int_0^\infty \frac{d}{dt}f_{n-1}(t)\,e^{-t/x}\,dt = \Big[f_{n-1}(t)\,e^{-t/x}\Big]_0^\infty + \frac{1}{x}\int_0^\infty f_{n-1}(t)\,e^{-t/x}\,dt.$$

Assuming that the integrated part vanishes at both limits, we see that (2) reduces to
$$x^2\phi_n = Lx\phi_{n-1}+M\phi_{n-1}+N\phi_{n-2}.$$

Since the leading coefficients in the ϕ's are still arbitrary, we may write this relation in the form

$$\phi_{n-2} = (1-a_n x)\,\phi_{n-1}+\beta_n x^2\phi_n. \tag{3}$$

The integrated part in the above equation will not, however, vanish at the lower limit when $n=1$; so that this case will require special consideration.

We have
$$\frac{d}{dt}f(t) = Af'(0)\,f(t)+Bf_1'(0)\,f_1(t);$$

therefore

$$\Big[f(t)\,e^{-t/x}\Big]_0^\infty + \frac{1}{x}\int_0^\infty f(t)\,e^{-t/x}\,dt = Af'(0)\int_0^\infty f(t)\,e^{-t/x}dt+Bf_1(0)\int_0^\infty f_1(t)\,e^{-t/x}dt.$$

If
$$f(t) = a_0+a_1t+\frac{a_2}{2!}t^2+\frac{a_3}{3!}t^3+\ldots,$$

then, still assuming that $t=\infty$ gives a zero value for $f(t)\,e^{-t/x}$, we have

$$-a_0+\phi_0 = xAa_1\phi_0+Bf_1'(0)\,x^2\phi_1$$

or
$$(1-xa_1)\,\phi_0 = a_0+\beta_1x^2\phi_1,$$

while, by (3),
$$\phi_0 = (1-a_2x)\,\phi_1+\beta_2x^2\phi_2,$$
$$\phi_1 = (1-a_3x)\,\phi_2+\beta_3x^2\phi_3,$$
$$\ldots \qquad \ldots \qquad \ldots.$$

Hence
$$\phi_0 = a_0 + a_1 x + a_2 x^2 + \ldots$$
$$= \frac{a_0}{1 - a_1 x - \beta_1 x^2 \phi_1/\phi_0}$$
$$= \frac{a_0}{1-a_1 x-}\,\frac{\beta_1 x^2}{1-a_2 x-}\,\frac{\beta_2 x^2}{1-a_3 x-\ldots}. \tag{4}$$

It may be noticed that the terms independent of x in all the functions ϕ are the same and equal to a_0. Moreover, it should be mentioned that (4) fails if any of the β's are zero, *i.e.*, if any of the persymmetrical determinants called a_{2n} in § 1 vanish.

We may now easily deduce the value of A_n. For $\frac{d^n}{dx^n} f(x) =$ a linear function in $f, f_1, \ldots, f_n$, where the coefficient in f_n is $A_n \frac{d^n}{dy^n} f_n(y)$, when $y = 0$. But
$$f_n(y) = \frac{a_0 y^n}{n!} + \ldots;$$
therefore this coefficient is $A_n a_0$. Thus
$$\frac{d^n}{dx^n} f(x) = A_n a_0 f_n(x) + \text{terms in } f_{n-1}(x) \ldots.$$

Similarly, $$\frac{d^{n+1}}{dx^{n+1}} f(x) = A_{n+1} a_0 f_{n+1}(x) + \ldots.$$

But
$$\frac{d^{n+1}}{dx^{n+1}} f(x) = A_n a_0 \frac{d}{dx} f_n(x) + \ldots$$
$$= A_n a_0 \{f_{n-1}(x) + a_n f_n(x) + \beta_n f_{n+1}(x)\} + \ldots.$$

Hence, equating coefficients of $f_{n+1}(x)$, we have
$$A_{n+1} = \beta_n A_n.$$

We see, then, that, if $\phi_0(x)$ is known as a continued fraction
$$\frac{e_0}{1-}\,\frac{e_1 x}{1-\ldots},$$
or, by Lemma I., as $$\frac{a_0}{1-a_1 x-}\,\frac{\beta_1 x^2}{1-a_2 x-\ldots},$$
we may obtain a series of functions $\phi_1, \phi_2, \ldots$; so that an identity
$$f(x+y) = A_0 f(x) f(y) + A_1 f_1(x) f_1(y) + \ldots$$
may be obtained, where the leading power of $f_n(x)$ is x^n.

Conversely, if the relation (1) is known, we may convert $\phi_0(x)$ into a continued fraction.

The connection is simpler when $f(x)$ is an even function of x; for then

$$a_1 = a_3 \dots = 0.$$

For instance, the "addition" theorem in Bessel's function

$$J_0(x+y) = J_0(x)\,J_0(y)+2J_1(x)\,J_1(y)+\dots$$

is connected with the chain-fraction form for $(1+x^2)^{-\frac{1}{2}}$, which is obtained from Gauss's formula for converting the quotient

$$F\{a,\ \beta+1,\ \gamma+1,\ x\}/F\{a,\ \beta,\ \gamma,\ x\}$$

into a continued fraction. Again, the relation

$$J_0\sqrt{(x^2+y^2)} = J_0(x)\,J_0(y)-2J_2(x)\,J_2(y)+\dots$$

is connected with the chain-fraction form for e^x, also obtainable from Gauss's formula.

A very general relation of the type of the Bessel formula may be got from Gauss's by putting $\beta = 0$, so that $f(x)$ is of the form

$$1+\frac{a}{\gamma}x+\frac{a(a+1)}{\gamma(\gamma+1)}\frac{x^2}{2!}+\frac{a(a+1)(a+2)}{\gamma(\gamma+1)(\gamma+2)}\frac{x^3}{3!}+\dots;$$

but is not of sufficient importance to work out here. The functions $f_1, f_2, \dots$ have all factorial coefficients and are absolutely convergent; but the relation (1) would not be valid unless the series of functions on the right-hand side were also convergent.

It is perhaps interesting to note that, if $f(x)$ is not algebraic, the series in (1) cannot be finite unless the fraction in (4) is algebraic. It then can be expressed as a series of partial fractions of the form $\dfrac{A}{1+ax}$, and the corresponding $f(x)$ would consist of a finite series of exponential functions of x, the simplest cases of (1) being

$$e^{x+y} = e^x.e^y \quad \text{and} \quad \cos(x+y) = \cos x\cos y-\sin x\sin y.$$

The elliptic functions $\operatorname{cn} x$ and $\operatorname{dn} x$ lend themselves to the formula of this section in a very elegant way. We have

$$\operatorname{cn}(x+y) = \frac{\operatorname{cn} x\operatorname{cn} y-\operatorname{sn} x\operatorname{dn} x\operatorname{sn} y\operatorname{dn} y}{1-k^2\operatorname{sn}^2 x\operatorname{sn}^2 y}$$

$$= \operatorname{cn} x\operatorname{cn} y-\operatorname{sn} x\operatorname{dn} x\operatorname{sn} y\operatorname{dn} y+k^2\operatorname{sn}^2 x\operatorname{cn} x\operatorname{sn}^2 y\operatorname{dn} y-\dots;$$

so that the series $f_1, f_2, \dots$ are

$$-\operatorname{sn} x\operatorname{dn} x,\quad \frac{1}{2}\operatorname{sn}^2 x\operatorname{dn} x,\quad -\frac{1}{3!}\operatorname{sn}^3 x\operatorname{cn} x,\quad \frac{1}{4!}\operatorname{sn}^3 x\operatorname{dn} x,\quad \dots;$$

so that $A_1 = -1,\quad A_2 = 2^2k^2,\quad A_3 = -(3\,!)^2k^2,\quad A_4 = -(4\,!)^2k^4,\quad \ldots,$

and $\quad \beta_1 = -1,\quad \beta_2 = -2^2k^2,\quad \beta_3 = -3^2,\quad \beta_4 = -4^2k^2,\quad \ldots,$

while the α's of (4) are all zero, since $\operatorname{cn} x$ is even. Hence we obtain the formula (1) of § 3.

Similarly, $\quad \sec^n(x+y) = (\cos x \cos y - \sin x \sin y)^{-n};$

so that the functions $f_1, f_2, \ldots$ are

$$\sec^{n+1} x \sin x,\qquad \frac{1}{2}\sec^{n+2} x \sin^2 x,\qquad \frac{1}{3\,!}\sec^{n+3} x \sin^3 x,\qquad \ldots,$$

while $\quad A_1 = n,\quad A_2 = 2\,!\,n(n+1),\quad A_0 = 3\,!\,(n+1)(n+2),\quad \ldots,$

and $\quad \beta_1 = n,\quad \beta_2 = 2(n+1),\quad \beta_3 = 3(n+2),\quad \ldots,$

whence $$\int_0^\infty \sec^n t\, e^{-t/x}\,dt = \frac{x}{1-}\,\frac{nx^2}{1-}\,\frac{2(n+1)x^2}{1-\ldots}.$$

This formula is, however, obviously absurd, since the integrand will pass through an infinity of infinite values. We may correct the formula by taking the hyperbolic secant, and derive the relation

$$\int_0^\infty \operatorname{sech}^n t\, e^{-t/x}\,dt = \frac{x}{1+}\,\frac{nx^2}{1+}\,\frac{2(n+1)x^2}{1+}\,\frac{3(n+2)x^2}{1+\ldots}.$$

The convergency of the fraction is readily established by Lemma IV.

The identity $\left\{\dfrac{1+x+y}{(1+x)(1+y)}\right\}^{-n} = \left\{1-\dfrac{xy}{(1+x)(1+y)}\right\}^{-n}$

gives a formula of the type § 4 (1), but the corresponding integral

$$\int_0^\infty (1+t)^{-n} e^{-t/x}\,dt,$$

i.e., the asymptotic series

$$x - nx^2 + n(n+1)x^3 - \ldots,$$

may be represented as a continued fraction more simply by Gauss's formula

$$\frac{1}{\gamma}F\{\alpha, \beta+1, \gamma+1, x\}/F\{\alpha, \beta, \gamma, x\}$$
$$= \frac{1}{\gamma-}\,\frac{\alpha(\gamma-\beta)x}{\gamma+1-}\,\frac{(\beta+1)(\gamma-\alpha+1)x}{\gamma+2-}\,\frac{(\alpha+1)(\gamma-\beta+1)x}{\gamma-3-\ldots},$$

by making x and γ infinite, while x/γ remains finite. If $\beta = 0$, we have

$$1+\alpha x+\alpha(\alpha+1)x^2+\ldots = \frac{1}{1-}\,\frac{\alpha x}{1-}\,\frac{x}{1-}\,\frac{(\alpha+1)x}{1-}\,\frac{2x}{1-}\,\frac{(\alpha+2)x}{1-}\,\frac{3x}{1-\ldots};$$

so that

$$\int_0^\infty (1+t)^{-n} e^{-t/x} dt = \frac{x}{1+}\,\frac{nx}{1+}\,\frac{x}{1+}\,\frac{(n+1)x}{1+}\,\frac{2x}{1+}$$

$$= \frac{x.}{1+nx-}\,\frac{nx^2}{1+(n+2)x-}\,\frac{2(n+1)x^2}{1+(n+4)x-}\,\frac{3(n+2)x^2}{1+\dots}.$$

If we put m for $\frac{1}{x}$ and $-n$ for n, we have

$$\int_0^\infty (1+t)^n e^{-mt} dt = \frac{1}{m-n+}\,\frac{n}{m-n+2+}\,\frac{2(n+1)}{m-n+4+}\,\frac{3(n+2)}{m-n+6+\dots}.$$

$$\text{This integral} = \int_{-1}^\infty (1+t)^n e^{-mt} dt - \int_{-1}^0 (1+t)^n e^{-mt} dt$$

$$= \int_0^\infty u^n e^{-mu} e^m du - \int_0^1 u^n e^{-mu} e^m du$$

$$= \frac{e^m}{m^{n+1}}\Gamma(n+1) - e^m\left\{\frac{1}{n+1} - \frac{m}{1(n+2)} + \frac{m^2}{2!\,(n+3)} - \dots\right\}$$

$$= \frac{e^m}{m^{n+1}}\Gamma(n+1) - \frac{1}{\gamma} F\{\alpha, \beta+1, \gamma+1, x\} / F\{\alpha, \beta, \gamma, x\},$$

where $\qquad \gamma = n+1, \quad \alpha = \gamma, \quad \beta = \infty, \quad x\beta = -m.$

This ratio of hypergeometric series is therefore

$$\frac{1}{n+1-}\,\frac{(n+1)m}{n+2+}\,\frac{m}{n+3-}\,\frac{(n+2)m}{n+4+\dots};$$

so that we have $\frac{e^m}{m^{n+1}}\Gamma(n+1)$ expressed as the sum of two continued fractions.

An alternative form of the identity is

$$\frac{e^m\Gamma(n+1)}{m^{n+1}} = \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{m^2}{(n+1)(n+2)(n+3)} + \dots$$

$$+ \frac{1}{m-n+}\,\frac{n}{m-n+2+}\,\frac{2(n+1)}{m-n+4+}\,\frac{3(n+2)}{m-n+6+\dots}.$$

6. The integral $\int_0^\infty \frac{t}{\sinh t} e^{-t/x} dt$ may be expressed as an asymptotic series in odd powers of x; so that we may assume it equivalent to a continued fraction of the form

$$\frac{x}{1-}\,\frac{e_1 e_2 x^2}{1-}\,\frac{e_3 e_4 x^2}{1-}\dots.$$

Calling this ϕx, we may notice that

$$\phi\frac{x}{1-x}-\phi\frac{x}{1+x}=\int_0^\infty\frac{t}{\sinh t}\left\{e^{-t/x(1-x)}-e^{-t/x(1+x)}\right\}dt=2\int_0^\infty te^{-t/x}\,dt=2x^2$$

Hence $$\frac{x}{1-x-}\,\frac{e_1e_2x^2}{1-x-}\,\frac{e_3e_4x^2}{1-x-}\cdots=2x^2+\frac{x}{1+x-}\,\frac{e_1e_2x^2}{1+x-}\cdots. \qquad (1)$$

Dividing by x, and using Lemmas I. and II.,

$$\frac{1}{1-x-}\,\frac{e_1e_2x^2}{1-x-}\cdots=1+\frac{f_1x}{1-}\,\frac{f_2x}{1-}\cdots,$$

where $$e_1=1,\quad e_2+e_3=e_4+e_5=\ldots$$

and $$f_1=e_1,\quad f_2=e_1+e_2,\quad f_2f_3=e_2e_3,\quad f_3+f_4=e_3+e_4\ldots.$$

But now $$\frac{f_1}{1-}\,\frac{f_2x}{1-}\cdots=2-\frac{f_1}{1+}\,\frac{f_2x}{1+}\cdots,$$

from (1); so that, by Lemma II.,

$$f_2+f_3=0=f_4+f_5=f_6+f_7=\ldots:$$

Hence $$e_1=1,\quad f_2=e_1+e_2,\quad f_2^2=e_2(e_2-1),$$

and therefore $e_2(e_2-1)=(1+e_2)^2$ or $3e_2=-1$; so that $e_2=-\frac{1}{3}$, $e_3=\frac{4}{3}$, while $f_2=-f_3=e_1+e_2=\frac{2}{3}$. Again, $f_4f_5=e_4e_5$; therefore

$$-f_4^2=e_4(1-e_4).$$

But $$f_4=e_3+e_4-f_3=e_4+\tfrac{4}{3}+\tfrac{2}{3}=e_4+2;$$

therefore $(e_4+2)^2=e_4(e_4-1)$ or $5e_4=-4$, and $e_5=\frac{9}{4}$.

In this way all the successive e's can be found, and, by induction, we have

$$e_{2n}=-\frac{n^2}{2n+1},\qquad e_{2n+1}=\frac{(n+1)^2}{2n+1},$$

$$f_{2n}=-f_{2n+1}=\frac{n(n+1)}{2n+1}.$$

Hence $$\int_0^\infty\frac{t}{\sinh t}e^{-t/x}\,dt=\frac{x}{1+}\,\frac{1^4x^2}{3+}\,\frac{2^4x^2}{5+}\,\frac{3^4x^2}{7+}\cdots.$$

Changing x into $\dfrac{x}{x+2}$, and t into $\frac{1}{2}t$, we have

$$\int_0^\infty\frac{t}{e^t-1}e^{-t/x}\,dt=\frac{2x}{a+2+}\,\frac{1^4x^2}{3(x+2)+}\,\frac{2^4x^2}{5(x+2)+}\cdots.$$

From this, by application of Lemmas I. and II., we get

$$B_2-B_4x^2+B_6x^4-\ldots=\frac{1}{6+}\,\frac{1.2^2.3x^2}{10+}\,\frac{2.3^2.4x^2}{14+}\,\frac{3.4^2.5x^2}{18+}\cdots.$$

7. The fraction $\frac{1}{1-x+}\,\frac{x^2}{2(1-x)+}\,\frac{3^2x^2}{2(1-x)+}\,\ldots$, which we shall write $f(x)$, has been discussed by Prof. Muir (*Phil. Trans.*, 1877). It may be reduced to a definite integral form in the following manner:—

By Lemma I., we may write the fraction in the form

$$\frac{1}{1-}\,\frac{e_1 x}{1-}\,\frac{e_2 x}{1-}\,\ldots,$$

where

$$e_1 = 1,\ e_2 = -\frac{1}{2},\ e_2+e_3 = 1,\ e_3e_4 = -\frac{3^2}{4},\ e_4+e_5 = 1,\ e_5e_6 = -\frac{5^2}{8},\ \ldots,$$

i.e., $\quad e_3 = \frac{3}{2},\quad e_4 = -\frac{3}{2},\quad e_5 = \frac{5}{2},\quad e_6 = -\frac{5}{2}.$

Moreover, by Lemma I.,

$$f(x) = \frac{1}{1-}\,\frac{e_1 x}{1-e_2x-}\,\frac{e_2e_3x^2}{1-(e_3+e_4)x-}\,\ldots.$$

But evidently $e_{2n-1}+e_{2n} = 0$; therefore

$$f(x) = \frac{1}{1-}\,\frac{x}{1+\frac{1}{2}x+u},$$

where u is an even function of x.

Thus $$f(x) = \frac{1+\frac{1}{2}x+u}{1-\frac{1}{2}x+u},$$

and therefore $f(x)f(-x) = 1$, as is shewn by Prof. Muir in *Phil. Trans.*, 1877.

If we put $\log f(x) = \phi(x)$, we have $\phi(x)+\phi(-x) = 0$; so that $\phi(x)$ is an odd function, while, since

$$f\left(\frac{x}{1+x}\right) = \frac{1+x}{1+}\,\frac{x^2}{2+}\,\frac{3^2x^2}{2+}\,\ldots,$$

we see that $\frac{1}{1+x}f\left(\frac{x}{1+x}\right)$ is an even function and $= \frac{x}{1-x}f\left(-\frac{x}{1-x}\right)$.

Hence $$\phi\left(\frac{x}{1+x}\right)-\log(1+x) = \phi\left(-\frac{x}{1-x}\right)-\log(1-x),$$

and therefore $\phi\left(\frac{x}{1+x}\right)+\phi\left(\frac{x}{1-x}\right) = 2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots\right).$

Assume $\phi(x) = \int^{\infty}\psi(t)e^{-t/x}dt$, where evidently $\psi(t)$ is even. Then

$$\int_0^{\infty}\psi(t)(e^t+e^{-t})e^{-t/x}dt = 2\left(x+\frac{x^3}{3}+\ldots\right),$$

which series $$= 2\int_0^{\infty}\left(t+\frac{t^3}{3!}+\frac{t^4}{5!}+\ldots\right)e^{-t/x}dt.$$

Hence
$$\psi(t)\cosh t = \frac{1}{t}\sinh t,$$

and
$$\phi(x) = \int_0^\infty \frac{\tanh t}{t}\, e^{-t/x}\, dt.$$

8. A fraction
$$\frac{e_0}{1-}\,\frac{e_1 x}{1-}\,\frac{e_2 x}{1-\dots} \qquad (1)$$

cannot in general be reduced to the form

$$\frac{e_0}{1-e_1x\mp\beta_1x^2-}\;\frac{\gamma_1 x^4}{1-a_2x\mp\beta_2x^2-}\;\frac{\gamma_2 x^4}{1-a_3x\mp\beta_3x^2-\dots}, \qquad (2)$$

but it may be shewn that this reduction is possible if

$$e_2+e_3 = 0 = e_6+e_7 = \text{in general } e_{4n+2}+e_{4n+3}.$$

Now, it can easily be seen that the first convergent of (2) implies neglect of x^4 and higher powers in equating (2) to the equivalent power series; in other words, this first convergent of (2) is the fourth convergent of (1). The second convergent of (2) implies the neglect of x^8, and is therefore the eighth convergent of (1).

Again, by Lemma I., if

$$1-e_1x = D_1, \quad 1-(e_2+e_3)x = D_3, \quad 1-(e_4+e_5)x = D_5, \quad \dots,$$

we have

$$4n\text{-th convergent of (1)} = 2n\text{-th convergent of } \frac{e_0}{D_1-}\,\frac{e_1e_2x^2}{D_3-}\,\dots,$$

which
$$= \frac{\dfrac{e_0}{D_1}}{1-}\;\frac{\dfrac{e_1e_2x^2}{D_1D_3}}{1-}\;\frac{\dfrac{e_3e_4x^2}{D_3D_5}}{1-};$$

therefore the $4n$-th convergent of (1) = n-th convergent of

$$\frac{\dfrac{e_0}{D_1}}{1-\dfrac{e_1e_2x^2}{D_1D_3}-}\;\frac{\dfrac{e_1e_2e_3e_4x^4}{D_1D_3^2D_5}}{1-\dfrac{e_3e_4x^2}{D_3D_5}-\dfrac{e_5e_6x^2}{D_5D_7}-}\;\frac{\dfrac{e_5e_6e_7e_8x^4}{D_5D_7^2D_9}}{1-\dfrac{e_7e_8x^2}{D_7D_9}-\dfrac{e_9e_{10}x^2}{D_9D_{11}}-}\;\dots. \qquad (3)$$

This fraction should be identical, convergent by convergent, with (2).

Hence $D_3 = 1$, *i.e.*, $e_2+e_3 = 0$, and (3) becomes

$$\frac{e_0}{D_1-e_1e_2x^2-}\;\frac{\dfrac{e_1e_2e_3e_4x^4}{D_5}}{1-\dfrac{e_3e_4x^2}{D_5}-\dfrac{e_5e_6x^2}{D_5D_7}-}.$$

This second constituent must be simplified by multiplying numerator and

denominator by $D_5 D_7$, but, in order to be equivalent to the second constituent of (2), we must have $D_7 = 1$, *i.e.*, $e_6+e_7 = 0$.

Finally, $D_{11} = D_{15} = D_{4n+3} = 1$, and (3) becomes

$$\frac{e_0}{1-e_1x-e_1e_2x^2-}\;\frac{e_1e_2e_3e_4x^4}{1-(e_4+e_5)x-(e_3e_4+e_5e_6)x^2-}\;\frac{e_5e_6e_7e_8x^4}{1-(e_8+e_9)x-(e_7e_8+e_9e_{10})x^2-}. \tag{4}$$

If $\phi(x) = \int_0^\infty \frac{t\,e^{-t/x}}{\cosh t}\,dt = \frac{f_0x^2}{1-}\,\frac{f_1x^2}{1-}\,\frac{f_2x^2}{1-\dots} = \frac{f_0x^2}{1-f_1x^2-}\,\frac{f_1f_2x^4}{1-(f_2+f_3)x^2-\dots}$,

then $$\phi\left(\frac{x}{1-x}\right) = \frac{f_0x^2}{(1-x)^2-f_1x^2-}\,\frac{f_1f_2x^4}{(1-x)^2-(f_2+f_3)x^2-\dots},$$

which is a fraction of the type just discussed.

Hence, if $$\phi\left(\frac{x}{1-x}\right) = \frac{e_0x^2}{1-}\,\frac{e_1x}{1-}\,\frac{e_2x}{1-\dots},$$

we have $$e_2+e_3 = 0 = e_6+e_7 = e_{10}+e_{11} = \dots,$$

while $$e_1 = e_4+e_5 = e_8+e_9 = \dots = 2.$$

But $$\phi\left(\frac{x}{1-x}\right)+\phi\left(\frac{x}{1+x}\right) = 2\int_0^\infty t\,e^{-t/x}\,dt = 2x^2;$$

therefore $$\frac{e_0}{1-}\,\frac{e_1x}{1-}\dots+\frac{e_0}{1+}\,\frac{e_1x}{1+}\dots = 2.$$

By Lemma II., this implies $e_1+e_2 = 0 = e_3+e_4 = e_5+e_6 = \dots.$ Thus

$$\begin{aligned} e_1 &= -e_2 = e_3 = -e_4 = 2, \\ e_5 &= 4 = -e_6 = e_7 = -e_8, \\ e_9 &= 6 = -e_{10} = e_{11} = -e_{12}, \\ &\dots\quad \dots\quad \dots\quad \dots \end{aligned}$$

and $$\phi\left(\frac{x}{1-x}\right) = \frac{x^2}{1-2x+2^2x^2-}\,\frac{2^4x^4}{1-2x+(2^2+4^2)x^2-}\,\frac{4^4x^4}{1-2x+(4^2+6^2)x^2-};$$

so that $$\phi(x) = \frac{x^2}{1+(2^2-1)x^2-}\,\frac{2^4x^4}{1+(2^2+4^2-1)x^2-}.$$

Let $x^2/(1-x^2) = y$; then

$$\begin{aligned}\phi(x) &= \frac{y}{1+2^2y-}\,\frac{2^4y^2}{1+(2^2+4^2)y-}\,\frac{4^4y^2}{1+(4^2+6^2)y-} \\ &= \frac{y}{1+}\,\frac{2^2y}{1+}\,\frac{2^2y}{1+}\,\frac{4^2y}{1+}\,\frac{4^2y}{1+}\,\frac{6^2y}{1+}\,\frac{6^2y}{1+}.\end{aligned}$$

If $x < 1$, y is positive, and, by Lemma IV., the condition for convergency is satisfied.

9. If $\psi(x)$ denote $$2x^3+\int_0^\infty \frac{t^2}{e^t-1}e^{-t/x}\,dt,$$
which may be represented by the asymptotic series
$$x^2+x^3+3B_2x^4-5B_4x^6+\ldots,$$
then
$$\psi\left(\frac{x}{1+x}\right)=\psi(-x). \tag{1}$$

If, then, $$\psi(x)=\frac{x^2}{1-}\,\frac{e_1x}{1-}\,\frac{e_2x}{1-\ldots}=\frac{x^2}{1-ax-\beta x^2-\gamma x^3-\ldots},$$
we have
$$\frac{x^2}{(1+x)^2-ax(1+x)-\beta x^2-\gamma x^3-\ldots}=\frac{x^2}{1+ax-\beta x^2+\gamma x^3-\ldots},$$
and evidently $$\gamma=0,\quad a=1.$$
Let then $$\psi(x)=\frac{x^2}{1-e_1x-e_1e_2x^2-x^2\psi_1x},$$
so that, from (1),
$$(1+x)^2-e_1x(1+x)-e_1e_2x^2-x^2\gamma_1\left(\frac{x}{1+x}\right)=1+e_1x-e_1e_2x^2-x^2\psi_1(-x)$$
and $$\psi_1\left(\frac{x}{1+x}\right)=\psi_1(-x).$$

Again, if $$\psi_1(x)=\frac{e_1e_2e_3e_4x^2}{1-a_1x-\beta_1x^2-x^2\psi_2x},$$
we shall have $$\psi_2\left(\frac{x}{1+x}\right)=\psi_2(-x)\quad\text{and}\quad x_1=1.$$
Proceeding in this way, we see that $\psi(x)$ takes the form (4) in § 8, where
$$e_1=e_4+e_5=e_8+e_9=\ldots=1$$
and $$e_2+e_3=e_6+e_7=\ldots=0.$$
The actual calculation of the e's may be effected by considering the relation
$$\psi(x)-\psi(-x)=2x^3; \tag{2}$$
so that $$\frac{e_0}{1-}\,\frac{e_1x}{1-}\ldots=2x+\frac{e_0}{1+}\,\frac{e_1x}{1+}\ldots.$$
By using Lemma II., and putting
$$\frac{e_0}{1-}\,\frac{e_1x}{1-}\ldots=1+\frac{f_1x}{1-}\,\frac{f_2x}{1-\ldots},$$
we have $$\frac{f_1}{1-}\,\frac{f_2x}{1-}\ldots+\frac{f_1}{1+}\,\frac{f_2x}{1+\ldots}=1,$$

whence $\quad f_1 = 1, \quad f_2+f_3 = 0, \quad f_4+f_5 = 0, \quad \ldots,$

where $\quad f_1 = e_0 e_1, \quad f_2 = e_1+e_2, \quad f_2 f_3 = e_2 e_3, \quad f_3+f_4 = e_3+e_4, \quad \ldots .$

The general values of the e's may be deduced inductively, viz.,

$$e_{4n+1} = \frac{(n+1)^2}{2n+1}, \quad e_{4n+2} = -\frac{n+1}{2} = -e_{4n+3}, \quad e_{4n+4} = -\frac{(n+1)^2}{2n+3}.$$

Since $$\psi(x) = \frac{x^2}{1-x-e_1 e_2 x^2-}\ \frac{e_1 e_2 e_3 e_4 x^4}{1-x-(e_3 e_4+e_5 e_6)\, x^2-\ldots},$$

we get, by writing $x^2/(1-x) = y$,

$$\psi(x) = \frac{y}{1-e_1 e_2 y-}\ \frac{e_1 e_2 e_3 e_4 y^2}{1-(e_3 e_4+e_5 e_6) y-\ldots}$$

$$= \frac{y}{1-}\ \frac{e_1 e_2 y}{1-}\ \frac{e_3 e_4 y}{1-}\ \frac{e_5 e_6 y}{1-\ldots} \quad \text{(by Lemma I.)}$$

$$= \frac{y}{1+}\ \frac{y}{2+}\ \frac{1^3 y}{3+}\ \frac{1^3 y}{2+}\ \frac{2^3 y}{5+}\ \frac{2^3 y}{2+}\ \frac{3^3 y}{7+}\ \frac{3^3 y}{2+\ldots}.$$

Moreover I find that

$$3B_2 x^2 - 5B_4 x^4 + \ldots = \frac{x^2}{2+}\ \frac{1^2.2x^2}{3+}\ \frac{1.2^2 x^2}{2+}\ \frac{2^2.3x^2}{5+}\ \frac{2.3^2 x^2}{2+\ldots}.$$

The types of definite integral to which the methods of this memoir are applicable are evidently very limited in number. Those treated of in § 2 to § 5 depend upon the fact that each integral is the leader of a series of functions connected by a difference equation of the second order. Such integrals as $\int_0^\infty (1+t^2)^n e^{-t/x}\, dt$ depend on difference equations of the third order, and would not be readily adaptable to the above methods, but it is not impossible that some form of convergent approximation may exist, depending on algebraic fractions in x.

The integrals treated of in the later sections depend upon relations of the type § 9, (1) and (2), and may have an analogue of the general form $\int_0^\infty t^n e^{-t/x} (e^t - 1)\, dt$, but I have not succeeded in finding a general law for the form of continued fraction when n is greater than 2.

THE THEORY OF INTEGRAL EQUATIONS

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CONTENTS.

I. *Sketch of the Subject.*

Although equations involving definite integrals are to be found in the works of Abel and Gauss, it was not until quite recently that a general theory of the subject began to be constructed.

1. In the solution of a dynamical problem* Abel found it necessary to determine a function $\phi(t)$ such that, for a given arbitrary function $f(s)$,

$$f(s) = \int_0^s \frac{\phi'(t)\,dt}{\sqrt{(s-t)}}.$$

This he succeeded in doing, showing that the more general equation

$$f(s) = \int_0^s \phi(t)(s-t)^{-n}\,dt \quad (0 < n < 1)$$

is satisfied by taking

$$\phi(s) = \frac{\sin n\pi}{\pi}\int_0^s f'(t)(s-t)^{n-1}\,dt.$$

This equation has lately been examined by Goursat,† who gives the solution for the case in which $f(0)$ is not zero. Abel's formula was generalised by Sonine,‡ but it was left for Volterra§ to give the complete solution of

* "Solution de quelques problèmes à l'aide d'intégrales définies": *Magazin for Naturvidenskaberne*, Aargang 1, Bind II., Christiania, 1823; *Coll. Works* (Sylow and Lie), Vol. I., p. 11.

† *Acta Math.*, 1903.

‡ *Ibid.*, 1884; see also *ibid.*, 1879.

§ *Ann. di Mat.*, [2], XXV., 1897. The first result is also given by Le Roux, *Ann. éc. norm.*, [2], XII., 1895, p. 293.

the general equation

$$f(y)-f(a) = \int_a^y \phi(x)\, H(x, y)\, dx. \tag{1}$$

His results are as follows.

If $f(y)$ and $f'(y)$ remain finite and continuous for values of y lying between a and $a+A$, and $H(x, y)$ and $\dfrac{\partial H}{\partial y}$, $= H_2(x, y)$, are always finite for $y > x > a$, $a+A > y > a$, and are integrable, and if the lower limit of the absolute value of $h(y) = H(y, y)$ is different from zero, there will exist one, and only one, finite and continuous function ϕ, which satisfies the functional equation for values of y between a and $a+A$, and this function will be given by

$$\phi(y) = \frac{f'(y)}{h(y)} - \frac{1}{h(y)} \int_a^y f'(x) \sum_0^\infty S_i(x, y)\, dx,$$

where
$$S_0(x, y) = H_2(x, y)/h(x) \tag{2}$$

and
$$S_i(x, y) = \int_y^x S_{i-j}(x, \xi)\, S_{j-1}(\xi, y)\, d\xi,$$

the term "integrable" being understood to include the conditions for a change in the order of integration in the multiple integrals which occur.

If the function $H(x, y)$ becomes infinite for $y = x$, so that the equation may be written

$$f(y)-f(a) = \int_a^y \phi(x) \frac{G(x, y)}{(y-x)^\lambda}\, dx \quad (\lambda < 1), \tag{3}$$

and if the function $G(x, y)$ satisfies the same kind of conditions as $H(x, y)$, and the functions $G_2(x, y)$ and $g(y)$ are defined as before, then there exists one, and only one, finite and continuous function ϕ which satisfies the functional equation for values of y between a and $a+A$, and this will be given by

$$\phi(z) = \frac{\sin \lambda\pi}{\pi} \frac{1}{g(z)} \int_a^z f'(x) \sum_0^\infty T(x, z)\, dx,$$

which
$$S_0(y, z) = \frac{\sin \lambda\pi}{\pi} \frac{1}{g(z)} \int_y^z G_2(y, \xi) \left(\frac{\xi-y}{z-\xi}\right)^{1-\lambda} \frac{d\xi}{z-y},$$

$$T_0(x, z) = (z-x)^{\lambda-1},$$

$$T_i(x, z) = \int_x^z S_0(\xi, z)\, T_{i-1}(x, \xi)\, d\xi.$$

2. The equation
$$f(s) = \int_a^b \kappa(s, t)\,\phi(t)\,dt, \tag{4}$$
which also dates back from Abel,* has been called by Hilbert an integral equation of the first kind. The functions $f(s)$ and $\kappa(s, t)$ are supposed to be known, and the function $\phi(t)$ is to be determined. The process of passing from $\phi(t)$ to $f(t)$ may be regarded as a transformation, and the function $\kappa(s, t)$ may be called the generating function† of the transformation; the theory has been developed from this point of view by Pincherle.‡

An important case in which a solution of the equation can be found is furnished by Fourier's double integral, which states that, if
$$g(s) = \int_0^\infty f(t) \cos \frac{\pi st}{2}\, dt,$$
then
$$f(s) = \int_0^\infty g(t) \cos \frac{\pi st}{2}\, dt;$$
but the function $g(s)$ must be subject to certain restrictions, and this is usually the case for an integral equation of the first kind.

The beautiful simplicity of the above formula and of a number of others of a similar character§ led mathematicians to seek a general theory of such recurrence formulæ. An analogous reciprocal formula for the

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* "Sur les fonctions génératrices et leurs déterminantes": *Collected Works* (Sylow and Lie), tome II., xi.

† In Germany the word *Kern* is used.

‡ "Sur certaines opérations fonctionnelles": *Acta Math.*, x.

§ For example, if
$$g(s) = \int_0^\infty h(x)\, x^{1-s}\, dx,$$
then
$$2\pi i\, h(y) = \int_{a-\infty i}^{a+\infty i} g(s)\, y^s\, ds$$
(Riemann, "Ueber die Anzahl der Primzahlen unter einer gegebene Grösse," 1859: *Gesammelte Werke*, Weber, p. 140).

If
$$g(s) = \int_c e^{st} h(t)\, dt,$$
then
$$h(t) = 2\pi i \int_0^\infty e^{-ts} g(s)\, ds$$
(Pincherle, *Mem. della Accad. della Sc. dell' Ist. di Bologna*, Serie 4, t. VIII.).

If
$$f(s) = \int_0^\infty J_n(st)\, \phi(t)\, t\, dt,$$
then
$$\phi(s) = \int_0^\infty J_n(st) f(t)\, t\, dt$$
(Hankel, *Math. Ann.*, Bd. VIII., p. 482, 1875; Sonine, *Math. Ann.*, Bd. XVI., p. 47, 1880).

Laplace transformation was found by Petzval,* and accordingly the integral equation of the first kind was studied in connection with the solution of linear differential equations by means of definite integrals.†

The connection with differential equations depends upon the possibility of constructing a relation

$$P_s\{\kappa(s, t)\} = Q_t\{g(s, t)\}, \tag{5}$$

where P_s is an operator such that

$$P_s\int_a^b \kappa(s, t)\phi(t)dt = \int_a^b P_s\kappa(s, t)\phi(t)dt$$

and

$$P_s\{f(s)\} = 0.$$

The function $\phi(t)$ must then be such that the integral

$$\int_a^b Q_t\{g(s, t)\}\phi(t)\,dt$$

can be directly evaluated to the value zero. If Q_t has the form $\Sigma a_r(t)\dfrac{d^n}{dt^n}$, the function ϕ is a solution of the equation adjoint to $Q_t = 0$; for then the quantity under the integral sign is a perfect differential. It appears to be generally true that, if a definite integral (4) exists, then there is a corresponding relation of the form (5), but the operators P_s and Q_t will not always contain differentiations only: indeed P may contain a number of integrations, and either operator may contain the operator

$$\Delta f(x) \equiv f(x+1) - f(x).$$

Another method of treating the equation

$$f(s) = \int_a^b \kappa(s, t)\phi(t)dt$$

was suggested by Volterra.‡ If $a < s < b$ and $\kappa(s, t)$ is symmetrical in s and t, the solution for an arbitrary f can be found if it is possible to determine a function $\lambda(t, z)$ such that, for $a < t < z < b$, the integral

$$\int_a^z \lambda(t, z)\kappa(s, t)\,dt$$

is independent of z.

We shall return to the discussion of the integral equation of the first

* *Integration der linearen Differentialgleichungen*, pp. 472, 473.

† A list of references is given in Pincherle's book: *Le operazioni distributive e le loro applicazioni all' analisi*, Bologna, 1901.

‡ "Sopra una problema di elettrostatica," *Rom. Linc. Trans.*, Ser. 3, Vol. VIII.

kind in § 2, and shall pass on now to the equation which owes its origin to Neumann's method of solving the problem of Dirichlet.*

The form in which Neumann's problem is presented by Poincaré† is as follows :—

Let S be a closed plane curve without double points. We propose to find a double layer of moment ϕ, such that its potential W satisfies the following conditions :—

It is harmonic inside and outside S, and its values V and V' at two points inside and outside S and very close to it are connected by the relation

$$V-V' = \lambda(V+V')+2\Phi, \tag{6}$$

where Φ is a given function.

If we put $\lambda = -1$, we obtain Neumann's problem, but it is convenient to retain this parameter, as it gives additional clearness to the subsequent work.

If, now, $x = x(s)$, $y = y(s)$ are the equations of the curve, the potential theory gives us the equations

$$V(s) = \quad \pi\phi(s)+\int_s \phi(t)\frac{\partial}{\partial t}\tan^{-1}\frac{y(t)-y(s)}{x(t)-x(s)}\,dt,$$

$$V'(s) = -\pi\phi(s)+\int_s \phi(t)\frac{\partial}{\partial t}\tan^{-1}\frac{y(t)-y(s)}{x(t)-x(s)}\,dt,$$

$\phi(t)$ being the strength of the double layer.

Substituting these values in (6), we are led at once to an equation of the form

$$f(s) = \phi(s)+\mu\int_s \kappa(s, t)\,\phi(t)\,dt, \tag{7}$$

which has been called by Hilbert an integral equation of the second kind. Many other problems in mathematical physics lead to equations of a similar character; so that we are justified in devoting considerable attention to their study.

If we write the equation

$$f(s) = \phi(s)-\lambda\int_a^b \kappa(s, t)\,\phi(t)\,dt \tag{7 bis}$$

in the symbolical form $f(s) = (1-\lambda S_\kappa)\,\phi(s)$, and apply the method of

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* "Ueber die Methode des arithmetischen Mittels": *Leipzig. Abh.*, Bd. XIII., 1887.

† "La méthode de Neumann et le problème de Dirichlet": *Acta Math.*, Bd. XX., 1896–97; "Sur les équations de la physique mathématique": *Rend. del circolo Palermo*, t. VIII., 1894.

Neumann,* we are led to the expansion

$$\phi(s) = [1+\lambda S_\kappa+\lambda^2 S_\kappa^2+\ldots]f(s).$$

If $f(s)$ and $\kappa(s, t)$ are finite and integrable, the above series will have a finite radius of convergence different from zero, and will therefore give the solution for certain values of λ. Performing the operations, and changing the order of integration, the solution becomes

$$\phi(s) = f(s)+\lambda\int_a^b K(s, t) f(t)\, dt, \tag{8}$$

where

$$K(s, t) = \kappa(s, t)+\lambda\int_a^b \kappa(s, r)\,\kappa(r, t)\, dr+\lambda^2\int_a^b\int_a^b \kappa(s, r)\,\kappa(r, \xi)\,\kappa(\xi, t)\, dr\, d\xi+\ldots. \tag{9}$$

The function $K(s, t)$ is called the solving function of the integral equation. The similarity of the equations (7) and (8) shows that the relation between it and $\kappa(s, t)$ is a reciprocal one, and it also follows that

$$\kappa(s, t) = K(s, t)-\lambda\int_a^b \kappa(s, r)\, K(r, t)\, dr\,; \tag{10}$$

so that $K(s, t)$ is the solution of (7) corresponding to $f(s) = \kappa(s, t)$.

It is noteworthy that the function $K(s, t)$ is also the solving function for the equation†

$$\xi(t) = \psi(t)-\lambda\int_a^b \kappa(s, t)\,\psi(s)\, ds, \tag{11}$$

the solution of this equation being given by

$$\psi(t) = \xi(t)+\lambda\int_a^b K(s, t)\,\xi(s)\, ds. \tag{12}$$

The disadvantage of the series obtained by Neumann's method is that it only converges for certain values of λ. This difficulty may be surmounted, as Plemelj shows, by the method of continuation; but a much better method was discovered by Fredholm‡ in 1900, in which the function $K(s, t)$ is exhibited as the ratio of two power series which are convergent for all values of λ.

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* Cf. Kellogg, "Zur Theorie der Integralgleichung $A(s, t)-\mathrm{A}(s, t) = \mu\int_0^1 A(s, r)\,\mathrm{A}(r, t)\, dr$": *Göttinger Nachrichten*, 1902, pp. 165–175.

† J. Plemelj, "Zur Theorie der Fredholmschen Funktionalgleichung": *Monatshefte für Mathematik und Physik*, Bd. xv., 1904.

‡ Ivar Fredholm, "Sur une nouvelle méthode pour la résolution du problème de Dirichlet": *Oefversigt af kongl. vet. akad. Förh. Stockholm*, 1900; also *Acta Math.*, 1903.

In this method the equation (7) is treated as the limit of a series of linear equations

$$f_1 = \phi_1 - \lambda \sum_1^n \kappa_{1r}\phi_r, \qquad f_2 = \phi_2 - \lambda \sum_1^n \kappa_{2r}\phi_r, \qquad \dots,$$

and similarly for equation (10). The formula obtained for K is then

$$K(s, t) = -\frac{\Delta(\lambda;\, s, t)}{\delta(\lambda)}, \tag{13}$$

where $$\Delta(\lambda;\, s, t) = -\kappa(s, t) + \lambda\Delta_1(s, t) - \lambda^2\Delta_2(s, t) + \lambda^3\Delta_3(s, t) - \dots, \tag{14}$$

$$\delta(\lambda) = 1 - \delta_1\lambda + \delta_2\lambda^2 - \dots, \tag{15}$$

$$\Delta_h(s, t) = \frac{1}{h!}\int_a^b \dots \int_a^b ds_1 \dots ds_h \begin{vmatrix} \kappa(s, t) & \kappa(s, s_1) & \dots & \kappa(s, s_h) \\ \kappa(s_1, t) & \dots & \dots & \dots \\ \kappa(s_h, t) & \dots & \dots & \kappa(s_h, s_h) \end{vmatrix}, \tag{16}$$

$$\delta_h = \frac{1}{h}\int_a^b \Delta_{h-1}(s, s)\, ds, \qquad \Delta_0(s, t) = \kappa(s, t), \tag{17}$$

$$\delta'(\lambda) = \int_a^b \Delta(\lambda;\, s, s)\, ds. \tag{18}$$

If the last formula is combined with the previous one for $K(s, t)$, viz.,

$$K(s, t) = \kappa(s, t) + \lambda\kappa_1(s, t) + \lambda^2\kappa_2(s, t) + \dots, \tag{19}$$

where $$\kappa_1(s, t) = \int_a^b \kappa(s, r)\,\kappa(r, t)\, dr, \qquad \kappa_n(s, t) = \int_a^b \kappa(s, r)\,\kappa_{n-1}(r, t)\, dr,$$

we get $$-\log \delta(\lambda) = a_1\lambda + \tfrac{1}{2}a_2\lambda^2 + \tfrac{1}{3}a_3\lambda^3 + \dots, \tag{20}$$

where $$a_1 = \int_a^b \kappa(s, s)\, ds, \qquad a_n = \int_a^b \kappa_{n-1}(s, s)\, ds.$$

Plemelj has used this result to obtain simpler expressions for $\delta(\lambda)$ and $\Delta(\lambda;\, s, t)$,

$$\delta(\lambda) = 1 - \frac{\lambda}{1!}a_1 + \frac{\lambda^2}{2!}\begin{vmatrix} a_1 & 1 \\ a_2 & a_1 \end{vmatrix} - \frac{\lambda^3}{3!}\begin{vmatrix} a_1 & 1 & 0 \\ a_2 & a_1 & 2 \\ a_3 & a_2 & a_1 \end{vmatrix} + \dots, \tag{21}$$

$$\Delta(\lambda;\, s, t) = -\kappa(s, t) + \frac{\lambda}{1!}\begin{vmatrix} \kappa(s, t) & 1 \\ \kappa_1(s, t) & a_1 \end{vmatrix} - \frac{\lambda^2}{2!}\begin{vmatrix} \kappa(s, t) & 2 & 0 \\ \kappa_1(s, t) & a_1 & 1 \\ \kappa_2(s, t) & a_2 & a_1 \end{vmatrix} + \dots. \tag{22}$$

If the quantity $\delta(\lambda)$ which corresponds to the determinant of the system of linear equations does not vanish, there will be a unique solution of the integral equation, and this is given by the formulæ (8) and (13);

but, if $\delta(\lambda) = 0$, the homogeneous equation

$$0 = \phi(s) - \lambda \int_a^b \kappa(s, t)\, \phi(t)\, dt \tag{23}$$

will possess a solution different from zero, and then certain conditions have to be satisfied by the function $f(s)$ in order that the equation (7) may possess a solution.* If λ_0 is a p-fold root of $\delta(\lambda)$, it can be shown that in the neighbourhood of λ_0 the function $K(s, t)$ has the form

$$K(s, t) = \frac{P(s, t)}{\lambda - \lambda_0} + F(s, t), \tag{24}$$

where $F(s, t)$ is finite for $\lambda = \lambda_0$ and $P(s, t)$ has the form

$$P(s, t) = \Phi_1(s, \lambda_0)\, \Psi_1(t, \lambda_0) + \ldots + \Phi_p(s, \lambda_0)\, \Psi_p(t, \lambda_0). \tag{25}$$

The functions $\Phi_1, \ldots, \Phi_p$; $\Psi_1, \ldots, \Psi_p$ are linearly independent solutions of the homogeneous equations

$$\left.\begin{array}{l} \phi(s) - \lambda_0 \int_a^b \kappa(s, \theta)\, \phi(\theta)\, d\theta \\[2ex] \psi(t) - \lambda_0 \int_a^b \kappa(s, t)\, \psi(s)\, ds \end{array}\right\} \tag{26}$$

respectively, and are connected by the orthogonal properties

$$\int \Phi_r(t, \lambda_0)\, \Psi_s(t, \lambda_0)\, dt = \begin{Bmatrix} 0 & (r \neq s) \\ 1 & (r = s) \end{Bmatrix}. \tag{27}$$

It is useful to write the equation (7) in the symbolical form

$$f(s) = S_\kappa \phi(s) ;$$

it can then be shown that the operations S_κ form a group, the law of multiplication being $S_\kappa S_l = S_g$, where

$$g(s, t) = \lambda\kappa(s, t) + \lambda l(s, t) - \lambda^2 \int_a^b \kappa(s, r)\, l(r, t)\, dr. \tag{28}$$

When the function $\kappa(s, t)$ is symmetrical† in s and t, many developments are suggested by the theory of the quadratic form $\Sigma \kappa_{pq} x_p x_q$, which becomes in the limit the double integral

$$\int_a^b \int_a^b \kappa(s, t)\, x(s)\, x(t)\, ds\, dt.$$

The function $\delta(\lambda)$ then corresponds to the determinantal equation of the quadratic form, and Sylvester's theorem that the roots of such an equation

* See Fredholm, *Acta Math.*, XXVII. ; also Erhard Schmidt, Inaugural Dissertation, Göttingen, 1905, p. 18.

† Hilbert, *Göttinger Nachrichten*, 1904.

are all real still remains true. The fact that a quadratic form can be expressed as a sum of squares also gives an important result, which may be stated as follows :—

Let $x(s)$ and $y(s)$ be two arbitrary functions such that $\int_a^b \{x(s)\}^2 ds$ and $\int_a^b \{y(s)\}^2 ds$ remain less than fixed quantities; then there exists an expansion

$$\int_a^b\int_a^b K(s, t)\, x(s)\, x(t)\, ds\, dt = \sum_n \frac{1}{\lambda_n}\int_a^b \psi_n(s)\, x_n(s)\, ds \int_a^b \psi_n(s)\, y_n(s)\, ds. \tag{29}$$

The function $\psi_n(s)$ is a solution of the homogeneous equation

$$0 = \psi_n(s) - \lambda_n \int_a^b \kappa(s, t)\, \psi_n(t)\, dt, \tag{30}$$

and may be defined by the equation*

$$\psi_n(s)\, \psi_n(t) = \frac{\Delta\{\lambda_n;\, s,\, t\}}{\delta'(\lambda_n)}. \tag{31}$$

These functions also possess the important orthogonal property

$$\int_a^b \psi_r(s)\, \psi_\kappa(s)\, ds = \begin{Bmatrix} 0 & (r \neq s) \\ 1 & (r = s) \end{Bmatrix}. \tag{32}$$

It can be shown that every symmetrical function must possess at least one function ψ,† and that a new solution of equation (7) may be obtained in terms of these functions in the form‡

$$\phi(s) = f(s) + \lambda \sum_n \frac{\psi_n(s)}{\lambda_n - \lambda}\int_a^b f(t)\, \psi_n(t)\, dt. \tag{33}$$

The definitions of the corresponding functions for an unsymmetrical *Kern* are given by Schmidt (*l.c.*): they are

$$\left.\begin{aligned} \phi_n(s) &= \lambda_n \int_a^b \kappa(s, t)\, \psi_n(t)\, dt \\ \psi_n(s) &= \lambda_n \int_a^b \kappa(t, s)\, \phi_n(t)\, dt \end{aligned}\right\}. \tag{34}$$

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* For an m-fold multiple root

$$\left\{\frac{\dfrac{\partial^{m-1}\Delta(\lambda;\, s,\, t)}{\partial\lambda^{m-1}}}{\dfrac{\partial^m \delta(\lambda)}{\partial\lambda^m}}\right\}_{\lambda=\lambda_h} = \psi_h(s)\,\psi_h(t) + \ldots + \psi_{h+m-1}(s)\,\psi_{h+m-1}(t).$$

The functions $\psi_n(s)$ are called by Hilbert the "Eigenfunktionen" of the "Kern" $\kappa(s, t)$, and the corresponding quantities λ_n the "Eigenwerte."

† If κ is such that no solution of $\int_a^b \kappa(s, t)\, g(t) = 0$ exists, there are an infinite number of functions ψ (Hilbert).

‡ Erhard Schmidt, *l.c.*

The quantities λ_n are shown to be all real, and the following important expansion theorem is given :—

If $\kappa(s, t)$ is such that to every function $a(s)$ and to every small quantity ϵ there corresponds a function $\beta(s)$ such that

$$\int_a^b \left[a(s) - \int_a^b \kappa(s, t)\,\beta(t)\,dt\right]^2 ds < \epsilon,$$

then every function $g(s)$ which can be defined by an integral of the form

$$g(s) = \int_a^b \kappa(s, t)\, h(t)\, dt \tag{85}$$

can be expanded in an absolutely and uniformly convergent series of the form $\Sigma A_n \phi_n(s)$, and a corresponding theorem holds for the functions ψ.

Another theorem for a quadratic form which has its analogue in the present theory is Gauss's variation problem, and Hilbert shows that the variation of the double integral

$$\int_a^b \int_a^b \kappa(s, t)\, x(s)\, x(t)\, ds\, dt,$$

subject to the condition $$\int_a^b \{x(s)\}^2 = 1, \tag{86}$$

leads to the integral equation

$$0 = x(s) - \int_a^b \kappa(s, t)\, x(t)\, dt.$$

The theory of integral equations has an important application to the theory of linear differential equations of the second order.* If the Green's function corresponding to a given set of boundary conditions for the differential equation

$$L(u) = \frac{d}{dx}\left(p\,\frac{du}{dx}\right) + q(u) = 0 \tag{87}$$

is taken as the generating function of an integral equation, the equation

$$f(x) = \int_a^b G(x, \xi)\,\phi(\xi)\,d\xi, \tag{88}$$

in which $f(x)$ is a function which can be differentiated twice, and which satisfies the given boundary conditions, will be given by

$$\phi(x) = -L\{f(x)\}\,; \tag{89}$$

* Hilbert, "Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen" (Zweite Mitteilung): *Göttinger Nachrichten*, 1904; Mason, Inaugural Dissertation, Göttingen, 1903; Andrae, Inaugural Dissertation, Göttingen, 1903.

also the Green's function for the differential equation

$$\Lambda(u) = L(u) + \lambda u = 0 \tag{40}$$

is the solving function for the integral equation of the second kind

$$f(x) = \phi(x) - \lambda \int_a^b G(x, \xi)\, \phi(\xi)\, d\xi.$$

The functions $\psi_n(x)$ corresponding to the function $G(x, \xi)$ are solutions of the equations $L(u) + \lambda_n u = 0$, and so the well known expansions in terms of trigonometrical, Legendre, Bessel, and other functions are found to be particular cases of the general expansion theorem given above.

Another important application of integral equations is connected with Riemann's problem in the theory of functions of a complex variable, but for this I must refer to the papers of Hilbert * and Kellogg.†

The theory of integral equations depending on several variables and involving multiple integrals can be treated in a similar way and have similar applications. The formulæ are given in the dissertation of Andrae.

Fredholm has shown that the system of equations

$$\phi_\lambda(x) + \int_0^1 \sum_{\nu=1}^{n} f_{\lambda\nu}(x, y)\, \phi_\nu(y) = \psi_\lambda(x)$$

can be reduced to one integral equation of the second kind by the following artifice :—

We define a function $F(x, y)$ for the values between 0 and n by the n^2 conditions

$$F(x, y) = f_{\lambda\nu}\{x-\lambda+1,\ y-\nu+1\} \quad \text{for } 0 < \begin{matrix} x-\lambda+1 \\ y-\nu+1 \end{matrix} < 1,$$

and a function Ψ by the n conditions

$$\Psi(x) = \psi_\lambda(x-\lambda+1) \quad \text{for } 0 < x-\lambda+1 < 1.$$

If then the determinant of the equation

$$\Phi(x) + \int_0^n F(x, y)\, \Phi(y)\, dy = \Psi(x)$$

is different from zero, the solution is given by

$$\phi_\lambda(x-\lambda+1) = \Phi(x) \quad \text{for } 0 < x-\lambda+1 < 1.$$

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* "Über eine Anwendung der Integralgleichungen auf ein Problem der Funktionentheorie," Heidelberg Congress, 1904, p. 233 : *Göttinger Nachrichten*, Dritte Mitteilung, 1905.

† "Unstetigkeiten bei der linearen Integralgleichungen mit Anwendung auf ein Problem von Riemann" : *Math. Ann.*, Bd. LX., p. 424.

This remark is of some importance because an integral equation of the second kind in which the integral is taken along a complex path can be reduced to two equations of the above type by equating the real and imaginary parts, and so can be solved by means of Fredholm's formula.

Another equation which can be reduced to an integral equation of the second kind is the equation

$$f(s) = \phi(s) - \lambda \int_a^s \kappa(s, t)\, \phi(t)\, dt,$$

which has been solved by Volterra* by the method of Neumann. In order to reduce it to Fredholm's form, we define a function $f(s, t)$ which is equal to $\kappa(s, t)$ for $s > t$ and equal to zero for $b > t \geqslant s$. When Fredholm's formula is applied to the equation, it is found that the quantity $\delta(\lambda) \equiv 1$, and the solution reduces to that given by the method of Neumann: since $\delta(\lambda)$ is never zero, this solution will hold for all values of λ. The above equation is of importance in connection with the theory of linear differential equations, as will be shown in § 8 of this paper.

II. *Solution of the Integral Equation of the First Kind.*

The integral equation

$$f(s) = \int_a^b \kappa(s, t)\, \phi(t)\, dt \tag{1}$$

is not in general soluble for a perfectly arbitrary function f: thus, if $\kappa(s, t)$ is a polynomial in S, the function $f(s)$ given by an equation of this kind can only be a polynomial.

In order, therefore, that the above equation may be soluble, we must restrict $f(s)$ to belong to a special class of functions associated with the generating function κ. There are two cases, however, in which the equation can be reduced to an integral equation of the second kind, and then $f(s)$ can have a high degree of arbitrariness.

1. Kellogg † has shown that the integral equation

$$f(s) = \int_0^1 \phi(t) \left[a \cot \pi(s-t) + S(s, t) \right] dt, \tag{2}$$

where $S(s, t)$ is finite for $s = t$ and the integral is supposed to have

* *Annali di Matematica*, 25_2, 1897.

† *Math. Ann.*, Bd. LVIII.

its principal value, can be solved by means of Hilbert's formula

$$\left.\begin{aligned} \phi(s) &= \int_0^1 f(t)\cot\pi(s-t)\,dt + \int_0^1 \phi(t)\,dt \\ f(s) &= -\int_0^1 \phi(t)\cot\pi(s-t)\,dt + \int_0^1 f(t)\,dt \end{aligned}\right\}. \tag{3}$$

For, if we multiply the above equation by $-\dfrac{1}{a}\cot\pi(r-s)$ and integrate from 0 to 1, we get

$$f_1(r) = \phi(r) + \int_0^1 K(r, t)\,\phi(t)\,dt,$$

where
$$f_1(r) = -\frac{1}{a}\int_0^1 f(s)\cot\pi(r-s)\,ds,$$

$$K(r, t) = -1 - \frac{1}{a}\int_0^1 S(s, t)\cot\pi(r-s)\,ds.$$

Now this is an integral equation of the second kind, and so can be solved by means of Fredholm's formula.

2. When the function $\kappa(s, t)$ has a finite discontinuity at the point $s = t$. If we integrate the equation

$$f'(s) = \phi(s) + \int_a^b \kappa(s, t)\,\phi(t)\,dt \tag{4}$$

between a and s, it may be written

$$f(s) - f(a) = \int_a^b R(s, t)\,\phi(t)\,dt,$$

where the function $R(s, t)$ is such that

$$\begin{aligned} R(s, t) &= 1 + \int_a^s \kappa(s, t)\,ds \quad \text{when } t < s \\ &= \int_a^s \kappa(s, t)\,ds \qquad \text{when } t > s. \end{aligned}$$

"Accordingly, the integral equation

$$F(s) = \int_a^b R(s, t)\,\phi(t)\,dt, \tag{5}$$

in which $F(a) = R(a, t) = 0$, where $F(s)$ possesses a finite first derivate between a and b, and $R(s, t)$ has a discontinuity $+1$ at the point $s = t$, but possesses a finite first derivate for all other values of a and b, can be reduced to equation (4) and so solved by means of Fredholm's formula."

8. We will now consider the equation

$$f(s) = \int_a^b \kappa(s, t)\,\phi(t)\,dt, \tag{6}$$

supposing that $\kappa(s, t)$ remains finite and integrable for all values of t between a and b, and for values of s within a certain range.

We shall first examine the case in which $\kappa(s, t)$ is a symmetrical function of its arguments, and shall write the equation in the symbolical form

$$f(s) = S_\kappa \phi(s). \tag{6 bis}$$

The problem before us is to discover the inverse operation

$$\frac{1}{S_\kappa} f(s) = \phi(s).$$

Now we know that for an arbitrary $f(s)$ this can have no meaning, but for a function $f(s)$ which can be expanded in absolutely and uniformly convergent series of the functions $\psi_n(s)$ associated with $\kappa(s, t)$ it has, and we may prove the following theorem :—

THEOREM.—If $f(s)$ can be expanded in an absolutely and uniformly convergent series of the functions $\psi_n(s)$ that satisfy the homogeneous equations

$$\psi_n(s) - \lambda_n \int_a^b \kappa(s, t)\,\psi_n(t)\,dt = 0 \quad (n = 1, 2, \ldots),$$

then it is possible to determine a function $\phi(t)$ such that

$$f(s) \sim \int_a^b \kappa(s, t)\,\phi(t)\,dt$$

may be less than any arbitrary small quantity ϵ.

This may be regarded as the converse of the expansion theorem quoted in § 1 : the proof is as follows :—

Let $f(s)$ and $\kappa(s, t)$ be finite and integrable for values of s and t lying between a and b : then the series

$$F(t, x) = xS_\kappa f(t) - \frac{x^3}{1!} S_\kappa^3 f(t) + \frac{x^5}{2!} S_\kappa^5 f(t) - \ldots$$

and

$$H(s, x) = f(s) - \frac{x^2}{1!} S_\kappa^2 f(s) + \frac{x^4}{2!} S_\kappa^4 f(s) - \ldots$$

are absolutely and uniformly convergent for all finite values of x. For, if f is the maximum value of $|f(t)|$ and k the maximum value of $|K(s, t)|$, the quantity

$$|S_\kappa^n f(t)| < (b-a)^n f k^n ;$$

and so each series becomes less than an exponential series.

Now consider the function

$$\phi(t) = 2\int_0^M F(t, x)\,dx,$$

where M is very large. We have

$$\int_a^b \kappa(s, t)\,\phi(t)\,dt = 2\int_a^b \kappa(s, t)\,dt \int_0^M F(t, x)\,dx = 2\int_0^M dx \int_a^b \kappa(s, t)\,F(t, x)\,dt.$$

Now the series $F(t, x)$, being uniformly convergent, can be integrated term by term: accordingly we have

$$2\int_a^b \kappa(s, t)\,F(t, x)\,dt = \Sigma\, 2(-1)^n \frac{x^{2n+1}}{(n)!}\, S_\kappa^{2n+2} f(t) = -\frac{d}{dx} H(s, x);$$

therefore

$$\int_a^b K(s, t)\,\phi(t)\,dt = -\int_0^M \frac{d}{dx}[H(s, x)]\,dx = f(s) - H(s, M).$$

We must now find another expression for $H(s, M)$ which will indicate its value for large values of M. Since

$$f(s) = \Sigma a_n \psi_n(s),$$

and this series is uniformly convergent, we have

$$\int_a^b \kappa(s, t)\,f(t)\,dt = \Sigma a_n \int_a^b \kappa(s, t)\,\psi_n(t)\,dt = \Sigma \frac{a_n}{\lambda_n}\psi_n(s),$$

and this series is also uniformly convergent, since the λ's are all real and arranged in ascending order of magnitude. Operating again with S_κ, we have

$$S_\kappa^2 f(s) = \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^2}\psi_n(s);$$

similarly,

$$S_\kappa^r f(s) = \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^r}\psi_n(s).$$

Substituting in the series for $H(s, x)$, we have

$$H(s, x) = \sum_0^{\infty} (-1)^r \frac{x^{2r}}{r!} S_\kappa^r f(s) = \sum_0^{\infty} (-1)^r \frac{x^{2r}}{r!} \sum_1^{\infty} \frac{a_n}{\lambda_n^{2r}}\psi_n(s).$$

Now

$$\sum_m^{\infty} \left|\frac{a_n}{\lambda_n^r}\psi_n(s)\right| < \frac{1}{|\lambda_m^r|} \sum_m^{\infty} |a_n \psi_n(s)|,$$

and, since the series $\sum_0^{\infty} a_n \psi_n(s)$ is absolutely convergent, we can find a number m such that $\sum_m^{\infty} |a_n \psi_n(s)| < \epsilon$; therefore

$$\sum_m^{\infty} \left|\frac{a_n}{\lambda_n^r}\psi_n(s)\right| < \frac{\epsilon}{|\lambda_m^r|},$$

and so $$H(s, x) = \sum_0^\infty (-1)^r \frac{x^{2r}}{r!} \sum_1^m \frac{a_n}{\lambda_n^{2r}} \psi_n(s) + \eta,$$

where $$|\eta| = \sum_0^\infty \frac{x^{2r}}{r!} \left| \sum_m^\infty \frac{a_n}{\lambda_n^{2r}} \psi_n(s) \right| < \sum_0^\infty \epsilon \frac{x^{2r}}{\lambda_m^{2r} r!} < \epsilon e^{x^2/\lambda_m^2}.$$

Rearranging the terms, we have

$$H(s, x) = \sum_1^m a_n e^{-x^2/\lambda_n^2} \psi_n(s) + \eta\,;$$

and, making ϵ zero, we have finally

$$H(s, x) = \sum_1^\infty a_n e^{-x^2/\lambda_n^2} \psi_n(s).$$

Now, all the quantities λ_n are real; therefore it will be possible to find a number p such that

$$\left| \sum_p^\infty a_n e^{-x^2/\lambda_n^2} \psi_n(s) \right| < \left| \sum_p^\infty a_n \psi_n(s) \right| < \frac{\epsilon}{2},$$

and, by choosing M large enough, we can make

$$\left| \sum_1^p a_n e^{-M^2/\lambda_n^2} \psi_n(s) \right| < \frac{\epsilon}{2}\,;$$

therefore $|H(s, M)|$ will be less than $\frac{1}{2}\epsilon + \frac{1}{2}\epsilon$, *i.e.*, less than ϵ. The integral $\int_a^b \kappa(s, t)\, \phi(t)\, dt$ will then differ from $f(s)$ by a quantity less than ϵ, and so we should expect the solution of the integral equation to be given by

$$\phi(t) = 2\int_0^\infty F(t, x)\, dx. \tag{7}$$

Next consider the case in which $\kappa(s, t)$ is not necessarily symmetrical: we shall assume that it is finite and integrable for values of s lying between c and d, and for values of t lying between a and b.

Let the integral equation be

$$f(s) = \int_a^b \kappa(s, t)\, \phi(t)\, dt, \tag{8}$$

and consider the function

$$g(s, r) = \int_a^b \kappa(s, t)\, \kappa(r, t)\, dt. \tag{9}$$

It is evidently finite and integrable within the range c to d, and is a symmetrical function of r and s. Accordingly, the integral equation

$$\psi(s) = \lambda \int_c^d g(s, r)\, \psi(r)\, dr$$

will possess a number of solutions $\psi_n(s)$, and we may enunciate the following theorem:—

If $f(s)$ can be expanded in an absolutely and uniformly convergent series $\Sigma a_n \psi_n(s)$, then a function $\phi(t)$ can be determined so that

$$\int_a^b \kappa(s, t)\, \phi(t)\, dt$$

shall differ from $f(s)$ by as small a quantity as we please.

To prove this we construct the functions

$$F(t, x) = x^1 \int_c^d \kappa(s, t)\, ds \int_a^b \kappa(s, r)\, dr \int_c^d \kappa(\xi, r) f(\xi)\, d\xi$$

$$- \frac{x^3}{1!} \int_c^d\int_a^b\int_c^d\int_a^b\int_c^d\int_a^b\int_c^d (\ldots) \ldots + \frac{x^5}{2!} \int_c^d\int_a^b \ldots - \ldots,$$

$$H(s, x) = f(s) - \frac{x^2}{1!} \int_c^d g(s, r)\, dr \int_c^d g(r, \xi) f(\xi)\, d\xi$$

$$+ \frac{x^4}{2!} \int_c^d\int_c^d\int_c^d\int_c^d (\ldots)(\ldots) \ldots - \ldots,$$

$$\phi(t) = 2\int_0^M F(t, x)\, dx.$$

The proof then proceeds as before; for it is easy to see that these series are absolutely and uniformly convergent for all finite values of x, and are connected by the relation

$$2\int_a^b \kappa(s, t)\, F(t, x)\, dt = -\frac{d}{dx} H(s, x).$$

The method just given is really of a much wider application; for it will sometimes apply to equations in which the integrals are taken along a complex path, or to integrals in which the limits are infinite, as, for instance, in the case of Fourier's theorem,

$$f(s) = \int_{-\infty}^{\infty} e^{ist} \phi(t)\, dt,$$

$$\phi(t) = \frac{1}{2\pi} \int_{-}^{\infty} e^{-itx} f(x)\, dx.$$

If either of the functions $f(s)$ or $\kappa(s, t)$ become infinite for a value of s independent of t, we may make both sides finite by multiplying by a suitable function of s, and our method will still apply.

It should be noticed that, if the equation (8) can be solved for the

generating function $\kappa(s, t)$, it can also be solved when the generating function is the solving function $K(s, t)$ of the equation

$$\psi(s) = \chi(s) - \lambda \int_a^b \kappa(s, t)\, \chi(t)\, dt.$$

For, since

$$\chi(s) = \psi(s) + \lambda \int_a^b K(s, t)\, \psi(t)\, dt,$$

we have

$$\int_a^b \kappa(s, t)\, \chi(t)\, dt = \int_a^b K(s, t)\, \psi(t)\, dt.$$

Accordingly, if a function $\chi(t)$ can be determined so that

$$f(s) = \int_a^b \kappa(s, t)\, \chi(t)\, dt,$$

the solution of

$$f(s) = \int_a^b K(s, t)\, \psi(t)\, dt$$

will, at the same time, be given by

$$\psi(s) = \chi(s) - \lambda \int_a^b \kappa(s, t)\, \chi(t)\, dt.$$

III. *Reduction of the General Linear Differential Equation of the n-th Order to an Integral Equation of Volterra's Type.*

Let it be assumed that the coefficients in the differential equation

$$P_x(y) \equiv \frac{d^n y}{dx^n} + C_1(x) \frac{d^{n-1} y}{dx^{n-1}} + \ldots + C_n(x)\, y = 0$$

are such that for a given range of values of x lying on a curve joining the points a and $a+A$ in the complex x plane the function $C_s(x)$ can be differentiated $n-s$ times with regard to x, and that all the quantities $C_s(x)$ and the derivatives assumed are finite and integrable for these values.

If, then, we multiply the equation by $x^r\ (r<n)$, and integrate along the curve from a to an intermediate point x, the result of the integration will be

$$\Big\{ R_x[y(x)\, x^r] \Big\}_a^x + \int_a^x y(x)\, p_x\{x^r\}\, dx = 0,$$

where $p_x(u) = 0$ is the equation adjoint to $P_x(y) = 0$, and R_x is the

bilinear concomitant. This result follows at once from the relation*

$$uP_x(y)-yp_x(u) = \frac{d}{dx}\{R_x(y,u)\}.$$

Giving r the values 0, 1, 2, ..., $n-1$, we obtain n equations from which the quantities $\frac{d^{n-1}y}{dx^{n-1}}, \ldots, \frac{dy}{dx}$ may be eliminated. The elimination may be performed by multiplying the equations by

$$x^{n-1}, \quad -(n-1)\,x^{n-2}, \quad \frac{(n-1)(n-2)}{2!}\,x^{n-3}, \quad \ldots, \quad (-1)^{n-1}$$

respectively and adding. For, if this is done, we get

$$\Big\{R_t[y(t),\ (x-t)^{n-1}]\Big\}_{t=a}^{x} + \int_a^x y(t)\,p_t\{(x-t)^{n-1}\}\,dt = 0,$$

and, since $R_t(v, w)$ is of the form

$$v\frac{d^{n-1}w}{dt^{n-1}} + d_1\frac{d^{n-2}w}{dt^{n-2}} + \ldots,$$

all the terms except those arising from $v\frac{d^{n-1}w}{dt^{n-1}}$ will vanish when $t = x$, and so the equation takes the form

$$(n-1)!\,y(x) - Q_{n-1}(x) + \int_a^x y(t)\,p_t\{(t-x)^{n-1}\}\,dt = 0$$

or

$$Q_{n-1}(x) = y(x) + \frac{1}{(n-1)!}\int_a^x p_t\{(t-x)^{n-1}\}\,y(t)\,dt, \tag{2}$$

where Q_{n-1} is a polynomial of degree $n-1$.

This is an equation of Volterra's type, and is exactly equivalent to the given differential equation; for the polynomial depends on the value of the $n-1$ derivates of y at the point a, and so contains n arbitrary constants. An equation of a more general character had previously been obtained by Dini,† but it is not so simple as the one just given.

If the path of integration and all the coefficients are real, we may

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* Forsyth's *Theory of Differential Equations*, Vol. IV., p. 252.

† *Ann. di Mat.*, Ser. 3, t. II., 1899, pp. 297–324; *ib.*, t. III., 1899, pp. 125–183; *ib.*, t. XI., Ser. 3, p. 385.

apply the method of Neumann and obtain a converging series*

$$y(x) = Q_{n-1}(x) - \int_a^x p_t \left\{ \frac{(t-x)^{n-1}}{(n-1)!} \right\} Q_{n-1}(t)\, dt$$
$$+ \int_a^x dt\, p_t \left\{ \frac{(t-x)^{n-1}}{(n-1)!} \right\} \int_a^t p_s \left\{ \frac{(s-t)^{n-1}}{(n-1)!} \right\} Q_{n-1}(t)\, dt - \ldots .$$

The series may also be written in the form

$$y(x) = Q_{n-1}(x) - \int_a^x S(x, t)\, Q_{n-1}(t)\, dt, \tag{3}$$

where

$$S(x, t) = s_0(x, t) - s_1(x, t) + s_2(x, t) - \ldots,$$
$$s_i(x, t) = \int_x^t s_{i-1}(x, \xi)\, s_0(\xi, t)\, d\xi,$$
$$s_0(x, t) = \frac{1}{(n-1)!} p_t \{(t-x)^{n-1}\}.$$

The series for S is certainly absolutely and uniformly convergent; for it is what we should obtain by the application of Fredholm's formula, the determinant being in this case unity.

If, on the other hand, we want to pass from a point a to any other point in the complex plane, we must choose a path of integration which does not pass through a singularity of one of the coefficients. We shall consider then the general equation

$$f(z) = \phi(z) + \int_a^z \kappa(z, \zeta)\, \phi(\zeta)\, d\zeta. \tag{4}$$

Let z and ζ be expressed as functions of the arcs s, σ of the curve joining a to the two points z and ζ respectively, and write

$$\kappa(z, \zeta)\, d\zeta = [F_1(s, \sigma) + iF_2(s, \sigma)]\, d\sigma,$$
$$\phi(z) = \phi_1(s) + i\phi_2(s),$$
$$f(z) = f_1(s) + if_2(s).$$

We then have

$$\left. \begin{aligned} f_1(s) &= \phi_1(s) + \int_0^s F_1(s, \sigma)\, \phi_1(\sigma)\, d\sigma - \int_0^s F_2(s, \sigma)\, \phi_2(\sigma)\, d\sigma \\ f_2(s) &= \phi_2(s) + \int_0^s F_2(s, \sigma)\, \phi_1(\sigma)\, d\sigma + \int_0^s F_1(s, \sigma)\, \phi_2(\sigma)\, d\sigma \end{aligned} \right\} . \tag{5}$$

* The series is proved to be convergent for the general integral equation of this kind in Volterra's paper.

We now adopt Fredholm's artifice and define a function $F(s, \sigma)$ for values of s and σ between 0 and $2a$ as follows :—

$$
\begin{aligned}
F(s,\sigma) &= F_1(s,\sigma) && (s>\sigma) && (\sigma<a) && (s<a)\\
&= 0 && (s \leqslant \sigma) && (\sigma<a) && (s<a)\\
&= -F_2(s,\sigma-a) && (s>\sigma-a) && (\sigma>a) && (s<a)\\
&= 0 && (s\leqslant \sigma-a) && (\sigma>a) && (s<a)\\
&= F_2(s-a,\sigma) && (s-a>\sigma) && (\sigma<a) && (s>a)\\
&= 0 && (s-a\leqslant\sigma) && (\sigma<a) && (s>a)\\
&= F_1(s-a,\sigma-a) && (s>\sigma) && (\sigma>a) && (s>a)\\
&= 0 && (s\leqslant\sigma) && (\sigma>a) && (s>a),
\end{aligned}
$$

and two functions $g(s)$ and $h(s)$ such that $g(s)$, $h(s) = f_1(s)$, $\phi_1(s)$ or $f_2(s-a)$, $\phi_2(s-a)$, according as a is greater or less than s; the two equations may be replaced by the single equation

$$g(s) = h(s) + \int_0^{2a} F(s, \sigma) h(\sigma) d\sigma, \tag{6}$$

and this may be solved by means of Fredholm's formula.

When we proceed to calculate $\delta(\lambda)$ by means of formula (21) it is found that all the quantities a_r are zero; for in the integral

$$a_n = \iint \dots \int_0^{2a} F(s, s_1) F(s_1, s_2) \dots F(s_{n-1}, s)\, ds_1 \dots ds_{n-1}$$

one at least of the factors F will be such that its second argument is greater than its first, and will therefore be zero from the definition of F. The quantity $\delta(\lambda)$ reduces to unity, and so the method will certainly succeed. The solving function as given by formula (22) now reduces to the series obtained by applying the method of Neumann to equation (6), and, if we write down the series separately for s greater and less than a. multiply the first by $\sqrt{(-1)}$, and add, we are led to a result exactly analogous to formula (3). Accordingly we may apply the method of Neumann directly to equation (4), and the series obtained will certainly converge.

The theorem just proved is an existence theorem of a very general character. It states that a solution involving n arbitrary constants exists at any point in the plane which can be reached by means of a curve at every point of which the quantity $p_t\{(x-t)^{n-1}\}\, dt$ is finite and integrable. Thus it is not necessary for all the coefficients in a linear differential equation to be continuous in order for a solution to exist.

IV. *The Effect of Varying the Limits in an Integral Equation of the Second Kind.*

The quantities which occur in the integral equation

$$f(s) = \phi(s) - \lambda \int_a^b \phi(t)\, \kappa(s, t)\, dt \tag{1}$$

may be regarded as functions of b. Now, if $f(s)$ is independent of b, this equation may be written

$$\phi(s, b) - \phi(s, a) = \lambda \int_a^b \kappa(s, t)\, \phi(t, b)\, dt;$$

hence the solution of the equation

$$\psi(b) - \psi(a) = \int_a^b \chi(t)\, \phi(t, b)\, dt \tag{2}$$

for $\psi(b) = \phi(s, b)$ is given by $\chi(t) = \lambda\kappa(s, t)$.

The equation (2) is similar to equation (1) of § 1, and so can be solved by the method of Volterra. We can therefore construct a function $\kappa(s, t)$, such that a given function $\phi(s, b)$ may be the solution of equation (1) for a certain range of values of b.

Again, if $\kappa(s, t)$ and $\phi(t, b)$ are supposed to be known, we may solve the equation (2) for any function $\psi(b)$ given by an equation of the form

$$\psi(b) = \int_{s_1}^{s_2} \phi(s, b)\, \theta(s)\, ds.$$

For, since the equation is linear, the solution will be

$$\chi(t) = \lambda \int_{s_1}^{s_2} \kappa(s, t)\, \theta(s)\, ds. \tag{3}$$

Next, consider the effect of varying b in equation (1). If we differentiate the formula (17) of § 1, viz.,

$$\delta_h = \frac{1}{(h)!} \int_a^b \cdots \int_a^b ds_1 \ldots ds_h \begin{vmatrix} \kappa(s_1, s_1) & \kappa(s_1, s_2) & \ldots & \kappa(s_1, s_n) \\ \kappa(s_2, s_1) & & & \\ \vdots & & & \end{vmatrix},$$

we find that

$$\frac{d\delta_h}{db} = \frac{1}{(h-1)!} \int_a^b \cdots \int_a^b ds_2 \ldots ds_h \begin{vmatrix} \kappa(b, b) & \kappa(b, s_2) & \ldots & \kappa(b, s_n) \\ \kappa(s_2, b) & & & \\ \vdots & & & \end{vmatrix}$$

$$= \Delta_{h-1}(b, b).$$

Accordingly $$\frac{d}{db}[\delta(\lambda)] = -\lambda\kappa(b, b)+\lambda^2\Delta_1(b, b)-\dots$$
$$= \lambda\Delta(\lambda;\ b, b) = -\lambda\delta(\lambda)K(b, b)$$

or $$\frac{d}{db}[\log\delta(\lambda)] = -\lambda K(b, b). \tag{4}$$

If we substitute in formula (19) of § 1, and integrate with regard to b, we shall obtain

$$\log\delta(\lambda) = -\lambda\int_a^b \kappa(s, s)\,ds - \lambda^2\int_a^b ds\int_a^s \kappa(s, r)\,\kappa(r, s)\,dr$$
$$-\lambda^3\int_a^b dt\int_a^t\int_a^t \kappa(t, r)\,\kappa(r, s)\,\kappa(s, t)\,dr\,ds-\dots, \tag{5}$$

which may be identified with formula (20).

Again, if we differentiate the formula (19), since

$$\kappa_n(s, t) = \int_a^b \kappa(s, r)\,\kappa_{n-1}(r, t)\,dr,$$

we have

$$\frac{d}{db}\kappa_n(s, t) = \kappa(s, b)\,\kappa_{n-1}(b, t)+\int_a^b \kappa(s, r)\,\frac{d}{db}\kappa_{n-1}(r, t)\,dr$$
$$= \kappa(s, b)\,\kappa_{n-1}(b, t)+\int_a^b \kappa(s, r)\,\kappa(r, b)\,\kappa_{n-2}(b, t)\,dr$$
$$+\int_a^b \kappa(s, r)\int_a^b \kappa(r, \xi)\,\frac{d}{db}\kappa_{n-2}(\xi, t)\,d\xi;$$

hence

$$\frac{d}{db}\kappa_n(s, t) = \kappa(s, b)\,\kappa_{n-1}(b, t)+\kappa_1(s, b)\,\kappa_{n-2}(b, t)+\dots+\kappa_{n-1}(s, b)\,\kappa(b, t).$$

Therefore $$\frac{d}{db}K(s, t) = \Sigma\lambda^n\left\{\Sigma\kappa_r(s, b)\,\kappa_{n-r+1}(b, t)\right\}$$
$$= \lambda K(s, b)\,K(b, t). \tag{6}$$

We can now find the effect on a solution of equation (1). If $f(s)$ is independent of b, the formula

$$\phi(s) = f(s)+\lambda\int_a^b K(s, t)\,f(t)\,dt$$

will give $$\frac{\partial}{\partial b}\phi(s) = \lambda K(s, b)\,f(b)+\lambda^2\int_a^b K(s, b)\,K(b, t)\,f(t)\,dt$$
$$= \lambda K(s, b)\,\phi(b). \tag{7}$$

V. *A certain Linear Partial Integral Equation.*

It was remarked in § 1 that, if the function $\kappa(s, t)$ is the Green's function for the linear differential equation $L_s(u) = 0$, the functions $\psi_n(s)$ connected with $\kappa(s, t)$ will be solutions of

$$L_s(u) + \lambda_n u = 0,$$

and so the product $\psi_n(s)\psi_n(t)$, and the functions $\kappa(s, t)$, $\kappa_n(s, t)$, $K(s, t)$, which can in general all be expressed as a series of products of this kind, will be solutions of the partial differential equation

$$L_s(u) = L_t(u). \tag{1}$$

In the general case when the function $\kappa(s, t)$ is no longer symmetrical, the functions $\kappa(s, t)$, $\kappa_n(s, t)$ and $K(s, t)$ are all solutions of the partial integral equation

$$\int_a^b \kappa(s, x) f(x, t)\, dx = \int_a^b \kappa(x, t) f(s, x)\, dx, \tag{2}$$

which bears a striking analogy to (1).

To prove this we recall the fact that the relation between the functions $\kappa(s, t)$ and $K(s, t)$ is a reciprocal one; so that, since by formula (10) of § 1 we have

$$\kappa(s, t) = K(s, t) - \lambda \int_a^b \kappa(s, r) K(r, t)\, dr,$$

we must also have

$$K(s, t) = \kappa(s, t) + \lambda \int_a^b K(s, r) \kappa(r, t)\, dr;$$

for $\kappa(s, t)$ is the solving function for the equation (8). Combining these two equations, we get

$$\int_a^b \kappa(s, r) K(r, t)\, dr = \int_a^b K(s, r)\, \kappa(r, t)\, dr;$$

so that $K(s, t)$ is a solution of (2). Equating the coefficients of λ^n in this equation, we see that $\kappa_n(s, t)$ is also a solution of (2).

If we seek a solution of the form $\phi(x)\psi(t) = f(x, t)$, we are led to the equation

$$\psi(t) \int_a^b \kappa(s, x)\, \phi(x)\, dx = \phi(s) \int_a^b \kappa(x, t)\, \psi(x)\, dx,$$

which requires that
$$\phi(s) - \lambda \int_a^b \kappa(s, x)\, \phi(x)\, dx = 0,$$

$$\psi(t) - \lambda \int_a^b \kappa(x, t)\, \psi(x)\, dx = 0.$$

Now these equations can be satisfied for the same value* of λ, if λ is a root of the equation $\delta(\lambda) = 0$. Accordingly to each root of $\delta(\lambda) = 0$ there corresponds a harmonic solution of the form $\phi(s)\,\psi(t)$.

If we can obtain a solution of (2) involving an arbitrary constant μ, then the function obtained by multiplying by an arbitrary function of μ and integrating will also be a solution. The solving function $K(s, t, \lambda)$ of the integral equation

$$f(s) = \phi(s) - \lambda \int_a^b \kappa(s, t)\,\phi(t)\,dt$$

is such a solution. Accordingly (2) is satisfied by

$$f(s, t) = \int_{\lambda_1}^{\lambda_2} K(s, t, \lambda)\,\chi(\lambda)\,d\lambda.$$

If, however, another such function can be readily obtained, it may sometimes be used to obtain a convenient expression for $K(s, t)$.

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ON THE MONOGENEITY OF A FUNCTION DEFINED BY AN ALGEBRAIC EQUATION

By H. F. BAKER.

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1. The present note offers a proof that, if $P_1, \ldots, P_n$ be (single-valued) integral functions of a single variable x, and the equation

$$y^n + P_1 y^{n-1} + \ldots + P_n = 0$$

have no root in common with an equation of the same form and lower degree, then all the roots are capable of derivation by analytical continuation of any one of them, so that, according to Weierstrass's use of the word, the various roots constitute together a single monogenic function.* The statement implies a certain form for the roots of the equation, the nature of which will appear in the course of the proof; and it is sufficient to exclude common roots of the two equations for every point of any two-dimensional area.

It may well be that a proof of this proposition has been published already; but, even in the case when $P_1, \ldots, P_n$ are integral polynomials in x, I am not aware, among the various proofs given, of any which seems to have quite the simplicity of this one; in particular, the proof given in Weierstrass's lectures is based upon the theory of rational functions whose poles are all at one place. The theorem includes clearly the similar result when $P_1, \ldots, P_n$ are single-valued functions having only poles for finite singularities; we have only to replace y by η/μ, where μ is a certain polynomial in x.

2. We assume, what is a particular case of Weierstrass's implicit-function theorem (*Vorbereitungssatz*), that, if $f(x, y)$ be a convergent power series in x and y, vanishing when $x = 0$, $y = 0$, there being, when x alone is put zero, a series remaining beginning with a term in y^n, then all the roots y of $f(x, y) = 0$ which vanish when $x = 0$ are given by an equation

$$\varpi(x, y) = y^n + p_1 y^{n-1} + \ldots + p_n = 0,$$

* The theorem is stated and used in a proof that functions of one variable with an algebraical addition theorem are elliptic functions (or particular cases of such): *Proc. Camb. Phil. Soc.*, Vol. XII., 1903, p. 236.

wherein $p_1, p_2, \ldots, p_n$ are power series in x vanishing for $x = 0$. A particular case is the result that the y-equation

$$a_1 x + y + (x, y)_2 + \ldots = 0,$$

arising when $n = 1$, has one, and only one, solution vanishing when $x = 0$, expressible in a form

$$y = -a_1 x + a_2 x^2 + a_3 x^3 + \ldots.$$

3. We give also, for the sake of the nature of the proof, a demonstration of the well known form possible for the roots of an equation $\varpi(x, y) = 0$ for the neighbourhood of the origin $x = 0$. For this there is no loss of generality in assuming that $\varpi(x, y)$ is incapable of being written as a product of factors of the same form (*i.e.*, with converging power series coefficients vanishing for $x = 0$), since otherwise we could deal with each factor in turn. We have then

$$\varpi'_y(x, y) = \partial\varpi(x, y)/\partial y = ny^{n-1} + (n-1)p_1 y^{n-2} + \ldots + p_{n-1};$$

form the Sylvester y-resultant of $\varpi(x, y)$ and $\varpi'_y(x, y)$, which, being a rational integral polynomial in $p_1, \ldots, p_n$, is a power series; if this power series vanish identically, $\varpi(x, y)$ and $\varpi'_y(x, y)$ have a factor

$$y^h + q_1 y^{h-1} + \ldots + q_h,$$

obtainable by the rational method of greatest common divisor, wherein $q_1, \ldots, q_h$ are rational in $p_1, \ldots, p_n$, and therefore all of the form $x^{-\lambda}P(x)$, where $P(x)$ is a convergent power series in x; as, however, all the roots of $\varpi(x, y) = 0$ vanish with x, and the roots of $y^h + q_1 y^{h-1} + \ldots + q_h = 0$ are chosen from those of $\varpi(x, y) = 0$, this can only be so if $\lambda = 0$ and $P(x)$ vanish with x. We have, however, assumed that $\varpi(x, y)$ is not divisible by any factor of the form then arising. The Sylvester y-resultant is therefore not identically zero. It vanishes for $x = 0$, since both $\varpi(x, y)$ and $\varpi'_y(x, y)$ vanish for $x = 0$, $y = 0$; but a region can be put about $x = 0$ within which no other zeros of this resultant are found: this region, taken circular, we call, momentarily, the domain of the origin. If x_0 be a point within this domain other than the origin, and y_0 any one of the corresponding roots of $\varpi(x_0, y) = 0$, we have $\varpi'_y(x_0, y) \neq 0$; put then $x = x_0 + \xi$, $y = y_0 + \eta$ in $\varpi(x, y)$, so obtaining

$$\frac{\partial\varpi(x_0, y_0)}{\partial x_0}\xi + \frac{\partial\varpi(x_0, y_0)}{\partial y_0}\eta + \ldots = 0,$$

and hence, by the assumed theorem of § 2 above, a power series

$$y = y_0 + A_1(x - x_0) + A_2(x - x_0)^2 + \ldots, \tag{A}$$

converging in a certain region about x_0, this representing the only function satisfying $\varpi(x, y) = 0$ and reducing to y_0 when $x = x_0$. A precisely similar form is possible for each of the other roots of $\varpi(x_0, y) = 0$. Let r be the least of the n radii of convergence of these n series belonging to x_0. Putting a small circle about the origin and another circle just within the outer circumference of the domain of the origin, and considering the closed annulus so determined, and the value of r for each point x_0 of this annulus, we say that a real number greater than zero exists such that r is everywhere greater than this real number.

For let the circle about x_0, of radius r, be called the *proper* region of x_0; the statement is that the lower limit ρ of the radii of the proper regions, for points x_0 within the closed annulus, is greater than zero. Let an area contained in the closed annulus be called *suitable* if it be contained in the proper region of some point within or upon the boundary of itself; the statement will be justified if it can be shown that the closed annulus can be divided into a number of finite areas each of which is suitable. Let the annulus be divided, for example, by means of n concentric circles and n equidistant radii, into equal sub-regions; if all these $n(n+1)$ sub-regions are not suitable, let an unsuitable one be again sub-divided by n concentric circles and n equidistant radii; and so on continually. The statement is that this subdivision will not need to be continued indefinitely in order that all the sub-regions may be suitable. For an indefinitely continued sequence of regions, each contained in the preceding, and a definite sub-multiple of its area, must have a limiting point, and the proper region of this limiting point will contain all of the sequence of sub-regions which arise beyond a certain stage in the process of subdivision.*

Returning then to the series (A) expressing the root of $\varpi(x, y) = 0$, which reduces to y_0 when $x = x_0$, let x_1 be a point within the closed annulus spoken of above, and within the circumference of convergence of (A), but at a less distance from it than the lower limit ρ established above for the distance r; let y_1 be the value of the series (A) at $x = x_1$; there exists then one root of $\varpi(x, y) = 0$, reducing to y_1 when $x = x_1$, expressible by a power series in $x - x_1$, converging for $|x - x_1| < \rho$, and therefore forming a continuation of the series (A) *beyond the circle of convergence of* (A).

It is thus clear that any root (y_0) of $\varpi(x, y) = 0$ can be continued

* I have used this phraseology for some time in expounding Goursat's proof of Cauchy's fundamental theorem for complex integrals, and in other of the many applications of the principle. *E.g.*, in the *Proc. Lond. Math. Soc.*, Vol. I., 1903, p. 24. The principle is nearly identical with that called in the theory of aggregates the "Heine-Borel Theorem."

completely round the closed annulus referred to above, back to the neighbourhood of x_0; it may not, however, after one circuit, resume its value; it may change into another root of $\varpi(x_0, y) = 0$; let it resume its original value after μ circuits. Put then $x = t^\mu$, so that the phase of t increases by 2π when the phase of x increases by $2\pi\mu$, and consider, just as we have considered the equation $\varpi(x, y) = 0$, the equation $\varpi(t^\mu, y) = 0$, and the root of this reducing to y_0 when $t = t_0$, where $t_0^\mu = x_0$. By precisely the above reasoning this root is a single-valued function of t within the annulus, and developable, as a power series in $t - t_0$, about any point t_0. Thus (by Laurent's theorem) it is capable of a representation $\sum_{-\infty}^{\infty} a_m t^m$, valid for the whole of the annulus. If, however, M be greater than the modulus of this series for $|t| = R$, we have $|a_{-m}| < MR^m$; as all the roots of $\varpi(x, y) = 0$ vanish for $x = 0$, it follows that the negative powers of t (and the zero power) are absent from the series. Consider now the μ roots of $\varpi(x, y) = 0$ given by the series $\sum_1^m a_m t^m$ for

$$t = t, \quad t = te^{2\pi i/\mu}, \quad \ldots, \quad t = te^{[2\pi i(\mu-1)]/\mu};$$

denote them by $y_1, \ldots, y_\mu$; if r be a positive integer, the sum

$$y_1^r + y_2^r + \ldots + y_\mu^r$$

(or, indeed, any rational integral symmetrical polynomial in $y_1, \ldots, y_\mu$) arises as a convergent power series in t, which is, in fact, a single-valued function of x, and is therefore a converging power series in x, manifestly vanishing for $x = 0$; thus $y_1, \ldots, y_\mu$ are the roots of an equation

$$y^\mu + q_1 y^{\mu-1} + \ldots + q_\mu = 0,$$

whose left side therefore divides $\varpi(x, y)$, and is therefore, as we have assumed $\varpi(x, y)$ not to have such factor for $\mu < n$, identical with it. All the roots of $\varpi(x, y) = 0$ in the neighbourhood of the origin are thus shewn to be expressible by a single series in t, with $x = t^n$.

4. Consider now an expression

$$y^n + P_1 y^{n-1} + \ldots + P_n,$$

wherein $P_1, \ldots, P_n$ are (single-valued) integral functions of x. We first prove a lemma similar in form to one given by Gauss in the *Disquisitiones Arithmeticæ*, that, if a decomposition be possible,

$$y^n + P_1 y^{n-1} + \ldots + P_n = (y^\mu + H_1 y^{\mu-1} + \ldots + H_\mu)(y^\nu + K_1 y^{\nu-1} + \ldots + K_\nu),$$

in which $H_1, \ldots, H_\mu, K_1, \ldots, K_\nu$ are rational functions of integral functions (and therefore are single-valued functions with no finite singularities other than poles), then such a decomposition is possible in which $H_1, \ldots, H_\mu$, $K_1, \ldots, K_\nu$ are integral functions. For the assumed decomposition can be put into the form

$$C(y^n+P_1y^{n-1}+\ldots+P_n) = (Ry^\mu+R_1y^{\mu-1}+\ldots+R_\mu)(Sy^\nu+S_1y^{\nu-1}+\ldots+S_\nu),$$

where $R, R_1, \ldots, R_\mu, S, S_1, \ldots, S_\nu$ are integral functions, and C is an integral polynomial. Consider any simple factor of C; from $CP_n = R_\mu S_\nu$ it follows that this factor must divide R_μ or S_ν; from

$$CP_{n-1} = R_\mu S_{\nu-1}+S_\nu R_{\mu-1}$$

it follows that, if this factor divide R_μ, it must divide S_ν or $R_{\mu-1}$; and so on. Suppose this factor divides

$$R_\mu, R_{\mu-1}, \ldots, R_{\mu-h+1} \quad \text{and} \quad S_\nu, S_{\nu-1}, \ldots, S_{\nu-k+1};$$

then compare the coefficients of y^{h+k}, giving $CP_{\mu+\nu-h-k} = R_{\mu-h}S_{\nu-k}+$terms in which the R factor has a suffix greater than $\mu-h$, or the S factor has a suffix greater than $\nu-k$; this factor of C must then divide either $R_{\mu-h}$ or $S_{\nu-k}$; proceeding thus, we see that this factor of C either divides every one of $R, R_1, \ldots, R_\mu$ or every one of $S, S_1, \ldots, S_\nu$; it may then be divided out; and so for every factor of C. We thus obtain an equation of the form

$$y^n+P_1y^{n-1}+\ldots+P_n = (Ry^\mu+\ldots+R_\mu)(Sy^\nu+\ldots+S_\nu),$$

in which every one of $R, \ldots, R_\mu, S, \ldots, S_\nu$ is an integral function; and $RS = 1$ implies that the inverse of each of R, S is an integral function; they may then be divided throughout. We thus obtain an equation of the form first put down, with every one of $H_1, \ldots, H_\mu, K_1, \ldots, K_\nu$ an integral function.

5. If now the equation

$$P(x, y) = y^n+P_1y^{n-1}+\ldots+P_n = 0,$$

wherein $P_1, \ldots, P_n$ are integral functions, has a root common with an equation of the same form and lower degree

$$Q(x, y) = y^h+Q_1y^{h-1}+\ldots+Q_h = 0,$$

this common root will satisfy $H(x, y) = 0$, where

$$H(x, y) = y^\mu+H_1y^{\mu-1}+\ldots+H_\mu$$

is the highest common factor of $P(x, y)$ and $Q(x, y)$, and $H_1, \ldots, H_\mu$ are rational functions of the integral functions $P_1, \ldots, P_n, Q_1, \ldots, Q_h$. The

existence of such a factor, however, implies another, and an equation

$$y^n+P_1y^{n-1}+\ldots+P_n = (y^\mu+H_1y^{\mu-1}+\ldots+H_\mu)(y^\nu+K_1y^{\nu-1}+\ldots+K_\nu),$$

wherein $K_1, \ldots, K_\nu$ are also rational functions of integral functions. We have shewn that the existence of such an equation requires an equation of the same form in which $H_1, \ldots, H_\mu, K_1, \ldots, K_\nu$ are actually integral functions.

6. We consider then an equation

$$P(x, y) = y^n+P_1y^{n-1}+\ldots+P_n = 0,$$

$P_1, \ldots, P_n$ being integral functions, in which the left side is incapable of being written as a product of factors of the same form; it has, therefore, no root common with an equation of the same form and lower degree.

Then the Sylvester y-resultant of $P(x, y) = 0$ and $\partial P(x, y)/\partial y = 0$ does not vanish identically, This resultant is a rational polynomial in the integral functions $P_1, \ldots, P_n$, and therefore also an integral function; therefore no two of its zeros are within an unassignable nearness of one another, and the number of them within any assigned finite portion of the plane is finite. Taking then any circle not including any zero of this resultant, there exists about any interior point of this circle a power series development for any root of $P(x, y) = 0$, just as in § 3, and, as there, this development for any one root can be continued over the whole of this circle. It follows that the root is representable over the whole of the circle by a single development about its centre. Taking next a circle whose centre is at one of the points x_0 where the resultant vanishes, but not including any other such point, the equations $P(x_0, y) = 0$, $\partial P(x_0, y)/\partial y = 0$ have at least one root in common. If y_1 be a root of the former equation which is not a root of the latter, there exists, as before, a root of $P(x, y) = 0$ which reduces to y_1 when $x = x_0$, and is representable as a power series in $x-x_0$. If y_0 be a root of the former equation which is also a root of the latter equation, put $x-x_0 = \xi$, $y-y_0 = \eta$. The roots of $P(x, y) = 0$ which reduce to y_0 when $x = x_0$ are then given (by Weierstrass's theorem, § 2) by an equation

$$\varpi(\xi, \eta) = \eta^\mu+p_1\eta^{\mu-1}+\ldots+p_\mu = 0,$$

wherein $p_1, \ldots, p_\mu$ are power series in ξ vanishing for $\xi = 0$, the number μ being the exponent of the lowest power of η in $P(x_0, y_0+\eta)$, it being impossible that $y^n+P_1y^{n-1}+\ldots+P_n$ should vanish identically when x is put equal to x_0; the solutions of the equation $\varpi(\xi, \eta) = 0$ are then expressible as before by a set of series of the form $\xi = t^\lambda$, $\eta = P(t)$, there being as many such pairs as irreducible factors of $\varpi(\xi, \eta)$ of the same form.

7. Taking now any root of $P(x, y) = 0$, expressed as a power series in $x-x'$, about any point x' which is not a zero of the Sylvester y-resultant of $P(x, y)$ and $\partial P(x, y)/\partial y$, let it be continued in all possible ways, by paths issuing from and again returning to the neighbourhood of x'. By reasoning as in § 3, it follows that such continuation is certainly possible over any region lying in the finite part of the plane so long as a point where the resultant vanishes is not included in this region. By what has been seen paths which do not enclose any of these exceptional points will lead back to the same value, but paths enclosing one or more of these may not do so. Notwithstanding that the number of such exceptional points increases indefinitely when the region of the plane considered is taken more and more extensive, the number of roots so obtainable from the original one by all possible continuations cannot, of course, exceed n, and must be a definite number, though we cannot experimentally obtain it by exhausting the infinite number of paths necessary to enclose all the exceptional points. Let $y_1, \ldots, y_m$ be the roots so obtainable, including the original one, and consider the function $y_1^r+\ldots+y_m^r$, where r is a positive integer; about every ordinary point x' this function is expressible as a power series in $x-x'$, and it can be continued over the whole finite plane by any path not passing through one of the exceptional points, and returns thereby always to the same value; about every exceptional point x' it is, however, also expressible as a power series in $x-x'$, since the possible determinations of any one root about this point enter symmetrically into the formation of the function; it is, moreover, never infinite for any finite value of x, since no root of the equation

$$y+P_1+\frac{P_2}{y}+\ldots+\frac{P_n}{y^{n-1}} = 0$$

can be so infinite. It is thus an integral function of x. Therefore $y_1, \ldots, y_m$ are the roots of an equation

$$y^m+H_1y^{m-1}+\ldots+H_m = 0,$$

where $H_1, \ldots, H_m$ are integral functions. Hence, by the hypothesis as to the polynomial $P(x, y)$, it follows that $m = n$, and the equation $P(x, y) = 0$ is satisfied by only one monogenic function; as was to be proved.

8. In conclusion, it is to be remarked that, if $P_1, \ldots, P_n$ be polynomials in x, and $P(x, y)$ be incapable of being written as a product of polynomials in x and y, that is, be irreducible in the ordinary sense, then it is equally incapable of being written as a product of factors $y^\mu+H_1y^{\mu-1}+\ldots+H_\mu$, $y^\nu+K_1y^{\nu-1}+\ldots+K_\nu$, wherein $H_1, \ldots, K_\nu$ are integral functions which are

not all polynomials. For, if $\eta = y/x^r$, the equation $P(x, y) = 0$ can be written

$$\eta^n + \frac{P_1}{x^r}\eta^{n-1} + \ldots + \frac{P_n}{x^{rn}} = 0.$$

Let r be taken positive and so large that every one of $\frac{P_1}{x^r}, \frac{P_2}{x^{2r}}, \ldots, \frac{P_n}{x^{nr}}$, vanishes when x increases indefinitely, as is possible when $P_1, P_2, \ldots, P_n$ are all polynomials; then, for every root y of $P(x, y) = 0$, the quotient y/x^r diminishes indefinitely for x infinite. Thence, if a certain number of these roots be the roots of an equation $y^\mu + H_1 y^{\mu-1} + \ldots + H_\mu = 0$, wherein $H_1, \ldots, H_\mu$ are integral functions, these integral functions must all be capable of being reduced, by division by a proper positive power of x, to a form in which they vanish for x infinite; so that they must all be polynomials. The theorem proved in § 7 thus shews that an irreducible algebraic equation with polynomial coefficients is satisfied by only one monogenic function of x.

ON THE EXPRESSION OF THE SO-CALLED BIQUATERNIONS AND TRIQUATERNIONS WITH THE AID OF QUATERNARY MATRICES

By J. BRILL.

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CLIFFORD defined his biquaternion as an expression of the form $q_1+\omega q_2$, where q_1 and q_2 are quaternions and ω is a symbol commutative with quaternions and possessing the property $\omega^2=0$. M. Combebiac has developed this idea by introducing another symbol μ, commutative with quaternions, possessing the property $\mu^2=1$, and connected with ω by the relations $\mu\omega=-\omega\mu=\omega$. To the expression $q_1+\omega q_2+\mu q_3$, where q_1, q_2, q_3 are quaternions, he gives the name triquaternion.*

It is well known that the theory of the binary matrix is equivalent to that of the quaternion. This, however, does not exhaust the possibilities of the representation of the quaternion analysis with the aid of matrices. Thus consider the quaternary matrix

$$\begin{matrix} t, & -x, & -y, & -z \\ x, & t, & -z, & y \\ y, & z, & t, & -x \\ z, & -y, & x, & t \end{matrix}$$

This may be written in the form

$$et+e_1x+e_2y+e_3z,$$

* G. Combebiac, "Calcul des triquaternions (nouvelle analyse géométrique)," Paris (*Thèse*), 1902. A brief summary will be found in "A Report on Recent Progress in the Quaternion Analysis," by A. Macfarlane, *Proceedings of the American Association for the Advancement of Science*, Vol. LI., 1902 (see p. 316).

where the e's are connected by the following multiplication table:*

$$\begin{array}{c|cccc} & e, & e_1, & e_2, & e_3 \\ \hline e & e, & e_1, & e_2, & e_3 \\ e_1 & e_1, & -e, & e_3, & -e_2 \\ e_2 & e_2, & -e_3, & -e, & e_1 \\ e_3 & e_3, & e_2, & -e_1, & -e \end{array}$$

It is further easily verified that, if we denote this matrix by the symbol q, its characteristic equation takes the degenerate form

$$q^2-2tq+t^2+x^2+y^2+z^2 = 0.$$

We will denote by the symbol m the general quaternary matrix

$$\begin{array}{cccc} a_1, & b_1, & c_1, & d_1 \\ a_2, & b_2, & c_2, & d_2 \\ a_3, & b_3, & c_3, & d_3 \\ a_4, & b_4, & c_4, & d_4 \end{array}$$

We now proceed to find a form for m such that we may have $mq = qm$ independently of the values of x, y, z, t. Equating the coefficients of x, y, z, t in the corresponding elements of the two products, we obtain a set of conditions which reduce to

$$\begin{aligned} a_1 &= b_2 = c_3 = d_4, \\ a_2 &= -b_1 = -c_4 = d_3, \\ a_3 &= b_4 = -c_1 = -d_2, \\ a_4 &= -b_3 = c_2 = -d_1. \end{aligned}$$

Thus the general form of m, commutative with any matrix whatsoever of the form q, is

$$\begin{array}{cccc} d, & -a, & -b, & -c \\ a, & d, & c, & -b \\ b, & -c, & d, & a \\ c, & b, & -a, & d \end{array}$$

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* Weyr has given a general method of realising a linear associative algebra of n units with the aid of the n-ary matrix. "Sur la réalisation des systèmes associatifs de quantités complexes à l'aide des matrices," *Prager Berichte*, 1887, 616–618.

We will denote this form of matrix by the symbol r. It is readily verified by direct multiplication that we have $qr = rq$. Like the matrix of the form q, the matrix of the form r is a skew matrix, but, though very similar, the two matrices are of distinct types.

It is easily verified that the characteristic equation of r takes the degenerate form

$$r^2 - 2dr + a^2 + b^2 + c^2 + d^2 = 0.$$

Further, we have $$r = ed + \epsilon_1 a + \epsilon_2 b + \epsilon_3 c;$$

where the relations of e, ϵ_1, ϵ_2, ϵ_3 are governed by the following multiplication table:—

	e,	ϵ_1,	ϵ_2,	ϵ_3
e	e,	ϵ_1,	ϵ_2,	ϵ_3
ϵ_1	ϵ_1,	$-e$,	$-\epsilon_3$,	ϵ_2
ϵ_2	ϵ_2,	ϵ_3,	$-e$,	$-\epsilon_1$
ϵ_3	ϵ_3,	$-\epsilon_2$,	ϵ_1,	$-e$

Our new matrices thus furnish us with a type of left-handed quaternions. We have thus arrived at the interesting result that the system of quaternary matrices includes a set of right-handed quaternions and a set of left-handed quaternions, any member of the one set being commutative with any member of the other set, although the members of each set are not in general commutative among themselves. This furnishes us with a failing case of Cayley's theorem.*

Among our second set of matrices we must search for forms suitable for the representation of ω and μ.† Suppose that r_1 and r_2 are two matrices of the latter group, having the characteristic equations

$$r_1^2 - 2d_1 r_1 + a_1^2 + b_1^2 + c_1^2 + d_1^2 = 0,$$
$$r_2^2 - 2d_2 r_2 + a_2^2 + b_2^2 + c_2^2 + d_2^2 = 0.$$

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* [*March 10th*, 1906.—The general theorem referred to (and named "Cayley's theorem" by Clifford, *Math. Papers*, p. 339) is that which states that, if m be an n-ary matrix, then the general type of matrix which is commutative with m may be written in the form

$$a_1 m^{n-1} + a_2 m^{n-2} + \dots + a_{n-1} m + a_n,$$

where the a's are scalars. I believe that Cayley only discussed particular cases, and am not aware of the existence of any general proof. It would appear that exceptional cases, similar to the above, occur in the case of most orders of matrices higher than the third. I have discovered some examples in the case of the senary matrix.]

† From the properties of these symbols it is clear that we cannot fairly style them scalars, as some writers seem inclined to do.

The characteristic equation of ur_1+vr_2, where u and v are scalars, will be

$$(ur_1+vr_2)^2-2(ud_1+vd_2)(ur_1+vr_2)$$
$$+(ua_1+va_2)^2+(ub_1+vb_2)^2+(uc_1+vc_2)^2+(ud_1+vd_2)^2=0.$$

Thus, in virtue of the above two equations, we deduce a third,

$$r_1r_2+r_2r_1-2d_2r_1-2d_1r_2+2(a_1a_2+b_1b_2+c_1c_2+d_1d_2)=0.$$

If, then, r_1 is to stand for ω, and r_2 for μ, we must have

$$d_1=0, \quad a_1^2+b_1^2+c_1^2+d_1^2=0,$$
$$d_2=0, \quad a_2^2+b_2^2+c_2^2+d_2^2=-1,$$
$$d_1=d_2=0, \quad a_1a_2+b_1b_2+c_1c_2+d_1d_2=0.$$

Thus we obtain the conditions

$$d_1=d_2=0, \tag{1}$$
$$a_1^2+b_1^2+c_1^2=0, \tag{2}$$
$$a_2^2+b_2^2+c_2^2=-1, \tag{3}$$
$$a_1a_2+b_1b_2+c_1c_2=0. \tag{4}$$

The only condition that remains to be expressed is $r_2r_1=r_1$. If we equate the elements of the matrices on the two sides of this equation, assuming the d's to be zero, we obtain the four conditions

$$b_1c_2-b_2c_1=a_1, \tag{5}$$
$$c_1a_2-c_2a_1=b_1, \tag{6}$$
$$a_1b_2-a_2b_1=c_1, \tag{7}$$
$$a_1a_2+b_1b_2+c_1c_2=0. \tag{8}$$

Of these (8) is the same as (4). Both (2) and (4) are readily deduced from (5), (6), and (7).

Further, from (5) and (6) we deduce

$$a_1b_2-a_2b_1=c_2(b_1b_2+a_1a_2)-c_1(a_2^2+b_2^2)=-c_1(a_2^2+b_2^2+c_2^2),$$

taking account of (4) or (8). Thus we have

$$c_1(a_2^2+b_2^2+c_2^2+1)=0,$$

and two similar relations in which a_1 and b_1 are respectively substituted for c_1. Thus we see that, except in the special case $a_1 = b_1 = c_1 = 0$, (8) may be taken as the direct consequence of (5), (6), and (7). Thus, so far as we are concerned, we may consider the eight quantities, defining our two matrices, as governed by the five conditions contained in (1), (5), (6), and (7). Thus three of the quantities may be chosen arbitrarily, and we will assume that

$$a_1 = 1, \qquad a_2 = b_2 = 0.$$

We shall then find that the set may be completed as follows:—

$$b_1 = i, \qquad c_1 = 0, \qquad c_2 = -i,$$

where i is the ordinary scalar imaginary.

Thus for ω we have the matrix

$$\begin{matrix} 0, & -1, & -i, & 0 \\ 1, & 0, & 0, & -i \\ i, & 0, & 0, & 1 \\ 0, & i, & -1, & 0 \end{matrix}$$

For μ we have the matrix

$$\begin{matrix} 0, & 0, & 0, & i \\ 0, & 0, & -i, & 0 \\ 0, & i, & 0, & 0 \\ -i, & 0, & 0, & 0 \end{matrix}$$

Given these forms, we may readily verify that they satisfy the conditions

$$\omega^2 = 0, \qquad \mu^2 = 1, \qquad \mu\omega = -\omega\mu = \omega.$$

We will now take as the symbolical form of our triquaternion, expressed as a matrix,

$$\begin{matrix} A_1, & B_1, & C_1, & D_1 \\ A_2, & B_2, & C_2, & D_2 \\ A_3, & B_3, & C_3, & D_3 \\ A_4, & B_4, & C_4, & D_4 \end{matrix}$$

We then have

$$A_1 = \quad t_1 - x_2 - i\,(y_2 - z_3), \qquad B_1 = -(x_1 + t_2) - i\,(z_2 + y_3),$$

$$C_1 = -y_1 + z_2 - i\,(t_2 - x_3), \qquad D_1 = -(z_1 + y_2) + i\,(x_2 + t_3),$$

$$\begin{aligned}
A_2 &= \quad x_1+t_2-i\,(z_2+y_3), & B_2 &= \quad t_1-x_2+i\,(y_2-z_3),\\
C_2 &= -(z_1+y_2)-i\,(x_2+t_3), & D_2 &= \quad y_1-z_2-i\,(t_2-x_3),\\
A_3 &= \quad y_1+z_2+i\,(t_2+x_3), & B_3 &= \quad z_1-y_2-i\,(x_2-t_3),\\
C_3 &= \quad t_1+x_2-i\,(y_2+z_3), & D_3 &= -x_1+t_2-i\,(z_2-y_3),\\
A_4 &= \quad z_1-y_2+i\,(x_2-t_3), & B_4 &= -(y_1+z_2)+i\,(t_2+x_3),\\
C_4 &= \quad x_1-t_2-i\,(z_2-y_3), & D_4 &= \quad t_1+x_2+i\,(y_2+z_3).
\end{aligned}$$

In conclusion we may remark that the preceding work suggests the expression of the general quaternary matrix with the aid of four quaternions in the form

$$q_1+\epsilon_1 q_2+\epsilon_2 q_3+\epsilon_3 q_4.$$

[*March* 10*th*, 1906.—This last point may be readily verified. In fact, if we denote the elements constituting the n-th row of the matrix by the symbols a_n, b_n, c_n, d_n, we have

$$\begin{aligned}
t_1-x_2-y_3-z_4 &= a_1, & -x_1-t_2-z_3+y_4 &= b_1,\\
-y_1+z_2-t_3-x_4 &= c_1, & -z_1-y_2+x_3-t_4 &= d_1,\\
x_1+t_2-z_3+y_4 &= a_2, & t_1-x_2+y_3+z_4 &= b_2,\\
-z_1-y_2-x_3+t_4 &= c_2, & y_1-z_2-t_3-x_4 &= d_2,\\
y_1+z_2+t_3-x_4 &= a_3, & z_1-y_2-x_3-t_4 &= b_3,\\
t_1+x_2-y_3+z_4 &= c_3, & -x_1+t_2-z_3-y_4 &= d_3,\\
z_1-y_2+x_3+t_4 &= a_4, & -y_1-z_2+t_3-x_4 &= b_4,\\
x_1-t_2-z_3-y_4 &= c_4, & t_1+x_2+y_3-z_4 &= d_4.
\end{aligned}$$

These equations arrange themselves, for solution, in groups of four. The solution of the first group gives

$$\begin{aligned}
4t_1 &= \quad a_1+b_2+c_3+d_4, & 4x_2 &= -a_1-b_2+c_3+d_4,\\
4y_3 &= -a_1+b_2-c_3+d_4, & 4z_4 &= -a_1+b_2+c_3-d_4.
\end{aligned}$$

The solution of the second group gives

$$\begin{aligned}
4t_2 &= a_2-b_1-c_4+d_3, & 4x_1 &= \quad a_2-b_1+c_4-d_3,\\
4y_4 &= a_2+b_1-c_4-d_3, & 4z_3 &= -a_2-b_1-c_4-d_3.
\end{aligned}$$

The solution of the third group gives

$$4t_3 = a_3+b_4-c_1-d_2, \qquad 4x_4 = -a_3-b_4-c_1-d_2,$$

$$4y_1 = a_3-b_4-c_1+d_2, \qquad 4z_2 = a_3-b_4+c_1-d_2.$$

Finally, the solution of the fourth group gives

$$4t_4 = a_4-b_3+c_2-d_1, \qquad 4x_3 = a_4-b_3-c_2+d_1,$$

$$4y_2 = -a_4-b_3-c_2-d_1, \qquad 4z_1 = a_4+b_3-c_2-d_1.]$$

REMARK ON THE EISENSTEIN-SYLVESTER EXTENSION OF FERMAT'S THEOREM*

By H. F. BAKER.

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1. Let N be any positive number, r any number prime to N, such that ω is the least positive integer for which $r^\omega \equiv 1 \pmod{N}$; further, s, t two other numbers prime to N, such that $st \equiv 1 \pmod{N}$; and put $sr^i \equiv \rho_i \pmod{N}$, where ρ_i is positive and less than N, so that $\rho_0 = \rho_\omega = \sigma$, where σ is the least positive residue of s; then, if $r\rho_{i-1} = \rho_i + a_i N$, for $i = 1, 2, \ldots, \omega$, we have

$$\begin{aligned} t \sum_{i=1}^{\omega} r^{\omega-i} a_i N &= t\Sigma r^{\omega-i}(r\rho_{i-1}-\rho_i) \\ &= t\left[(r^\omega\rho_0 - r^{\omega-1}\rho_1)+(r^{\omega-1}\rho_1 - r^{\omega-2}\rho_2]+\ldots+(r\rho_{\omega-1}-\rho_\omega)\right] \\ &= t(r^\omega\rho_0-\rho_\omega) = \sigma t(r^\omega - 1), \end{aligned}$$

so that
$$\sigma t(r^\omega-1)/N = t\sum_{i=1}^{\omega} r^{\omega-i} a_i.$$

Now let $1/\rho_i$ denote the least positive number x_i, such that

$$x_i\rho_i \equiv 1 \pmod{N}, \quad \text{or} \quad x_i sr^i \equiv 1 \pmod{N}, \quad \text{or} \quad x_i \equiv tr^{\omega-i} \pmod{N};$$

then, as $\sigma t \equiv 1 \pmod{N}$, we have

$$\frac{r^\omega - 1}{N} \equiv \sum_i \frac{a_i}{\rho_i} \pmod{N}.$$

It is proved at once, e being any positive number, that

$$(r^{e\omega}-1)/N \equiv e(r^\omega - 1)/N \pmod{N};$$

in particular, if N, in prime factors, be $p^\lambda q^\mu \ldots$, we may take $e\omega =$ least common multiple of $p^{\lambda-1}(p-1)$, $q^{\mu-1}(q-1)$, ...; or we may take $e\omega$ equal to the product of these numbers, namely, equal to $\phi(N)$, the number

* See Eisenstein, *Monatsber. Berlin. Akad.*, 1850, p. 41; Sylvester, *Comp. Rend.*, Vol. LII., 1861, pp. 161–163, p. 308, p. 817, and *Phil. Mag.*, Vol. XXI., 1861, p. 136; Stern, *Crelle*, Vol. C., 1887, p. 182; Mirimanoff, *Crelle*, Vol. CXV., 1895, p. 295; Glaisher, *Quart. Jour.*, Vol. XXXII., 1901, p. 1, p. 240. Dr. Glaisher's proof of Sylvester's theorem is easily modified to the case when the modulus is a composite integer.

of numbers less than N and prime to N; now these numbers may be arranged in $e = \phi(N)/\omega$ sets of ω each, according to the value of s above, namely, $(1, r, r^2, \ldots, r^{\omega-1})$, $(s, sr, sr^2, \ldots, sr^{\omega-1})$, $(s', s'r, s'r^2, \ldots, s'r^{\omega-1})$, ...; we thus obtain e alternative representations

$$\frac{r^{\omega}-1}{N} \equiv \Sigma \frac{a_i^{(0)}}{\rho_i^{(0)}} \equiv \Sigma \frac{a_i}{\rho_i} \equiv \Sigma \frac{a_i'}{\rho_i'} \equiv \ldots \pmod{N},$$

which, on addition, give

$$\frac{r^{\phi(N)}-1}{N} \equiv \Sigma \frac{a_i^{(0)}}{\rho_i^{(0)}} + \Sigma \frac{a_i}{\rho_i} + \Sigma \frac{a_i'}{\rho_i'} + \ldots \pmod{N},$$

where now, in the denominators, occur all numbers less than N and prime to N.

When r is positive and less than N, the number ρ_i in the formula $r\rho_{i-1} = \rho_i + a_i N$ is the positive residue of $r\rho_{i-1}$, so that a_i is zero, or positive and less than r, since $r\rho_{i-1} < rN$; if N' be the least positive number such that $N'N \equiv 1 \pmod{r}$, and $\rho_i = N - m_i$, we have

$$a_i N + N - m_i \equiv 0 \pmod{r},$$

and hence $$a_i \equiv N'm_i - 1 \pmod{r}$$

bearing in mind that $1/(N-m_i)$ means the least positive integer x such that

$$x(N-m_i) \equiv 1 \pmod{N},$$

it is easy to prove that $$\Sigma \frac{1}{N-m_i} \equiv 0 \pmod{N},$$

the summation extending to $\phi(N)$ terms; adding this sum to the preceding, we thus have

$$\frac{r^{\phi(N)}-1}{N} \equiv \sum_{i=1}^{\phi(N)} \frac{(N'm_i)}{N-m_i} \pmod{N},$$

where the denominators are all the numbers less than N and prime to N, and the numerator $(N'm_i)$ means the number of the set $1, 2, \ldots, r$ which is $\equiv N'm_i \pmod{r}$. This is exactly Sylvester's theorem, for the case, however, when the modulus is composite. The method of proof appears to be applicable to cases where the modulus is an algebraic integer.

2. Supposing in the preceding that $s = 1$, so that ρ_i is the least positive residue of $r^i \pmod{N}$, we have

$$\begin{aligned} r^i - \rho_i &= r^{i-1}(r-\rho_1) + r^{i-2}(r\rho_1 - \rho_2) + \ldots + r\rho_{i-1} - \rho_i \\ &= N(a_1 r^{i-1} + a_2 r^{i-2} + \ldots + a_i). \end{aligned}$$

Taking i successive equations of this form, we find, writing β_i for $a_i N$,

$$r^i = \begin{vmatrix} \beta_1, & \beta_2, & \beta_3, & \dots, & \beta_{i-1}, & \beta_i + \rho_i \\ -1, & \beta_1, & \beta_2, & \dots, & \beta_{i-2}, & \beta_{i-1} + \rho_{i-1} \\ 0, & -1, & \beta_1, & \dots, & \beta_{i-1}, & \beta_{i-2} + \rho_{i-2} \\ \cdot & \cdot & \cdot & \dots, & \cdot & \cdot \\ 0, & 0, & 0, & \dots, & -1, & \beta_1 + \rho_1 \end{vmatrix},$$

and it can be proved that, expanded in powers of N, this is capable of symbolic representation by the (finite) series

$$r^i = \rho_i + N\rho_i\Omega + N^2\rho_i\Omega^2 + \dots,$$

where
$$\Omega = \frac{a_1}{u} + \frac{a_2}{u^2} + \frac{a_3}{u^3} + \dots,$$

and, after expansion of $\rho_i\Omega^n$, the coefficient ρ_i/u^μ is to be replaced by $\rho_{i-\mu}$, where $\rho_0 = 1$ and $\rho_{-k} = 0$; in particular

$$\rho_i\Omega = \rho_i\left(\frac{a_1}{u} + \frac{a_2}{u^2} + \dots\right) = a_1\rho^{i-1} + a_2\rho^{i-2} + \dots,$$

$$\rho_i\Omega^2 = \rho_i\left(\frac{a_1^2}{u^2} + \frac{2a_1a_2}{u^3} + \frac{a_2^2 + 2a_1a_3}{u^4} + \dots\right)$$
$$= a_1^2\rho_{i-2} + 2a_1a_2\rho_{i-3} + (a_2^2 + 2a_1a_3)\rho_{i-4} + \dots .$$

This being understood, we may write

$$r^i = \frac{\rho_i}{1 - N\Omega}.$$

The numbers previously denoted by $1/\rho_1$, $1/\rho_2$, ... are the same as $\rho_{\omega-1}$, $\rho_{\omega-2}$, ...; this result therefore includes (Fermat's theorem and) Sylvester's theorem, and gives a representation of $(r^i - \rho_i)/N$ for modulus any power of N, where N is not necessarily a prime number, and r not necessarily positive or less than N (the coefficients of the various powers of N requiring, however, in general, reduction in regard to the modulus).

3. Some examples of § 1 will show how very simple the theorem really is.

(a) $N = 36$, $r = 5$; then $\omega = 6$, $\phi(N) = 12$. Also $N'N \equiv 1 \pmod r$ gives $N' = 1$.—Take first $s = 1$; then, in the previous notation,

$$\begin{aligned} r^{i-1} &= 1,\ 5,\ 5^2,\ 5^3,\ 5^4,\ 5^5, \\ \rho_{i-1} &= 1,\ 5,\ 25,\ 17,\ 13,\ 29, \\ r\rho_{i-1} &= 5,\ 25,\ 125,\ 85,\ 65,\ 145, \\ \rho_i &= 5,\ 25,\ 17,\ 13,\ 29,\ 1, \\ a_i &= 0,\ 0,\ 3,\ 2,\ 1,\ 4, \end{aligned}$$

$$\text{and}\quad 2 \equiv 434 = \frac{5^6-1}{36} \equiv \frac{3}{17}+\frac{2}{13}+\frac{1}{29}+4 \equiv \frac{3}{5^3}+\frac{2}{5^4}+\frac{1}{5^5}+4$$
$$\equiv 3.5^3+2.5^2+5+4$$
$$\equiv 3.17+2.25+5+4 = 110 \equiv 2 \pmod{36}.$$

Take next $s = 7$; then

$$\begin{aligned}
sr^{i-1} &= 7,\ 7.5,\ 7.5^2,\ 7.5^3,\ 7.5^4,\ 7.5^5,\\
\rho_{i-1} &= 7,\ 35,\ 31,\ 11,\ 19,\ 23,\\
r\rho_{i-1} &= 35,\ 175,\ 155,\ 55,\ 95,\ 115,\\
\rho_i &= 35,\ 31,\ 11,\ 19,\ 23,\ 7,\\
a_i &= 0,\ 4,\ 4,\ 1,\ 2,\ 3,
\end{aligned}$$

$$\text{and}\quad \frac{5^6-1}{36} \equiv \frac{4}{31}+\frac{4}{11}+\frac{1}{19}+\frac{2}{23}+\frac{3}{7} \equiv \frac{4}{7.5^2}+\frac{4}{7.5^3}+\frac{1}{7.5^4}+\frac{2}{7.5^5}+\frac{3}{7}$$
$$\equiv 31(4.5^4+4.5^3+5^2+2.5+3)$$
$$\equiv 31(4.13+4.17+25+10+3)$$
$$\equiv 31.158 = 4898 \equiv 2 \pmod{36}.$$

$$\text{Thus}\quad \frac{5^6-1}{36} \equiv \frac{3}{17}+\frac{2}{13}+\frac{1}{29}+4 \equiv \frac{4}{31}+\frac{4}{11}+\frac{1}{19}+\frac{2}{23}+\frac{3}{7} \pmod{36}$$

$$\text{and}\quad \frac{5^{\phi(N)}-1}{36} = \frac{5^{12}-1}{36} \equiv \frac{3}{17}+\frac{2}{13}+\frac{1}{29}+4+\frac{4}{31}+\frac{4}{11}+\frac{1}{19}+\frac{2}{23}+\frac{3}{7}$$
$$\equiv \frac{0}{N-1}+\frac{4}{N-5}+\frac{1}{N-7}+\frac{0}{N-11}+\frac{2}{N-13}+\frac{1}{N-17}$$
$$+\frac{3}{N-19}+\frac{2}{N-23}+\frac{4}{N-25}+\frac{3}{N-29}+\frac{0}{N-31}+\frac{4}{N-35},$$

where the numerators, chosen from 0, 1, ..., 4, are easily seen, if the general denominator be called $N-m$, to be of the form $\equiv N'm-1$, that is $\equiv m-1 \pmod 5$, and, as remarked, we may, if we wish, add unity to each of these numerators.

(*b*) $N = 89$, $r = 2$, $\omega = 11$, $\phi(N)/\omega = 8$.—There are 8 representations according to the value of s; the equation $89N' \equiv 1 \pmod 2$ gives $N' = 1$, and the numerators are 0 or 1 according as, in the denominators $N-m$, the number m is odd or even, that is according as the denominators are even or odd.

Take first $s = 1$; among the numbers r^i the only odd residues ρ_i are $39 \equiv 2^7$, $67 \equiv 2^9$, $45 \equiv 2^{10}$, $1 \equiv 2^0 \pmod{89}$, and we have

$$\frac{2^{88}-1}{89} \equiv 8\left(\frac{1}{39}+\frac{1}{67}+\frac{1}{45}+1\right) \equiv 6 \pmod{89}.$$

Take next $s=3$; the only odd residues are $3 \equiv 3.2^0$, $7 \equiv 3.2^5$, $23 \equiv 3.2^9$, and we have

$$\frac{2^{88}-1}{89} \equiv 8\left(\frac{1}{3}+\frac{1}{7}+\frac{1}{23}\right).$$

Take next $s=5$; the odd residues are $5 \equiv 5.2^0$, $71 \equiv 5.2^5$, $53 \equiv 5.2^6$, $17 \equiv 5.2^7$, $47 \equiv 5.2^{10}$, and we have

$$\frac{2^{88}-1}{89} \equiv 8\left(\frac{1}{5}+\frac{1}{71}+\frac{1}{53}+\frac{1}{17}+\frac{1}{47}\right).$$

These three expressions are respectively congruent to

$$8(2^4+2^2+2+1), \qquad 8.30(1+2^6+2^3), \qquad 8.18(1+2^6+2^5+2^4+2),$$

or 184, 16560, 16560, all of which are $\equiv 6$. The other five representations are similarly obtainable. The first of these here is given by Mirimanoff, without indication of the existence of the others.

4. The expression obtained by Sylvester's theorem in § 1, even with assigned $\rho_1, \rho_2, \ldots, \rho_\omega$, can be further modified thus:—We have $r^\omega = 1+MN > N$, where $0<r<N$, and can write

$$N = \beta_\omega+\beta_{\omega-1}r+\beta_{\omega-2}r^2+\ldots+\beta_1 r^{\omega-1} \quad (0 \leqslant \beta_i < r),$$

and hence
$$t\left(\frac{\beta_1}{r}+\frac{\beta_2}{r^2}+\ldots+\frac{\beta_{\omega-1}}{r^{\omega-1}}+\beta_\omega\right) \equiv 0 \pmod{N},$$

giving
$$\frac{\beta_1}{\rho_1}+\frac{\beta_2}{\rho_2}+\ldots+\frac{\beta_{\omega-1}}{\rho_{\omega-1}}+\frac{\beta_\omega}{\rho_\omega} \equiv 0 \pmod{N}.$$

For instance, when $N=36$, $r=5$, $s=1$, $N=1+2r+r^2$, and we have

$$\frac{1}{13}+\frac{2}{29}+1 \equiv 0 \pmod{36},$$

and, when $N=36$, $r=5$, $s=7$,

$$\frac{1}{19}+\frac{2}{23}+\frac{1}{7} \equiv 0 \pmod{36}.$$

When $N=89$, $r=2$, we have $N=1+2^3+2^4+2^6$; so that $\beta_5=1$, $\beta_7=1$, $\beta_8=1$, $\beta_{11}=1$, the others being zero; thus we have, for $s=1$, $s=3$, $s=5$ respectively,

$$\frac{1}{32}+\frac{1}{89}+\frac{1}{78}+1 \equiv 0, \quad \frac{1}{7}+\frac{1}{28}+\frac{1}{56}+\frac{1}{3} \equiv 0, \quad \frac{1}{71}+\frac{1}{17}+\frac{1}{34}+\frac{1}{5} \equiv 0$$
$$\pmod{89}.$$

ON ABSOLUTELY CONVERGENT IMPROPER DOUBLE INTEGRALS

By E. W. HOBSON.

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THE theory of those improper double integrals in which the domain of integration is limited has been developed on the basis of two distinct definitions: that of Jordan and that of de la Vallée-Poussin. In the first part of the present paper these two definitions are compared, and are shown to be completely equivalent to one another. These definitions only admit of the existence of such improper double integrals as are absolutely convergent. A definition of a less stringent character is required in order to admit of the existence of non-absolutely convergent improper double integrals, but in the present communication those double integrals only will be taken into account which exist in accordance with either of the two definitions above cited. A wider definition due to Lebesgue has also been considered. The second part of the paper is concerned with the conditions under which the double integrals can be replaced by repeated integrals. This matter has been elaborately treated by de la Vallée-Poussin, who has obtained, in the case of a function which is never negative, a necessary and sufficient condition for the equivalence in question. His theorem has, however, only been hitherto established under certain restrictive conditions, which impair the generality of the result. In the present communication, the theorem is established without any such restrictive assumption. The recent development of the theory of the measure of sets of points, by Borel and Lebesgue, has made this more general treatment of the question possible. A more general definition of regular convergence than that of de la Vallée-Poussin is here introduced, including the latter as a special case.

The definition here introduced is of a very general character, and is applicable to cases in which the functions of a sequence, and also the limiting function defined by the sequence, are not restricted to be limited functions, or to have definite values for every value of the variable, but may be indefinite between limits of indeterminacy, either of which may be finite or infinite. It is shown that Arzelà's "convergenza uniforme a

tratti in generale" is equivalent to that particular case of regular convergence except at the points of a set of zero measure, which arises when all the functions of the sequence are limited functions, and have definite values for each value of the variable.

A short discussion is given of the view which has been maintained by Schönflies, that every improper double integral can be replaced by the corresponding repeated integrals, and it is shown that this view is untenable.

As regards notation, the double integral of $f(x, y)$ is denoted by $\int f(x, y)(dx\,dy)$, in recognition of the fact that a proper double integral is defined as a single limit: the word "double" must be taken to refer to the two dimensions of the domain for which the integral is defined. On the other hand, a repeated integral $\int dx \int f(x, y)\,dy$, or $\int\int f(x, y)\,dx\,dy$ is properly represented by a double use of the sign of integration, since it is defined as a repeated limit.

The Definition of Improper Double Integrals.

1. Let G be a finite plane domain, that is a set of points (x, y) such that $|x|$, $|y|$ have definite upper limits when all the points of G are taken into account. Let it further be assumed that the domain G has a frontier which is of plane content zero, the term frontier being used in the sense employed by Jordan, as consisting of the set of points each of which is either a point of G which is also a limiting point of a sequence of points not belonging to G, or else a point not belonging to G which is also a limiting point of a sequence of points belonging to G. Let $f(x, y)$ be a function defined for all points of the domain G; this function may be replaced by a function defined for all points in a fundamental rectangle with sides parallel to the coordinate axes, and containing G. The function is defined to have the same value as $f(x, y)$ at all points of G, and to be zero at all points of the fundamental rectangle that do not belong to G. We may denote the function so extended by $f(x, y)$, as before. If the function $f(x, y)$ is such that at each point of a certain closed set K_∞ the function has an infinite discontinuity, the integral of $f(x, y)$ taken over the fundamental rectangle is said to be an improper double integral.

The following definition of an improper double integral is substantially that given by Jordan:—

Let $D_1, D_2, \ldots, D_n, \ldots$ denote a sequence of domains contained in the fundamental rectangle, each one of which consists of a finite number of connex closed portions with its frontier of zero plane content, and in which the number of portions may increase indefinitely with n. Let it be

assumed that the function $f(x, y)$ is such that the closed set of points K_∞ of infinite discontinuity of the function has zero content, and that none of the domains D_n contain, in their interiors or on their boundaries, any point of K_∞. Let the sequence $\{D_n\}$ be such that the content of D_n converges to that of the fundamental rectangle. Then, if $f(x, y)$ is integrable in every domain such as D_n, and the integrals

$$\int_{D_1} f(x, y)(dx\,dy), \quad \int_{D_2} f(x, y)(dx\,dy), \quad \ldots, \quad \int_{D_n} f(x, y)(dx\,dy), \quad \ldots$$

converge to a definite limit independent of the particular sequence $\{D_n\}$ chosen, subject to the conditions stated, this limit is said to define the improper double integral $\int f(x, y)(dx\,dy)$ of $f(x, y)$ in the domain G.

It has been shown by Jordan that whenever $\int_G f(x, y)(dx\,dy)$ exists, in accordance with the above definition, then $\int_G |f(x, y)|\,(dx\,dy)$ also exists, so that every improper double integral, so defined, is absolutely convergent.

The following definition, different from that of Jordan, has been given by de la Vallée-Poussin :—

Let $f_n(x, y)$ be that function which is such that $f_n(x, y) = f(x, y)$ at every point (x, y) at which $M_n \geqslant f(x, y) \geqslant -N_n$, where M_n, N_n are two positive numbers, and that $f_n(x, y) = M_n$ at every point where $f(x, y) > M_n$; and also $f_n(x, y) = -N_n$ at every point where $f(x, y) < -N_n$. If $f(x, y)$ be such that the proper integral $\int f_n(x, y)(dx\,dy)$ over the fundamental rectangle exists whatever positive values M_n, N_n may have, then, if the sequence

$$\int f_1(x, y)(dx\,dy), \quad \int f_2(x, y)(dx\,dy), \quad \ldots, \quad \int f_n(x, y)(dx\,dy), \quad \ldots$$

has a definite limit, provided the sequences $\{M_n\}$, $\{N_n\}$ have no upper limits, and if this limit is independent of the particular sequences $\{M_n\}$, $\{N_n\}$ chosen, subject to the condition stated, then this limit defines the improper double integral $\int f(x, y)(dx\,dy)$ over the fundamental rectangle.

It has been shown by Schönflies that when the integral exists, in accordance with this definition, the set of points K_∞ must have zero content.

It is easily seen that every improper integral so defined is absolutely convergent.

The theory of absolutely convergent improper integrals has been developed on two independent lines from the two definitions given above

as starting points. It will here be shown that the two definitions are completely equivalent to one another.*

2. In accordance with either of the definitions the existence of the absolutely convergent improper integral of $f(x, y)$ implies that of each of the two functions $f^+(x, y)$, $f^-(x, y)$, where $f^+(x, y)$ is defined by the conditions that $f^+(x, y) = f(x, y)$, at any point at which $f(x, y)$ is positive, and everywhere else $f^+(x, y) = 0$; and similarly

$$f^-(x, y) = -f(x, y),$$

where $f(x, y)$ is negative, and everywhere else $f^-(x, y) = 0$. It is consequently clear that, in order to show the complete equivalence of the two definitions, it is sufficient to consider the case in which $f(x, y)$ is everywhere either positive or zero. Let us then assume that the function $f(x, y)$, which is never negative, has an improper integral in accordance with Jordan's definition.

Let the set of points K_∞ be enclosed in a finite set of rectangles $\{\delta\}$, and let the remaining part of the fundamental rectangle consist of a set of non-overlapping rectangles $\{\eta\}$. The sum $\Sigma\delta$ can be chosen so small that the integral of $f(x, y)$ through the rectangles $\{\eta\}$ is less than the improper integral by less than an arbitrarily chosen positive number ρ.

Let N be a positive number not less than the upper limit of $f(x, y)$ in all the rectangles $\{\eta\}$, and let $f_n(x, y)$ be the function, corresponding to N, employed in de la Vallée-Poussin's definition. Let another set of non-overlapping rectangles $\{\delta'\}$ interior to the set $\{\delta\}$ also enclose all the points of K_∞, and let $\{\eta'\}$ be the finite set of rectangles complementary to $\{\delta'\}$. The integral of $f(x, y)$ over $\{\eta'\}$ lies between the value of the integral over $\{\eta\}$ and that of Jordan's improper integral, and therefore differs from the latter by less than ρ. It follows that the integral of $f(x, y)$ through the area obtained by removing the set $\{\delta'\}$ from the set $\{\delta\}$ is also $< \rho$; and, since $f_n(x, y) \leqslant f(x, y)$, we see that the integral of $f_n(x, y)$ over the same area is $< \rho$. From this we deduce that $\int f_n(x, y)\,(dx\,dy)$ taken through the rectangles $\{\delta\}$ is $< \rho + N\Sigma\delta'$; and, since this holds for an arbitrarily small value of $\Sigma\delta'$, N being fixed, we see that $\int f_n(x, y)\,(dx\,dy)$ taken over the rectangles $\{\delta\}$ is $\leqslant \rho$.

It now follows that the difference of the integrals of $f_n(x, y)$ taken over the fundamental rectangle and over the rectangles $\{\eta\}$ is $\leqslant \rho$; and, since ρ is arbitrarily small, N being sufficiently increased, it follows that

* Stolz seems to imply that the definition of de la Vallée-Poussin is in some sense more general than that of Jordan, which Stolz has himself adopted as the basis of his own treatment. See the *Grundzüge*, Vol. III., p. 124.

the integral of $f_n(x, y)$ over the fundamental rectangle has a definite limit when n is indefinitely increased, and that this limit is Jordan's improper integral. It has thus been shewn that a function which has an improper integral in accordance with Jordan's definition has one also in accordance with the definition of de la Vallée-Poussin; the integrals having the same value in the two cases.

To prove the converse, we assume that

$$\int_{\{\delta\}} f_n(x, y)(dx\,dy) + \int_{\{\eta\}} f(x, y)(dx\,dy)$$

has a definite limit as n is indefinitely increased and $\Sigma\delta$ is indefinitely diminished, the value of N being fixed, as before, for each set $\{\eta\}$. Since both the integrals are positive, it follows that $\int_{\{\eta\}} f(x, y)(dx\,dy)$, which increases as $\Sigma\delta$ is diminished, is less than a fixed finite number, and therefore has a definite upper limit. It has thus been shewn that there exists a special class of domains $\{D_n\}$ such that $\int_{D_n} f(x, y)(dx\,dy)$ has a definite upper limit as the content of D_n converges, with increasing n, to the content of the fundamental rectangle. These domains D_n are complementary to a finite set of rectangles enclosing the points K_∞. It remains to be shewn, in order to establish the existence of Jordan's improper integral, that, if any other set of domains $\{D'_n\}$ be chosen such that the content of D'_n converges to that of the fundamental rectangle, but such that D'_n is not restricted to be complementary to a finite set of rectangles $\{\delta\}$, then $\int_{D'_n} f(x, y)(dx\,dy)$ converges to the same limit as $\int_{D_n} f(x, y)(dx\,dy)$ does. Denoting the content of D'_n by $m(D'_n)$, and that of the fundamental rectangle by A, let D'_n be such that $A - m(D'_n) = \epsilon_n$. The domain D_n can be so chosen as to contain D'_n in its interior. For, since D'_n does not contain, in its interior or on its boundary, any points of K_∞, it follows* that for each point of K_∞ the distance from all the points of D_n has a minimum greater than zero. Hence each point of K_∞ can be enclosed in a rectangle which contains no points of D'_n in its interior or on its sides. The set K_∞ being closed, a finite set of these rectangles can, in accordance with a known extension of the Heine-Borel theorem, be chosen so as to enclose the whole set of points K_∞; and the complement of this finite set

* This is a consequence of the connexity of the domains D_n, D'_n. Jordan, in his definition (*Cours d'Analyse*, Vol. II., p. 76), does not explicitly state that D_n is made up of a finite number of connex portions. He describes it as "mesurable et parfait."

of rectangles may be taken as D_n. This may be done for each value of n. If $m(D'_n)$ converges to A, it is clear that $m(D_n)$, which as just chosen is $> m(D'_n)$, also converges to A. Also a number $n' > n$ can be determined such that $D'_{n'}$ encloses D_n in its interior; we have then

$$\int_{D_{n'}} f(x, y)(dx\,dy) \geqslant \int_{D'_{n'}} f(x, y)(dx\,dy) \geqslant \int_{D_n} (x, y)(dx\,dy)$$
$$\geqslant \int_{D'_n} f(x, y)(dx\,dy).$$

If then $\int_{D_n} f(x, y)(dx\,dy)$ converges to a definite limit $\int_A f(x, y)(dx\,dy)$, n may be taken so great that

$$\int_A f(x, y)(dx\,dy) - \int_{D_n} f(x, y)(dx\,dy) < \eta,$$

where η is an arbitrarily chosen positive number; then also

$$\int_A f(x, y)(dx\,dy) - \int_{D'_{n'}} f(x, y)(dx\,dy) < \eta,$$

and it thus follows that $\int_{D'_{n'}} f(x, y)(dx\,dy)$ also converges to the limit $\int_A f(x, y)(dx\,dy)$. It has now been shewn that the existence of Jordan's improper double integral is a necessary consequence of the existence of that of de la Vallée-Poussin; and the two definitions have thus been shewn to be completely equivalent.

Lebesgue's Definition of an Improper Integral.

3. A limited function $f(x, y)$ which is everywhere positive or zero in the fundamental rectangle being defined, the integral $\int f(x, y)(dx\,dy)$ has been defined by Lebesgue* as follows :—

Denoting by U the upper limit of $f(x, y)$ in its domain, let $u_0, u_1, u_2, \ldots, u_n$, where $u_0 = 0$, $u_n = U$, be a set of numbers such that $u_1 - u_0$, $u_2 - u_1$, $\ldots$, $u_n - u_{n-1}$ are all positive, the greatest of them being η. Let e_r denote the set of points (x, y) for which $f(x, y) = u_r$, and $\bar{e}_r$ the set of points for which $u_r < f(x, y) < u_{r+1}$, and let $m(e_r)$, $m(\bar{e}_r)$ denote

* See his memoir "Intégral, Longueur, Aire," *Annali di Matematica*, Ser. 3A, Vol. VIII., 1902.

the plane measure of e_r and $\bar{e}_r$ respectively. Consider the two sums

$$\sigma = \sum_{r=0}^{r=n} u_r m(e_r) + \sum_{r=0}^{r=n-1} u_r m(\bar{e}_r),$$

$$\sigma' = \sum_{r=0}^{r=n} u_r m(e_r) + \sum_{r=0}^{r=n-1} u_{r+1} m(\bar{e}_r).$$

It can be shewn that, if the number n be increased indefinitely, whilst at the same time η converges to zero, then σ and σ' converge to one and the same number; and that this number is independent of the mode in which the interval $(0, U)$ is divided by the set of numbers $u_1, u_2, \ldots, u_{n-1}$, and is independent of the mode in which the successive further sub-division of the interval $(0, U)$ proceeds, subject to the condition that η, the greatest of the differences $u_r - u_{r-1}$, converges to zero as the sub-division is continued indefinitely. The common limit of σ and σ' is defined to be the double integral $\int f(x, y)(dx\,dy)$, and it is shewn that the integral so defined always exists provided only that $f(x, y)$ is a *summable* function, *i.e.*, a function such that the set of points (x, y) for which $A < f(x, y) < B$ is a measurable set, for every pair of values of A and B.

The integral of a limited function which is not necessarily positive is then defined by

$$\int f(x, y)(dx\,dy) = \int f^+(x, y)(dx\,dy) - \int f^-(x, y)(dx\,dy),$$

where $f^+(x, y)$ is equal to $f(x, y)$ or 0 according as $f(x, y) \geqslant 0$ or < 0, and $f^-(x, y)$ is equal to $-f(x, y)$ or 0 according as $f(x, y)$ is negative or not. This definition being applicable to every summable function, it is wider than the ordinary Riemann definition of a double integral, and includes all the functions defined by ordinary processes. The condition that the plane measure of all the points of discontinuity of the function shall be zero, which is necessary for the existence of the Riemann integral, is not necessarily satisfied by a function which possesses a Lebesgue integral. It is not definitely known whether every limited function is summable or not. Lebesgue has shewn that, when the Riemann integral exists, the integral in accordance with his own definition also exists, and that the two are identical in value. When the Riemann integral does not exist, Lebesgue's integral lies between the upper and lower integrals of the function as defined by Darboux.

Lebesgue has extended his definition so as to afford a definition of an absolutely convergent improper integral. It is clearly sufficient to take the case of an unlimited function $f(x, y)$ which is nowhere negative in the fundamental rectangle. The definition is substantially as follows:—

Let $u_0, u_1, u_2, \ldots, u_n, \ldots$ be a sequence of increasing numbers, such

that $u_0 = 0$, and that u_n has no upper limit as the index n is indefinitely increased; also let the differences $u_1 - u_0$, $u_2 - u_1$, ..., $u_{n+1} - u_n$, ... be limited, having η as their upper limit. Consider the two series

$$\sigma = \sum_{r=0}^{\infty} u_r m(e_r) + \sum_{r=0}^{\infty} u_r m(\bar{e}_r),$$

$$\sigma' = \sum_{r=0}^{\infty} u_r m(e_r) + \sum_{r=0}^{\infty} u_{r+1} m(\bar{e}_r).$$

Since the difference of the two series is

$$\sum_{r=0}^{\infty} (u_{r+1} - u_r)\, m(\bar{e}_r),$$

which is less than $\eta \sum_{r=0}^{\infty} m(\bar{e}_r)$, it is clear that the two series are either both convergent or are both divergent. Let us suppose that the series are both convergent; it can then be shewn that they are still convergent when further numbers are interpolated between each consecutive pair of the numbers u_0, u_1, u_2, ..., and the corresponding new series are formed. It can then be further shewn that, as the process of successive sub-division of the interval $(0, \infty)$ proceeds in any manner consistent with the continual diminution of η to the limit zero, the sums σ, σ' both converge to a single number, for $\lim \eta = 0$; in fact σ constantly increases, and σ' constantly decreases. The value to which σ, σ' converge can be shewn to be independent of the original mode of sub-division of the interval $(0, \infty)$, and of the precise mode in which the further sub-division proceeds. The common limit of σ, σ', when it exists, is then defined to be the value of the improper integral $\int f(x, y)(dx\,dy)$.

In order that an improper integral may exist, it is necessary, though not sufficient, that $f(x, y)$ be a summable function, and also that the measure of those points (x, y) at which $f(x, y)$ is greater than or equal to an arbitrarily great number N shall be arbitrarily small for a sufficiently great value of N. For it is a necessary consequence of the convergence of the above series that $\sum_{r=n}^{\infty} \{m(e_r) + m(\bar{e}_r)\}$, which is the plane measure of the set of points at which $f(x, y) \geqslant u_n$, should have a value which converges to zero, as n and u_n are indefinitely increased. It is, however, not necessary that the content of the set K_∞ of all the points of infinite discontinuity should be zero; in fact it is even possible that the improper integral may exist whilst every point of the fundamental rectangle is a point of infinite discontinuity.

It will now be shewn that Lebesgue's definition of an improper integral can be replaced by one which differs from that of de la Vallée-Poussin,

only in the one respect that the convergent sequence of proper integrals $\int f_n(x, y)(dx\,dy)$ consists of Lebesgue integrals, which are not necessarily Riemann integrals.

From the condition of convergence of the second series, corresponding to an arbitrarily chosen positive number ϵ, we may determine s so that

$$\sigma' = \sum_{r=0}^{r=s} u_r m(e_r) + \sum_{r=0}^{s-1} u_{r+1} m(\bar{e}_r) + R,$$

where $R < \epsilon$, whilst at the same time η is so small that σ' differs from $\int f(x, y)(dx\,dy)$ by less than ϵ. Now let $u_s = N$, and let $f_n(x, y)$ be that function which $= f(x, y)$, for $f(x, y) < N$, and $= N$, for $f(x, y) \geqslant N$.

The Lebesgue proper integral $\int f_n(x, y)(dx\,dy)$ is then the limit when η converges to zero of the sum

$$\sum_{r=0}^{r=s} u_r m(e_r) + \sum_{r=0}^{s-1} u_{r+1} m(\bar{e}_r) + u_s \sum_{r=s+1}^{\infty} m(e_r) + u_s \sum_{r=s}^{\infty} m(\bar{e}_r),$$

and this sum is equal to

$$\sum_{r=0}^{r=s} u_r m(e_r) + \sum_{r=0}^{s-1} u_{r+1} m(\bar{e}_r) + S,$$

where $S < R < \epsilon$. Keeping $u_r = N$ fixed, we may now, if necessary, diminish η by interpolating further numbers between the pairs of numbers $u_0, u_1, u_2, \ldots, u_r, \ldots$, until we have the new sum which corresponds to

$$\sum_{r=0}^{r=s} u_r m(e_r) + \sum_{r=0}^{s-1} m(\bar{e}_r) + S$$

differing from $\int f_n(x, y)(dx\,dy)$ by less than ϵ, the part S not having been increased by any diminution of η. We thus find that σ' differs from $\int f_n(x, y)(dx\,dy)$ by less than ϵ, when N is sufficiently great and η sufficiently small; also σ' has been taken to differ from $\int f(x, y)(dx\,dy)$ by less than ϵ, η having been chosen sufficiently small. Since ϵ is arbitrarily small, it is clear that $\int f_n(x, y)(dx\,dy)$ converges to $\int f(x, y)(dx\,dy)$ as N is increased indefinitely.

It has therefore been shewn that de la Vallée-Poussin's definition of an improper double integral may be extended to the case in which the integrals $\int f_n(x, y)(dx\,dy)$ exist only in the sense defined by Lebesgue. This definition is then equivalent to that of Lebesgue. It is clear that Jordan's definition is only capable of extension, in the case in which K_∞ has zero content; for otherwise the measures of the domains D_n do not converge to that of the fundamental rectangle; and, in fact, in case K_∞ contains every point of the fundamental rectangle, no such domains as the D_n exist.

If the condition that K_∞ have zero content be satisfied, the whole of the reasoning in § 2 is applicable without essential change, and in that case Jordan's definition of an improper integral can be extended to the case in which the proper integrals $\int_{D_n} f(x, y)(dx\,dy)$ exist only in the sense defined by Lebesgue. Thus in this case all three definitions are equivalent to one another.*

The Regular Convergence of a Sequence of Functions.

Let $\phi_1(x)$, $\phi_2(x)$, ..., $\phi_n(x)$, ... be a sequence of functions defined for the interval (a, b). We shall suppose that, for each value of x, any one of these functions $\phi_n(x)$ has either a definite value or is multiple-valued, and is then regarded as indeterminate, between limits of indeterminacy,† of which the upper limit may be denoted by $\overline{\phi_n(x)}$, and the lower limit by $\underline{\phi_n(x)}$. For any value of x for which $\phi_n(x)$ is determinate, we have $\overline{\phi_n(x)} = \underline{\phi_n(x)}$. When either $\overline{\phi_n(x)}$ or $\underline{\phi_n(x)}$ is to be taken indifferently, we may use the notation $\overline{\underline{\phi_n(x)}}$. The consideration of a function $\phi_n(x)$ which, for a particular set of values of x, is indeterminate between limits of indeterminacy, as a single function, involves an extension of Dirichlet's definition of a function, which is justified by its convenience for use in investigations such as the present one. This extension is convenient when the functional value of $\phi_n(x)$ at a point x is defined by means of a limit, say $(\phi_n x) = \lim_{m=\infty} \psi_n(x, m)$, such that, for a particular value of x, $\lim_{m=\infty} \psi_n(x, m)$ has no single value, but may be multiple-valued between finite or infinite limits $\overline{\phi_n(x)}$, $\underline{\phi_n(x)}$. The function $\phi_n(x)$, for such a value of x, may be capable of having a finite number of values, or an infinite number, and possibly all values between $\overline{\phi_n(x)}$, $\underline{\phi_n(x)}$: but in the application of the theory we need only attend to these upper and lower limits of indeterminacy, it being indifferent whether $\phi(x)$ has all values between these limits or some values only. The fluctuation of $\phi_n(x)$ in any interval (α, β) is the excess of the upper limit of the numbers $\overline{\phi_n(x)}$ for all points of (α, β) over the lower limit of the numbers $\underline{\phi_n(x)}$ in the same interval. The saltus (*Sprung*) of $\phi_n(x)$ at the point x is the limit of the fluctuation in an interval $(x-\delta,\ x+\delta)$, when δ is indefinitely

* In the remainder of the paper, it will be assumed that all proper integrals exist in accordance with Riemann's definition.

† In the application of this definition made in this paper, all the functions $\phi_n(x)$ are limited functions. This is, however, not necessary for the validity of the definition. In general $\phi_n(x)$ may be unlimited, and the values of $\overline{\phi_n(x)}$, $\underline{\phi_n(x)}$ for particular values of x may have the improper values ∞ or $-\infty$.

diminished, and this saltus is $\geqslant \overline{\phi_n(x)} - \underline{\phi_n(x)}$. Riemann's theory of integration is applicable to such a function $\phi_n(x)$, in case it is limited in (a, b), just as in the case of a single-valued function.

For any fixed value of x, the numbers

$$\overline{\phi_1(x)},\ \overline{\phi_2(x)},\ \ldots,\ \overline{\phi_n(x)},\ \ldots;\quad \underline{\phi_1(x)},\ \underline{\phi_2(x)},\ \ldots,\ \underline{\phi_n(x)},\ \ldots$$

form a set which we may denote by G.

Let us consider the derivative G' of G; then, if G' is limited, since it is a closed set, it has a greatest value A and a least value B, and these numbers A and B are such that, for a given ϵ, there are an infinite number of values of n such that $|\overline{\phi_n(x)} - A| < \epsilon$, and also an infinite number of values of n such that $|\underline{\phi_n(x)} - B| < \epsilon$. If G' is unlimited in one direction or in both directions, either A or B or both may be regarded as having one of the improper values ∞, $-\infty$.

We now define a function $\phi(x)$ for the interval (a, b) in the following manner:—When, for a particular value of x, the numbers A and B are equal and finite, their value is taken to be that of $\phi(x)$. If A and B are unequal and finite, we regard $\phi(x)$ as multiple-valued, with $\overline{\phi(x)} = A$, $\underline{\phi(x)} = B$. If either A or B has one of the improper values $+\infty$, $-\infty$, the point x is taken to be a point of infinite discontinuity of $\phi(x)$. The function $\phi(x)$ is regarded as a single function, not necessarily limited, and it may have an improper integral in (a, b) in accordance with Harnack's definition of the improper integral of an unlimited function. This function $\phi(x)$ is said to be *the limiting function defined by the sequence* $\{\phi_n(x)\}$, and the functions $\phi_n(x)$ may be said to *converge*, in an extended sense of the term, to the function $\phi(x)$; and thus $\phi(x) = \lim\limits_{n=\infty} \phi_n(x)$.

In case the sequence $\{\overline{\phi_n(x)}\}$ is monotone and non-diminishing, so that, for every value of x and n, the condition $\overline{\phi_n(x)} \leqslant \overline{\phi_{n+1}(x)}$ is satisfied, the sequence $\{\overline{\phi_n(x)}\}$ has, for each particular value of x, either a definite upper limit A or the improper limit $+\infty$. If $\underline{\phi_n(x)} \geqslant \underline{\phi_{n+1}(x)}$, for every value of x and n, the sequence $\{\underline{\phi_n(x)}\}$ has, for each particular value of x, either a definite lower limit B or the improper lower limit $-\infty$.

Let a positive number ϵ and a positive integer n_1 be arbitrarily chosen, and let E be a set of points in (a, b) of which the measure is zero. Let us suppose that, for each point x_1 in (a, b) which does not belong to a certain component E_ϵ of E, this component depending on ϵ, an integer $m > n_1$, and also a neighbourhood $(x_1 - \delta, x_1 + \delta')$, can be determined, such that the four inequalities $|\overline{\underline{\phi(x)}} - \overline{\underline{\phi_n(x)}}| < \epsilon$ are all satisfied at every point in the interval $(x_1 - \delta, x_1 + \delta')$ which is in (a, b).

Then, provided this condition is satisfied for every value of ϵ, *and also* E *is such that each point of it belongs to* E_ϵ *for some sufficiently small value of* ϵ, *the convergence of the sequence* $\{\phi_n(x)\}$ *to* $\phi(x)$ *is said to be regular in* (a, b) *except for the set* E *of zero measure.**

If will be observed that, for a given ϵ, the integer n $(> n_1)$ depends in general upon the particular point x_1 which does not belong to E_ϵ. Moreover, since n_1 is arbitrary, there exists for a particular point x_1 an infinite number of values of n; the neighbourhood $(x_1-\delta,\ x_1+\delta')$ depending, however, in general upon the value of n chosen.

In case the sequence $\{\phi_n(x)\}$ is monotone and increasing, so that $\overline{\phi_n}(x) \leqslant \overline{\phi_{n+1}}(x)$, and also $\underline{\phi_n}(x) \leqslant \underline{\phi_{n+1}}(x)$, when the conditions

$$|\overline{\underline{\phi}}(x)-\overline{\underline{\phi_n}}(x)| < \epsilon$$

are satisfied for a particular value of n, they are also satisfied for every greater value of n. In the general case, however, this is no longer true.

It is easily seen that the set E_ϵ must, for each value of ϵ, be a non-dense closed set, although the set E is not necessarily non-dense, and may be everywhere dense in (a, b). For, if ξ be a limiting point of E_ϵ, then every neighbourhood of ξ contains points of E_ϵ, and it is impossible that the conditions $|\overline{\underline{\phi}}(x)-\overline{\underline{\phi_n}}(x)| < \epsilon$ can be satisfied for every point of such a neighbourhood. Therefore ξ must itself belong to E_ϵ, which must consequently be a closed set; and, since E_ϵ has the measure zero, it cannot contain all the points of any interval (α, β), and is therefore non-dense in (a, b).

The set E, which consists of the points which belong to any of the sets $E_{\epsilon_1}, E_{\epsilon_2}, \ldots, E_{\epsilon_n}, \ldots$, where $\epsilon_1, \epsilon_2, \ldots, \epsilon_n, \ldots$ is a sequence of descending values of ϵ converging to zero, is a set of the *first category*.

The set E contains every point at which $\phi(x)$ has not a definite finite value; for, since $\overline{\phi}(x)-\overline{\phi_n}(x)$, $\underline{\phi}(x)-\underline{\phi_n}(x)$ are both numerically less than ϵ, at a point which does not belong to E, for some value of n, it follows that $\overline{\phi}(x)-\underline{\phi}(x)$ is less than 2ϵ; and, since ϵ is arbitrarily small, it follows that $\overline{\phi}(x) = \underline{\phi}(x)$. It is clear that the points of infinite discontinuity of $\phi(x)$ belong to the set E_ϵ, whatever be the value of ϵ.

Let the numbers ϵ and n_1 be fixed; then, since E_ϵ is closed and has its content zero, all its points may be enclosed in the interiors of a finite set of intervals of which the sum is η, an arbitrarily small number; let

* The term *measure* of a set is throughout used in the sense employed by Borel and Lebesgue. The term *content* is used in the sense employed by Cantor and Harnack. The measure and the content of a closed set are identical, but this is not in general true of an unclosed set.

these intervals be excluded from (a, b). There remains a finite set of intervals such that, for each point x_1 in any of them, a neighbourhood $(x_1-\delta, x_1+\delta')$ can be found, for the whole of which the conditions $|\overline{\phi(x)}-\overline{\phi_n(x)}| < \epsilon$ are satisfied for some value of n $(> n_1)$ dependent on x_1. Let us consider these intervals $(x_1-\delta, x_1+\delta')$ for every point of the finite set of intervals which remain in (a, b). Each point of this finite set is in the interior of some of the intervals $(x_1-\delta, x_1+\delta')$; and therefore, by employing the Heine-Borel theorem, we see that a finite number of the intervals $(x_1-\delta, x_1+\delta')$ can be selected so that every point of (a, b) not interior to the excluded intervals is interior to one at least of these selected intervals. It follows that the conditions $|\overline{\phi(x)}-\overline{\phi_n(x)}| < \epsilon$ are satisfied at every point x of (a, b) not interior to the excluded intervals whose sum is η, when n has one of a finite number of values

$$n_1+p_1,\ n_1+p_2,\ \ldots,\ n_1+p_r.$$

The particular number n_1+p which must be taken for a point x depends upon the position of that point, but the same number n_1+p is applicable to all the points of one or more continuous intervals.*

In the particular case in which $\phi_1(x), \ldots, \phi_n(x), \ldots$ are all definite in value for each value of x, and for which $\phi_n(x) \leqslant \phi_{n+1}(x)$ for every value of x and n, the condition $\phi(x)-\phi_n(x) < \epsilon$ is satisfied for every point not interior to the intervals enclosing E_ϵ, the value of n being everywhere the same. For we may take as the value of n the greatest of the numbers n_1+p. In this case the definition is equivalent to the definition of regular convergence given † by de la Vallée-Poussin for the case he considered. The definition given above is much more general than that of de la Vallée-Poussin, but is requisite for the purpose of a complete treatment of the conditions under which an absolutely convergent improper integral can be replaced by a repeated integral.

The Repeated Improper Integrals.

5. The function $f(x, y)$ being defined, as explained in § 1, for the fundamental rectangle bounded by $x = a$, $x = b$, $y = c$, $y = d$, and it being assumed that the absolutely convergent improper integral

$$\int f(x, y)\,(dx, dy),$$

taken over the fundamental rectangle, exists, necessary and also sufficient

* It thus appears that regular convergence, except for a set E of zero measure, is closely related to Arzelà's "convergenza uniforme a tratti in generale," which I have considered in *Proceedings*, Ser. 2, Vol. 1, p. 380. In fact, in the case in which the functions are all single-valued at every point there is precise equivalence between the two definitions.

† *Liouville's Journal*, Ser. 4, Vol. VIII., 1892, pp. 435, 436.

conditions will now be investigated that the repeated integral

$$\int_a^b dx \int_c^d f(x, y)\,dy$$

may exist and have the same value as the double integral.

We shall consider a sequence $f_1(x, y)$, $f_2(x, y)$, ..., $f_n(x, y)$, ... of functions obtained from $f(x, y)$ as in de la Vallée-Poussin's definition given in § 1.

The integral $\int_c^d f_n(x, y)dy$ will be denoted by $\phi_n(x)$, where $\phi_n(x)$ may either have a determinate value or may have as limits of indeterminacy $\overline{\phi_n(x)}$, $\underline{\phi_n(x)}$, the upper and lower values of the integral $\int_c^d f_n(x, y)\,dy$, in accordance with Darboux's definition of the upper and lower integral of a limited function. The existence of $\int f_n(x, y)(dx\,dy)$ does not ensure the determinacy of $\phi_n(x)$ for all values of x. The integral $\int_c^d f(x, y)\,dy$ will be denoted by $\phi(x)$; a similar remark applies to the determinacy of $\phi(x)$, as in the case of $\phi_n(x)$. Moreover, $\phi(x)$ may have the improper value ∞ or $-\infty$, or may have one of these as a limit of indeterminacy; for $f(x, y)$ does not necessarily, for each value of x, possess either a proper or an improper integral in the interval (c, d). In de la Vallée-Poussin's investigation the restrictive assumptions are made that the functions $\phi_n(x)$ are everywhere definite and that $\phi(x)$ is everywhere finite or definite. Moreover, in part of his work it is assumed that the functions $\phi_n(x)$ are all essentially positive or zero.

It will first be shewn on the assumption of the existence of the double integral $\int f(x, y)\,(dx, dy)$ to be necessary, in order that

$$\int_a^b dx \int_a^d f(x, y)\,dy$$

may exist, that the sequence $\{\phi_n(x)\}$ should converge regularly to the limit $\phi(x)$, except for a set of points E of the first category and of zero measure.

When, for a fixed x, the function $f(x, y)$ has points of infinite discontinuity with respect to the variable y, in the interval (c, d), the value of $\overline{\int_c^d} f(x, y)dy$ or $\overline{\phi(x)}$ is the upper limit of $\overline{\int_c^d} f_n(x, y)\,dy$, that is of $\overline{\phi_n(x)}$, when all values of n are taken into account, and $\underline{\int_c^d} f(x, y)dy$, or $\underline{\phi(x)}$, is the limit of $\underline{\int_c^d} f_n(x, y)\,dy$, or $\underline{\phi_n(y)}$. In case the $\overline{\phi_n(x)}$ have no

upper limit, $\overline{\overline{\phi(x)}}$ has the improper value ∞, and a similar remark applies to $\underline{\phi}(x)$.

Since $f(x, y)$ is integrable in the fundamental rectangle, all the functions $f_n(x, y)$ have proper integrals in that domain. The proper integral $\int f_n(x, y)(dx, dy)$ is, by a known theorem, replaceable by the repeated integral $\int_a^b dx \int_c^d f_n(x, y)\,dy$, and thus $\phi_n(x)$ is integrable in the linear interval (a, b). It follows that the points of discontinuity of $\phi_n(x)$ form a set of linear measure zero. The set of all points of discontinuity of any of the functions $\phi_1(x)$, $\phi_2(x)$, ..., $\phi_n(x)$, ... is consequently, in accordance with the theory of measurable sets, also a set of zero measure. If $\phi(x)$ be integrable in the interval (a, b), its points of discontinuity must form a set of zero measure. Let us suppose that $\phi(x)$ is integrable in (a, b), and thus that $\int_a^b dx \int_c^d f(x, y)\,dy$ exists; and let us assume, if possible, that the set E_ϵ, referred to in the definition of regular convergence in § 4, has its measure greater than zero. Remove from E_ϵ those points at which one or more of the functions $\phi_1(x)$, $\phi_2(x)$, ..., $\phi_n(x)$, ... is discontinuous, and also remove all those points at which $\phi(x)$ is discontinuous; we have then left a set F_ϵ of measure equal to that of E_ϵ, and therefore, by hypothesis, greater than zero. At every point of F_ϵ all the functions $\phi_n(x)$ are definite and continuous, and $\phi(x)$ is also definite and continuous. If ξ be a point of F_ϵ, the number n ($> n_1$) can be so chosen that

$$|\phi(\xi)-\phi_n(\xi)| < \tfrac{1}{3}\epsilon;$$

also δ can then be so chosen that, for every point x in the interval $(\xi-\delta, \xi+\delta)$, the four inequalities

$$|\phi(\xi)-\overline{\underline{\phi(x)}}| < \tfrac{1}{3}\epsilon,$$

$$|\phi_n(\xi)-\overline{\underline{\phi_n(x)}}| < \tfrac{1}{3}\epsilon$$

are all satisfied. From these inequalities we deduce that the four inequalities $|\overline{\underline{\phi(x)}}-\overline{\underline{\phi_n(x)}}| < \epsilon$ are all satisfied for all points x in the interval $(\xi-\delta, \xi+\delta)$. But this is contrary to the hypothesis that ξ is a point belonging to E_ϵ. It therefore follows that, on the assumption that $f(x, y)$ has an improper integral in the fundamental rectangle, the repeated integral $\int_a^b dx \int_c^d f(x, y)\,dy$ cannot exist unless E_ϵ has the measure zero. Since this holds for every value of ϵ, we have obtained the following theorem :—

If $f(x, y)$ has an improper (absolutely convergent) integral in the

fundamental rectangle, a necessary condition for the existence of the repeated integral $\int_a^b dx \int_c^d f(x, y)\,dy$ *is that the convergence of* $\int_c^d f_n(x, y)\,dy$ *to* $\int_c^d f(x, y)\,dy$ *should be regular except for a set of points E of the first category and of zero measure.*

The special case of this theorem which arises when $f(x, y)$ is restricted to be everywhere positive or zero has been established by de la Vallée-Poussin * under certain restrictive hypotheses. He assumed that

$$\int_c^d f(x, y)\,dy \quad \text{and} \quad \int_c^d f_n(x, y)\,dy$$

both have definite finite values at all points x which do not belong to a set of points of zero content; this is equivalent to the assumption that all those points x, such that the set of points on the ordinate through x at which the saltus of $f(x, y)$ is $\geqslant \alpha$, where α is an arbitrarily chosen positive number, have content zero, form a set of linear content zero. It is true that the set of such points x forms a set of zero measure, but, as it is not necessarily non-dense in (a, b), the content is not necessarily zero. In a later memoir,† de la Vallée-Poussin states that he has not been able to remove the restrictive hypothesis made in the first memoir. He then proves that, when $f(x, y) \geqslant 0$, the double integral can be replaced by $\int_a^b dx \int_c^d mf(x, y)\,dy$, where $mf(x, y)$ denotes the minimum of the function $f(x, y)$ at the point (x, y); but he gives no general investigation of the conditions that the equality

$$\int_a^b dx \int_c^d mf(x, y)\,dy = \int_a^b dx \int_c^d f(x, y)\,dy$$

may hold.

6. Let it now be assumed that at every point (x, y) the function $f(x, y)$ is either positive or zero, but never negative. It will be shewn that in this case the condition of regular convergence of the sequence $\{\phi_n(x)\}$ to $\phi(x)$, at all points except a set of the first category and of zero measure, is sufficient to ensure that $\int_a^b dx \int_c^d f(x, y)\,dy$ exists and is equal to $\int f(x, y)\,(dx, dy)$; it being assumed that the double integral exists.

In this case the four inequalities $|\overline{\phi(x)} - \overline{\phi_n(x)}| < \epsilon$ are equivalent to the one $\overline{\phi(x)} - \underline{\phi_n(x)} < \epsilon$; and, if at any point x this is satisfied for a value of n, then it is also satisfied for all greater values of n. Including

* *Loc. cit.*, pp. 448–450. † *Liouville's Journal*, Ser. 5, Vol. v., 1899.

all the points of E_ϵ in the interior of intervals of a finite set, such that the sum of these intervals is the arbitrarily small number η, we see that the condition $\overline{\phi}(x)-\underline{\phi_n}(x) < \epsilon$ is satisfied for one and the same value of n ($> n_1$) at all points x not interior to the intervals whose sum is η. For we have only to take for n the greatest of the numbers

$$n_1+p_1,\ n_1+p_2,\ \ldots,\ n_1+p_r$$

defined in § 4.

The number ϵ being fixed, we can choose the number η so small that the double integral $\int f(x, y)(dx\,dy)$ over those rectangles of which the height is $d-c$ and the sum of the breadths η is less than an arbitrarily fixed positive number ζ; this follows from Jordan's definition of an improper double integral. The number η being fixed, a number m exists, such that, for $n \geqslant m$, we have $\overline{\phi(x)}-\underline{\phi_n(x)} < \epsilon$, except in the intervals which enclose E_ϵ. We have therefore

$$\overline{\underline{\int}} \phi(x)dx-\int \phi_n(x)\,dx < \epsilon(b-a-\eta) < \epsilon(b-a),$$

the integration being taken along the parts of (a, b) which remain when the enclosing intervals are removed. Hence we have

$$\overline{\underline{\int}} \phi(x)\,dx-\int f_n(x, y)\,(dx\,dy) < \epsilon(b-a),$$

where the double integral is taken over the fundamental rectangle with the exception of those parts of which the breadths are the enclosing intervals. Also, if ζ' is an arbitrarily chosen number, we can choose n so great that

$$\int f(x, y)(dx\,dy)-\int f_n(x, y)\,(dx\,dy) < \zeta',$$

where both the double integrals are taken over the same region as before. We now see that

$$\left|\overline{\underline{\int}} \phi(x)\,dx-\int f(x, y)(dx\,dy)\right| < \zeta'+\epsilon(b-a),$$

and from this we see that

$$\left|\int f(x, y)(dx\,dy)-\overline{\underline{\int}} \phi(x)\,dx\right| < \zeta+\zeta'+\epsilon(b-a),$$

where the double integral is taken over the fundamental rectangle, and the single integral over the points of (a, b) which remain when the intervals enclosing E_ϵ are removed. Now ζ' is arbitrarily small, and ζ, η converge together to zero. It follows that $\int_a^b \phi(x)\,dx$, whether definite or not, lies between $\int f(x, y)(dx\,dy) \pm \epsilon(b-a)$; and, since ϵ is arbitrarily

small, it follows that $\int_a^b \phi(x)\,dx$ exists as a definite proper or improper integral, and is equal to $\int f(x, y)\,(dx\,dy)$.

The following theorem has now been established:—

If $f(x, y)$ is never negative, and has an improper double integral in the fundamental rectangle, then the condition that the integrals $\int_c^d f_n(x, y)\,dy$ converge to $\int_c^d f(x, y)\,dy$ regularly, except for a set of points of the first category and zero measure, is a sufficient condition that $\int_a^b dx \int_c^d f_n(x, y)\,dy$ exists and is equal to $\int f(x, y)\,(dx\,dy)$.

The sufficiency of the same condition, for the case in which $f(x, y)$ is not restricted to have one sign only, does not appear to be capable of establishment, because it is in this case impossible to shew that the conditions $|\overline{\phi(x)}-\overline{\phi_n(x)}| < \epsilon$ are satisfied at all points except in the enclosing intervals, for one and the same value of n; it having been only established that it holds when n has one of a finite number of values.

Combining the present results with that of § 5, we see that—

If $f(x, y)$ is never negative, and has an absolutely convergent improper integral in the fundamental rectangle, the necessary and sufficient condition that $\int_a^b dx \int_c^d f(x, y)\,dy$ exists and is equal to $\int f(x, y)\,(dx\,dy)$ is that the sequence $\int_c^d f_n(x, y)\,dy$ $(n = 1, 2, 3, \ldots)$ converges regularly to $\int_c^d f(x, y)\,dy$, except for a set E of the first category and of zero measure.

It has also been shewn that *when $\int f(x, y)\,(dx\,dy)$ exists then, if $\int_a^b dx \int_c^d f(x, y)\,dy$ have a definite meaning, it is equal to the double integral.*

For it has been shewn that the repeated integral cannot have a definite meaning, $\phi(x)$ being integrable in (a, b), unless the convergence is of the kind specified.

7. Returning to the case in which $f(x, y)$ is not restricted to be of one sign, the following theorem will be established:—

If $f(x, y)$ have an absolutely convergent improper integral in the fundamental rectangle, a sufficient condition that $\int_a^b dx \int_c^d f(x, y)\,dy$ may

exist and have the same value as the double integral $\int f(x, y)(dx\,dy)$ *is that* $\int_c^d |f_n(x, y)|\,dy$ *should converge regularly to* $\int_c^d |f(x, y)|\,dy$, *except for a set of the first category and of zero measure.*

Using
$$f(x, y) = f^+(x, y) - f^-(x, y),$$
$$_n(x, y) = f_n^+(x, y) - f_n^-(x, y),$$

as in § 2, and denoting $\int_c^d f_n^+(x, y)\,dy$, $\int_c^d f_n^-(x, y)\,dy$ by $\phi_n^+(x)$, $\phi_n^-(x)$ respectively, we see that the condition stated in the theorem is that $\phi_n^+(x) + \phi_n^-(x)$ should converge regularly to $\phi^+(x) + \phi^-(x)$. In order that this condition may be satisfied, we must have

$$\overline{\phi^+(x)} + \overline{\phi^-(x)} - \underline{\phi_n^+(x)} - \underline{\phi_n^-(x)} < \epsilon,$$

for a sufficiently great value of n, at every point not interior to a finite set of intervals of arbitrarily small sum η enclosing the points of E_ϵ, a set of zero content. From this condition we deduce that

$$\overline{\phi^+(x)} - \underline{\phi_n^+(x)} < \epsilon \quad \text{and} \quad \overline{\phi^-(x)} - \underline{\phi_n^-(x)} < \epsilon,$$

at every point not in the interior of the intervals; and hence $\phi_n^+(x)$ converges regularly to $\phi^+(x)$, and also $\phi_n^-(x)$ converges regularly to $\phi^-(x)$, at all points except a set E of zero measure. It follows that

$$\int_a^b dx \int_c^d f^+(x, y)\,dy$$

exists and is equal to $\int f^+(x, y)(dx\,dy)$, and also that $\int_a^b dx \int_c^d f^-(x, y)\,dy$ exists and is equal to $\int f^-(x, y)(dx\,dy)$; and therefore $\int_a^b dx \int_c^d f(x, y)\,dy$ exists and is equal to $\int f(x, y)(dx\,dy)$.

The condition stated in the theorem, though sufficient, is not necessary; for the integral $\int_a^b \{\phi^+(x) - \phi^-(x)\}\,dx$ may exist only as a non-absolutely convergent improper integral, in which case $\int_a^b \{\phi^+(x) + \phi^-(x)\}\,dx$ does not exist. In this case, $\int_a^b dx \int_c^d |f(x, y)|\,dy$ not existing, the convergence of $\int_c^d |f_n(x, y)|\,dy$ to $\int_c^d |f(x, y)|\,dy$ cannot be regular. An example will be given below in which this case actually arises.

8. Whether the double integral $\int f(x, y)(dx\,dy)$ exist or not, the proof in § 5 suffices to shew that, if all the double integrals $\int f_n(x, y)(dx\,dy)$ exist, then it is a necessary condition for the existence of the repeated integral $\int_a^b dx \int_c^d f(x, y)\,dy$ that the integrals $\int_c^d f_n(x, y)\,dy$ should converge regularly to $\int_c^d f(x, y)(dy)$, except for a set of points x, of the first category and of zero measure.

Moreover, if it be known that $\int_c^d f(x, y)\,dy$ is a function of x which is limited in the interval (a, b), we can infer the existence of the double integral $\int f(x, y)(dx\,dy)$. For, since

$$\int f_n(x, y)(dx\,dy) = \int_a^b dx \int_c^d f_n(x, y)\,dy,$$

we have
$$\left| \int f_n(x, y)(dx\,dy) \right| < (b-a)\,U_n,$$

where U_n is the upper limit of $\left| \int_c^d f_n(x, y)\,dy \right|$ in the interval (a, b). It is thus seen that $\int f_n(x, y)(dx\,dy)$ cannot increase indefinitely in numerical value as n is indefinitely increased, since U_n does not increase indefinitely.

The following theorem has therefore been established :—

If all the functions $f_n(x, y)$ have double integrals in the fundamental rectangle, and $\int_c^d f_n(x, y)\,dy$ converges to $\int_c^d f(x, y)\,dy$ regularly, except for a set of points x of zero measure and of the first category, then, if $\int_c^d f(x, y)\,dy$ be a limited function of x in the interval (a, b), the double integral $\int f(x, y)(dx\,dy)$ exists and is equal to $\int_a^b dx \int_b^d f(x, y)\,dy$.

Combining this theorem with that of § 7, we have the following theorem :—

If all the functions $f_n(x, y)$ have double integrals in the undamental rectangle, and either $\int_c^d f(x, y)\,dy$ is limited in the interval (a, b) of x, or $\int_a^b f(x, y)\,dx$ is limited in the interval (c, d) of y, and if the conditions are

satisfied that each of the sequences

$$\int_c^d |f_n(x, y)|\, dy, \quad \int_a^b |f_n(x, y)|\, dx$$

converges to the limits

$$\int_c^d |f(x, y)|\, dy, \quad \int_a^b |f(x, y)|\, dx$$

in each case regularly, except for a set of points of zero measure and of the first category, then the double integral exists, and

$$\int f(x, y)\,(dx\,dy) = \int_a^b dx \int_c^d f(x, y)\, dy = \int_c^d dy \int_a^b f(x, y)\, dx.$$

9. It has been maintained* by Schönflies that an absolutely convergent improper double integral can always be replaced by either of the corresponding repeated integrals; no condition beyond that of the existence of the double integral in accordance with the definition of de la Vallée-Poussin being required for the validity of this equivalence. In the case in which the integrand is essentially positive or zero, this view could only be correct in case the regular convergence of the functions $\phi_n(x)$ to the limiting function $\phi(x)$ followed as a necessary consequence of the existence of the double integral. That Schönflies' view is incorrect can be shewn by means of an example.

Let the fundamental rectangle be bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$; and let the function $\psi(x)$ be defined† by the rule that, for every rational value of x of the form $\frac{2m+1}{2^n}$ $(n > 0)$, $\psi(x) = \frac{1}{2^n}$, and that, for every other value of x, $\psi(x) = 0$. Let

$$f(x, y) = \left| \frac{1}{y} \sin \frac{1}{y} \right| \psi(x);$$

then it is easily seen that the improper integral

$$\int \left| \frac{1}{y} \sin \frac{1}{y} \right| \psi(x)(dx\,dy),$$

taken over the rectangle, exists and has the value zero. The integral

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* See the *Bericht über die Mengenlehre*, pp. 198–202.

† This function $\psi(x)$ was first given by Du Bois Reymond, *Crelle's Journal*, Vol. XCII., p. 278. See also Stolz's *Grundzüge*, Vol. III., p. 149, where the above method of constructing a double integral which cannot be replaced by the repeated integral is indicated.]

$\int_0^1 \psi(x)\,dx$ exists and has the value zero. The repeated integral

$$\int_0^1 dx \int_0^1 \psi(x) \left| \frac{1}{y} \sin \frac{1}{y} \right| dy$$

does not exist; for $$\int_0^1 \psi(x) \left| \frac{1}{y} \sin \frac{1}{y} \right| dy$$

diverges for each value of x of the form $\dfrac{2m+1}{2^n}$, and is zero for other values of x. In this case the function $\phi(x)$ is infinite for the everywhere dense set of values $x = \dfrac{2m+1}{2^n}$; and therefore $\phi(x)$ is not integrable in the interval $(0, 1)$; therefore $\int_0^1 \phi(x)\,dx$ has no meaning. In this case, the other repeated integral

$$\int_0^1 dy \int_0^1 \psi(x) \left| \frac{1}{y} \sin \frac{1}{y} \right| dx$$

exists, and is equal to zero. The integral

$$\int \left(\frac{1}{y} \sin \frac{1}{y} \right) \psi(x) (dx\,dy)$$

can be replaced by the repeated integral

$$\int_0^1 dx \int_0^1 \left(\frac{1}{y} \sin \frac{1}{y} \right) \psi(x)\,dy;$$

for $\int_0^1 \frac{1}{y} \sin \frac{1}{y}\,dy$ exists as a non-absolutely convergent single integral and has a value A; hence in this case

$$\phi(x) = A\,\psi(x);$$

and therefore the repeated integral has the value zero, the same as that of the double integral. This is an instance in which the repeated integral exists and is equal to the double integral, although the sufficient condition given in § 5 is not satisfied.

Schönflies has given an example (*loc. cit.*, pp. 201, 202) intended to illustrate his theorem that the condition of regular convergence is unnecessary for the equality of the double integral and repeated integral of a function which is never negative. It will, however, be seen that the example does not bear out his contention. He defines the function $f(x, y)$ as follows:—The rectangle for which the function is defined is bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$, and in all points $x = \dfrac{2m+1}{2^s}$, $y \leqslant \dfrac{1}{2}$,

$f(x, y)$ has the improper value* $+\infty$, and everywhere else $f(x, y) = 0$. In this case the function $f_n(x, y)$ will be given by the conditions

$$f_n(x, y) = N_n,$$

at all points $x = \dfrac{2m+1}{2^s}$, $y \leqslant \dfrac{1}{2^s}$, and $f_n(x, y) = 0$, everywhere else.

It can be shewn that $\quad \int f_n(x, y)(dx\,dy) = 0,$

for every value of N_n, and thus $\int f(x, y)(dx\,dy)$ exists and $= 0$. The condition

$$\int f(x, y)\,dy - \int f_n(x, y)\,dy < \epsilon$$

is not satisfied for any of the everywhere dense set of values $x = \dfrac{2m+1}{2^s}$; and therefore the convergence of $\int f_n(x, y)\,dy$ to $\int f(x, y)\,dy$ is not regular. Schönflies maintains that, notwithstanding this, the repeated integral $\int_0^1 dx \int f(x, y)\,dy$ exists, and is also equal to zero; it will be shewn that this is not the case.

It is true that $\int_0^1 dx \int_0^1 f_n(x, y)\,dy$ is equal to zero, for every value of n, and thus that $\lim\limits_{n=\infty} \int_0^1 dx \int_0^1 f_n(x, y)\,dy$ is zero. But $\int_0^1 dx \int_0^1 f(x, y)\,dy$ is not equivalent to $\lim\limits_{n=\infty} \int_0^1 dx \int_0^1 f_n(x, y)\,dy$, but to $\int_0^1 dx \lim\limits_{n=\infty} \int_0^1 f_n(x, y)\,dy$, since $\int_0^1 f(x, y)\,dy$ is defined to be $\lim\limits_{n=\infty} \int_0^1 f_n(x, y)\,dy$, in accordance with de la Vallée-Poussin's definition of an improper single integral. Now $\int_0^1 f_n(x, y)\,dy$ is zero, unless $x = \dfrac{2m+1}{2^s}$, in which case it is $\dfrac{N_n}{2^s}$, and, for such values of x, $\lim\limits_{n=\infty} \int_0^1 f_n(x, y)\,dy$ or $\phi(x)$ is ∞, and thus $\int_0^1 \phi(x)\,dx$ does not exist, because $\phi(x)$ is ∞ at the everywhere-dense set of points $x = \dfrac{2m+1}{2^s}$, in the interval (0, 1). It has thus been shewn that the repeated integral has no existence, and, since the condition of regular convergence is not satisfied, this is in accordance with the theorem of § 6.

* It may be objected to this definition that the function is not properly defined at the points of the specified set, since the functional *values* are there regarded as having the improper values ∞. The extension of Dirichlet's definition of a function involved in the admission of improper functional values ∞ or $-\infty$, as distinct from functional limits, leads, however, to no difficulty in relation to the theory of integration, and may therefore be admitted without modifying the theory.

This example throws light on the error in Schönflies' proof (*loc. cit.*, pp. 199, 200) of the theorem that the existence of the double integral necessarily entails that of the repeated integral, and the equality of the two. He replaces the function $f_n(x, y)$ by the most nearly continuous function $\phi_n(x, y)$ and then also $\int \phi_n(x, y)dy$ by the most nearly continuous function $\Phi_n(x)$, and argues that

$$\int \Phi_1(x)dx, \quad \int \Phi_2(x)dx, \quad \ldots, \quad \int \Phi_n(x)dx, \quad \ldots$$

form a sequence which defines $\int \Phi(x)dx$, in accordance with de la Vallée-Poussin's definition of an improper single integral. To establish this, he relies upon the insufficient fact that $\Phi_{n+1}(x) \geqq \Phi_n(x)$, for every value of x, whereas $\Phi_n(x)$ in general differs from $\Phi_{n+1}(x)$, not merely for such values as are greater than some fixed number. In the above example

$$\int \Phi_1(x)dx, \quad \int \Phi_2(x)dx, \quad \ldots$$

are all zero, but $\int \Phi(x)dx$ is infinite. The error in the proof appears to depend essentially on an illegitimate identification of

$$\lim_{n=\infty} \int dx \int f_n(x, y)dy$$

with

$$\int dx \lim_{n=\infty} \int f_n(x, y)dy\,;$$

the former of these limits is always equal to $\int f(x, y)(dx\,dy)$, but the latter, which is the interpretation of $\int dx \int f(x, y)dy$, is not unconditionally equal to the former limit.

ON THE REDUCTION OF THE TERNARY QUINTIC AND SEPTIMIC TO THEIR CANONICAL FORMS

By A. C. DIXON *and* T. STUART.

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THIS paper gives a method for reducing ternary quantics of the degrees 5, 7 to their canonical forms.* It depends on Sylvester's extended dialytic method of elimination.

The original variables are taken to be x, y, z, the contragredient set λ, μ, ν; and the words contracubic, contraquartic have been coined to mean respectively cubic and quartic in λ, μ, ν. The symbol Δ has been used for the operator $\frac{\partial^2}{\partial x\,\partial\lambda}+\frac{\partial^2}{\partial y\,\partial\mu}+\frac{\partial^2}{\partial z\,\partial\nu}$, which, when repeated a suitable number of times, turns a quantic in λ, μ, ν into an operator; thus, if $f(\lambda, \mu, \nu)$ is of degree n,

$$\Delta^n f(\lambda, \mu, \nu)\, F(x, y, z) = n!\, f\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) F(x, y, z).$$

I. *Reduction of a Ternary Quintic to the Sum of Seven Fifth Powers.*

Let Q be a homogeneous quintic in x, y, z; it contains twenty-one coefficients, and there are twenty-one arbitrary constants in the form $\sum_{r=1}^{7} L_r^5$ where L_r is linear, say $\lambda_r x+\mu_r y+\nu_r z$; the problem is to reduce Q to this form, if possible. The number of independent contracubics, that is curves of the third class, touched by the seven lines $L_r = 0$ is three, and, if these three can be found, the seven lines are determined and the problem may be considered as solved.

* The general problem has been discussed by Serret (*Nouvelles Annales*, Vol. IV., pp. 145, 193, 433), Clifford (*Works*, p. 123), and Palatini. The last named (*Rom. Acc. L. Rend.*, Vol. XII.) has proved the possibility of the reduction in general for ternary quantics above the fourth order. The uniqueness of the solution for the quintic has been proved by Hilbert (*Liouville's Journal*, 1888), Richmond (*Quarterly Journal*, 1902), and Palatini. Lasker's paper (*Math. Annalen*, Vol. LVIII.) gives a valuable method.

Let U_1, U_2, U_3 be these three contracubics. We then have

$$\Delta^3 U_s L_r^3 = 0 \quad (s = 1, 2, 3;\ r = 1, 2, \ldots, 7);$$

and therefore

$$\Delta^3 U_s Q = 0 \quad (s = 1, 2, 3),$$

that is, U_1, U_2, U_3 are apolar to Q.

Of contracubics apolar to Q there are four, since any contracubic V contains ten coefficients, and these must satisfy six conditions if the identity

$$\Delta^3 V Q = 0$$

is to hold good.

Let V_1, V_2, V_3, V_4 be the four contracubics apolar to Q (giving the four cubic operators which destroy or annihilate Q). Then U_1, U_2, U_3 must be linear combinations of V_1, V_2, V_3, V_4 and the equations $U_1 = U_2 = U_3 = 0$ are equivalent to

$$\left\| \begin{matrix} V_1 & V_2 & V_3 & V_4 \\ a_1 & a_2 & a_3 & a_4 \end{matrix} \right\| = 0,$$

in which a_1, a_2, a_3, a_4 are constants.

The Jacobian of U_1, U_2, U_3 is then

$$\begin{vmatrix} \dfrac{\partial V_1}{\partial \lambda}, & \dfrac{\partial V_2}{\partial \lambda}, & \dfrac{\partial V_3}{\partial \lambda}, & \dfrac{\partial V}{\partial \lambda} \\ \dfrac{\partial V_1}{\partial \mu}, & \dfrac{\partial V_2}{\partial \mu}, & \dfrac{\partial V_3}{\partial \mu}, & \dfrac{\partial V_4}{\partial \mu} \\ \dfrac{\partial V_1}{\partial \nu}, & \dfrac{\partial V_2}{\partial \nu}, & \dfrac{\partial V_3}{\partial \nu}, & \dfrac{\partial V_4}{\partial \nu} \\ a_1, & a_2, & a_3, & a_4 \end{vmatrix} \equiv \Omega, \text{ say.}$$

Now the equations $U_1 = U_2 = U_3 = 0$ have a common solution λ_r, μ_r, ν_r. Hence Ω, a sextic, and its three first derivatives Ω_1, Ω_2, Ω_3, quintics, vanish for the same values of λ, μ, ν, that is,

$$\Delta^5 \Omega_s L_r^5 = 0 \quad (s = 1, 2, 3;\ r = 1, 2, \ldots, 7)$$

and

$$\Delta^5 \Omega_s Q = 0 \quad (s = 1, 2, 3).$$

These three equations are linear in a_1, a_2, a_3, a_4, the coefficients being known in terms of the coefficients in Q, and they therefore determine the ratios

$$a_1 : a_2 : a_3 : a_4$$

uniquely.* Thus the system of conditions

$$U_1 = U_2 = U_3 = 0$$

can be formed, and that in one way only, and the problem has one solution only.

To prove that the problem has been solved, suppose the contracubics U_1, U_2, U_3 to have been formed by the above method in such a way that

$$\Delta^3 U_1 Q = \Delta^3 U_2 Q = \Delta^3 U_3 Q = 0,$$
$$\Delta^5 \Omega_1 Q = \Delta^5 \Omega_2 Q = \Delta^5 \Omega_3 Q = 0,$$

where Ω_1, Ω_2, Ω_3 are the derivatives of Ω, the Jacobian of U_1, U_2, U_3.

Now Sylvester's method of forming E, the eliminant of U_1, U_2, U_3, is to eliminate linearly the twenty-one coefficients of a quintic R from the twenty-one equations included in

$$\Delta^3 U_1 R = \Delta^3 U_2 R = \Delta^3 U_3 R = 0,$$
$$\Delta^5 \Omega_1 R = \Delta^5 \Omega_2 R = \Delta^5 \Omega_3 R = 0,$$

which are known to be satisfied when

$$R \equiv (\lambda' x + \mu' y + \nu' z)^5,$$

where λ', μ', ν' are values that satisfy the equations

$$U_1 = U_2 = U_3 = 0.$$

(See Salmon, *Higher Algebra*, 4th ed., p. 85.)†

In our case the equations are satisfied by taking

$$R \equiv Q;$$

and therefore $E = 0$, so that $U_1 = U_2 = U_3 = 0$ have a common solution, say $(\lambda_1, \mu_1, \nu_1)$; hence a second form for R is $(\lambda_1 x + \mu_1 y + \nu_1 z)^5$ and this cannot generally coincide with Q, which is an arbitrary quintic. There are then two possible forms for R, and the first minors of the determinant E must vanish. The first derivatives of E taken with respect to the coefficients of U_1, U_2, U_3 must then vanish; hence the equations

$$U_1 = U_2 = U_3 = 0$$

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* If the three equations for a_1, a_2, a_3, a_4 are not independent, we are to take a set of values satisfying them; similarly, if there are more than four contracubics such as V_1, V_2, V_3, V_4, any four that are linearly independent may be used.

† Salmon's argument shews that the expression so formed contains the eliminant as a factor. It can contain no other factor on account of its degree, and it does not vanish identically, as may be proved by taking $U_1 = \lambda^3$, $U_2 = \mu^3$, $U_3 = \nu^3$, when $\Omega_1 = 54\lambda\mu^2\nu^2$, $\Omega_2 = 54\lambda^2\mu\nu^2$, $\Omega_3 = 54\lambda^2\mu^2\nu$, and the determinant does not vanish.

have a second common solution, say $(\lambda_2, \mu_2, \nu_2)$, and there is a third form for R, namely, $(\lambda_2 x+\mu_2 y+\nu_2 z)^5$. This argument may be repeated so long as we know that Q cannot be a mere linear combination of the forms $(\lambda_1 x+\mu_1 y+\nu_1 z)^5$, $(\lambda_2 x+\mu_2 y+\nu_2 z)^5$, ... ; and it therefore proves that the equations

$$U_1 = U_2 = U_3 = 0$$

have at least seven common solutions. Then we have an alternative, either

$$Q \equiv \sum_{r=1}^{7} \kappa_r (\lambda_r x+\mu_r y+\nu_r z)^5,$$

for certain values of κ_1, κ_2, ..., or else there is an eighth common solution. From the latter it would follow that U_1, U_2, U_3 were not linearly independent, which is untrue on account of their mode of formation; or else that they had a common quadratic factor, in which case Q, having an apolar conic, would not be the most general quintic; or, again, that they had a common linear factor, in which case the first polar with respect to Q of a certain point would have three apolar conics, and again Q would not be the most general quintic.

In each of these cases there is no difficulty in proving that Q can be reduced to the sum of six fifth powers. When there is an apolar conic the six lines are tangents to it and one may be chosen at will. When U_1, U_2, U_3 have a common linear factor three of the six lines pass through the point represented by that factor, and the other three are common tangents to the conics represented by the quadratic factors of U_1, U_2, U_3.

To form U_1, U_2, U_3 in practice it is simplest* to make use of the fact that there is a syzygy of the form

$$a_1 U_1+a_2 U_2+a_3 U_3 \equiv 0,$$

where a_1, a_2, a_3 are linear in λ, μ, ν; this is a known property of cubics with seven common points.

It follows that there is a syzygy

$$\beta_1 V_1+\beta_2 V_2+\beta_3 V_3+\beta_4 V_4 \equiv 0,$$

where β_1, β_2, β_3, β_4 are linear in λ, μ, ν; β_1, β_2, β_3, β_4 can be found when V_1, V_2, ... have been formed, and if

$$\beta_r \equiv g_r \lambda+h_r \mu+k_r \nu \quad (r = 1, 2, 3, 4),$$

we may take U_1, U_2, U_3 to be $\Sigma g_r V_r$, $\Sigma h_r V_r$, and $\Sigma k_r V_r$.

* This method would seem to be known, but we have not met with it in print.

II. *Reduction of a Ternary Septimic to the Sum of Twelve Seventh Powers.*

Let Q be a homogeneous septimic in x, y, z; it contains thirty-six coefficients, and the problem is to reduce it to the form $\sum_{r=1}^{12} L_r^7$, where

$$L_r = \lambda_r x + \mu_r y + \nu_r z;$$

so that the form proposed also contains thirty-six arbitrary constants. We shall shew how to form the independent contraquartics which are touched by the twelve lines $L_r = 0$; their number is three, and the former method can be used.

Let U_1, U_2, U_3 be the three contraquartics, so that

$$\Delta^4 U_s L_r^7 = 0 \quad (s = 1, 2, 3;\ r = 1, 2, \ldots, 12)$$

and

$$\Delta^4 U_s Q = 0 \quad (s = 1, 2, 3).$$

Let V_1, V_2, V_3, V_4, V_5 be the contraquartics apolar to Q; there are five, since the fifteen coefficients of the general contraquartic V are subjected to the ten conditions included in

$$\Delta^4 VQ = 0.$$

Hence the equations $U_1 = U_2 = U_3 = 0$ are equivalent to

$$\begin{matrix} V_1 & V_2 & V_3 & V_4 & V_5 \\ a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \end{matrix} = 0,$$

if $a_1, \ldots, a_5, b_1, \ldots, b_5$ are certain constants, which it is our object to find. To do this we form equations by Sylvester's method (Salmon, *Higher Algebra*, p. 86), but of the seventh degree* instead of the sixth.

Arrange the expressions U_1, U_2, U_3 in the forms $A_1\lambda^3 + B_1\mu + C_1\nu$, $A_2\lambda^3 + B_2\mu + C_2\nu$, $A_3\lambda^3 + B_3\mu + C_3\nu$. Then, since these all vanish for the values λ_r, μ_r, ν_r, it follows that

$$\Omega_1, \equiv \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix},$$

a septimic in λ, μ, ν, vanishes also. Arranging $V_1, V_2, \ldots$ in the same

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* As in Sylvester's *Collected Works*, Vol. I., pp. 76–9.

form, so that $V_1 = E_1\lambda^3+F_1\mu+G_1\nu$, ..., we have

$$\Omega_1 \equiv \begin{vmatrix} E_1 & E_2 & E_3 & E_4 & E_5 \\ F_1 & F_2 & F_3 & F_4 & F_5 \\ G_1 & G_2 & G_3 & G_4 & G_5 \\ a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \end{vmatrix},$$

and the condition $\Delta^7\Omega_1 Q = 0$ is one of those that must be satisfied by $a_1, a_2, \ldots, b_1, b_2, \ldots$. Five other conditions can be formed in the same way by resolving $U_1, \ldots, V_1, \ldots$ into parts containing respectively the factors $(\lambda, \mu^3, \nu)(\lambda, \mu, \nu^3)(\lambda, \mu^2, \nu^2)(\lambda^2, \mu, \nu^2)(\lambda^2, \mu^2, \nu)$. These six conditions may be written

$$\Delta^7\Omega_r Q = 0 \quad (r = 1, 2, \ldots, 6),$$

and are linear in the determinants $D_{12}, D_{13}, \ldots$ included in

$$\left\| \begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \end{matrix} \right\|.$$

Since only six of the ratios $D_{12} : D_{13} : \ldots$ are independent, the number of solutions of the six equations will be finite, and the problem of expressing Q in the form $\sum_{r=1}^{12} L_r^7$ appears to be a determinate one. We have still to find the number of solutions, and to prove that the conditions imposed upon U_1, U_2, U_3 are enough to ensure that the problem is solved.

Now the determinants $D_{12}, D_{13}, \ldots$ satisfy six linear conditions, and therefore can be expressed linearly in terms of four of them. They also satisfy the five equations

$$D_{23}D_{45}+D_{24}D_{53}+D_{25}D_{34} = 0, \tag{1}$$

$$D_{13}D_{45}+D_{14}D_{53}+D_{15}D_{34} = 0, \tag{2}$$

$$D_{12}D_{45}+D_{14}D_{52}+D_{15}D_{24} = 0, \tag{3}$$

$$D_{12}D_{35}+D_{13}D_{52}+D_{15}D_{23} = 0, \tag{4}$$

$$D_{12}D_{34}+D_{13}D_{42}+D_{14}D_{23} = 0, \tag{5}$$

which are now reduced to homogeneous quadratics in four unknowns. Of the eight sets of values which satisfy (1), (2), (3) we must, to satisfy (4), (5), reject the three which are given by

$$D_{45} = 0, \quad \frac{D_{14}}{D_{15}} = \frac{D_{24}}{D_{25}} = \frac{D_{34}}{D_{35}}:$$

there are left five determinations, all of which fulfil the conditions (1)–(5).

Choosing one of the five solutions, we put

$$U_1 \equiv D_{45}V_1 - D_{15}V_4 + D_{14}V_5,$$
$$U_2 \equiv D_{45}V_2 - D_{25}V_4 + D_{24}V_5,$$
$$U_3 \equiv D_{45}V_3 - D_{35}V_4 + D_{34}V_5;$$

it is to be proved that the contraquartics U_1, U_2, U_3 vanish for twelve common sets of values of λ, μ, ν, and that, if $L_1, L_2, \ldots, L_{12}$ are formed with these twelve sets, we may write

$$Q = \sum_{r=1}^{12} L_r^7.$$

We follow Sylvester's method of forming the eliminant of the three quartics U_1, U_2, U_3.

If λ_1, μ_1, ν_1 is a set of values of λ, μ, ν for which U_1, U_2, U_3 vanish, the conditions

$$\Delta^4 U_1 R \equiv 0, \quad \Delta^4 U_2 R \equiv 0, \quad \Delta^4 U_3 R \equiv 0,$$

equivalent to thirty, are satisfied by the septimic R when

$$R \equiv (\lambda_1 x + \mu_1 y + \nu_1 z)^7.$$

We also have $$\Delta^7 \Omega_r R = 0 \quad (r = 1, 2, \ldots, 6)$$

for the same value of R. In R there are thirty-six coefficients which may be eliminated from these thirty-six linear equations; the determinant thus formed coincides with the eliminant of U_1, U_2, U_3, unless it vanishes identically.

To test this point take

$$U_1 \equiv \lambda^4, \quad U_2 \equiv \mu^4, \quad U_3 \equiv \nu^4.$$

Then $$\Omega_1 \equiv \lambda\mu^3\nu^3, \quad \Omega_2 \equiv \lambda^3\mu\nu^3, \quad \Omega_3 \equiv \lambda^3\mu^3\nu,$$
$$\Omega_4 \equiv \lambda^3\mu^2\nu^2, \quad \Omega_5 \equiv \lambda^2\mu^3\nu^2, \quad \Omega_6 \equiv \lambda^2\mu^2\nu^3;$$

the conditions to which R is subjected actually involve thirty-six equations, and can only be satisfied when all the coefficients in R vanish. Hence the determinant of the thirty-six equations in general cannot vanish identically, but must be the eliminant.

Now in the present case the thirty-six conditions are satisfied by the supposition $R \equiv Q$. Hence the eliminant of U_1, U_2, U_3 is zero, and these three contraquartics vanish for the same set of values of λ, μ, ν, say λ_1, μ_1, ν_1. The argument used in the case of the quintic may be applied again, and it follows that the three vanish for at least twelve common sets

of values; if there are no more, then Q must be a linear combination of the seventh powers of the twelve corresponding linear expressions, and the desired reduction has been made.

If U_1, U_2, U_3 vanish for thirteen common sets of values, it is known* that there must be a syzygy

$$a_1 U_1 + a_2 U_2 + a_3 U_3 \equiv 0,$$

where a_1, a_2, a_3 are either constant or linear in λ, μ, ν. Now they cannot be constant on account of the way in which U_1, U_2, U_3 were constructed; if they are linear, then V_1, V_2, ..., V_5 are connected by a syzygy with linear coefficients, which is not true in a particular case that we shall examine: the conditions for such a syzygy are not satisfied identically, and therefore in general there is no such syzygy. When there is, Q must be a linear combination of the seventh powers of thirteen linear expressions; these are not all independent, but are determined by the solution of a system of equations such as

$$S_1/\lambda = S_2/\mu = S_3/\nu,$$

where S_1, S_2, S_3 are contracubics. This system of equations depends on twenty-three parameters, and thus the form of Q apparently involves thirty-six,† so that the problem of reducing Q to this form must be porismatic, while the reduction to twelve seventh powers is possible in general.

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* See Bacharach, *Math. Annalen*, Vol. XXVI.

† [*Added in proof.*—The same thing may be proved otherwise. Let

$$Q = \sum_{r=1}^{13} A_r L_r^7,$$

where L_1, L_2, ..., L_{13} are common tangents to the contraquartics U_1, U_2, U_3. Then L_1, ..., L_{13} form a special group, coresidual to a single tangent; such a group on a given contraquartic can be made to include 11 arbitrary tangents. Take any curve of the system

$$\lambda_1 U_1 + \lambda_2 U_2 + \lambda_3 U_3 = 0,$$

and let L_{14}, L_{15}, ..., L_{26} be a second special group of 13 tangents to it. Then in all we have 26 tangents, which are coresidual to two, and therefore touch one contraseptimic. There is therefore a syzygy

$$\sum_{r=1}^{26} A' L_r^7 = 0.$$

If $$A_1 : A_2 : \ldots : A_1' = A_1 : A_2 : \ldots : A_{13},$$

this gives a new expression for Q in the same form; of such expressions there will be a singly infinite number, since there are 12 conditions and 13 unknowns to satisfy them, namely, the ratios of λ_1, λ_2, λ_3 and the positions of L_{14}, ..., L_{26}.]

The special form of Q that we are to examine is

$$x^2y^2z^2(x+y+z).$$

Here V_1, V_2, V_3, V_4, V_5 are

$$\lambda^4,\quad \mu^4,\quad \nu^4,\quad \lambda^2\mu\nu-\lambda^3\mu-\lambda^3\nu-\lambda\mu^2\nu+\mu^3\lambda+\mu^3\nu,$$

$$\lambda^2\mu\nu-\lambda^3\mu-\lambda^3\nu-\lambda\mu\nu^2+\nu^3\lambda+\nu^3\mu,$$

and it is easily seen that when these five are multiplied by λ, μ, ν the fifteen products are linearly independent.

ON FUNCTION SUM THEOREMS CONNECTED WITH THE SERIES $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$.

By L. J. ROGERS.

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1. The transformation of the series

$$x+\frac{x^2}{2^2}+\frac{x^3}{3^2}+\ldots,$$

or more generally the integral

$$-\int_0 \log(1-x)\frac{dx}{x},$$

where x is a real quantity, has been considered in Bertrand's *Calcule Intégral* § 270 (1870), and connections are there established between this function of x and the same functions of its co-anharmonic ratios $1-x$, $\frac{1}{x}$, $\frac{1}{1-x}$, $\frac{x}{x-1}$, $\frac{x-1}{x}$. It will, however, be more convenient and will lead to conciser results if we take the function

$$-\tfrac{1}{2}\int_0 \left\{\frac{\log(1-x)}{x}+\frac{\log x}{1-x}\right\}dx, \tag{1}$$

which, if x is real and not greater than unity, may be represented by

$$x+\frac{x^2}{2^2}+\frac{x^3}{3^2}+\ldots+\tfrac{1}{2}\log x\log(1-x),$$

the logarithms being taken in their real principal values.

Calling this function Lx, we see immediately that

$$Lx+L(1-x)=L1, \tag{2}$$

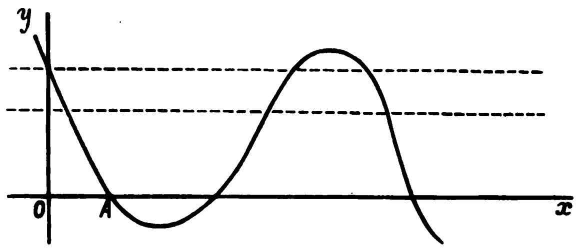
and, since $\log x\log(1-x)$ has zero for its limiting value when $x=0$, we see that $L1=\frac{1}{6}\pi^2$.

The equation (2) gives us a connection between *two* L functions whose arguments depend on *one* variable. This relation, we may shew, is a particular case of a linear connection between n^2+1 such functions whose arguments depend on n independent variables.

Let

$$y = 1-p_1x+p_2x^2-\ldots\pm p_nx^n = (1-\mu_1x)(1-\mu_2x)\ldots(1-\mu_nx),$$

where the μ's are all real and positive, represent a curve in rectangular



Cartesian coordinates, and let us suppose for the present that this curve cuts the line $y = 1$ in n real points, including (0, 1). In this case it is evident that the curve will cut the line $y = 1-m$ in n real points, provided $m < 1$; so that we may put

$$m-p_1x+p_2x^2-\ldots\pm p_nx^n = m(1-\lambda_1x)(1-\lambda_2x)\ldots(1-\lambda_nx),$$

where the λ's are also real.

Now consider the sum of the n^2 terms

$$\Sigma\Sigma dL\frac{\mu_s}{\lambda_r}-\Sigma\Sigma dL\frac{\lambda_r}{\mu_s}, \tag{3}$$

where in the positive terms $\mu_s < \lambda_r$, and in the negative terms $\lambda_r < \mu_s$, while r and s have all values from 1 to n. Since

$$2dLx \equiv \log x\, d\log(1-x)-\log(1-x)\, d\log x,$$

we have

$$\begin{aligned} 2\Sigma\Sigma dL\frac{\mu_s}{\lambda_r} &= \Sigma\Sigma\left[(\log\mu_s-\log\lambda_r)\, d\left\{\log(\lambda_r-\mu_s)-\log\lambda_r\right\}\right. \\ &\qquad \left.-\left\{\log(\lambda_r-\mu_s)-\log\lambda_r\right\} d(\log\mu_s-\log\lambda_r)\right] \\ &= \Sigma\Sigma\left[(\log\mu_s-\log\lambda_r)\, d\log(\lambda_r-\mu_s)\right. \\ &\qquad -\log(\lambda_r-\mu_s)\, d(\log\mu_s-\log\lambda_r) \\ &\qquad \left.-\log\mu_s\, d\log\lambda_r+\log\lambda_r\, d\log\mu_s\right], \end{aligned}$$

while

$$\begin{aligned} -2\Sigma\Sigma dL\frac{\lambda_r}{\mu} &= -\Sigma\Sigma\left[(\log\lambda_r-\log\mu_s)\, d\log(\mu_s-\lambda_r)\right. \\ &\qquad -\log(\mu_s-\lambda_r)\, d(\log\lambda_r-\log\mu_s) \\ &\qquad \left.-\log\lambda_r\, d\log\mu_s+\log\mu_s\, d\log\lambda_r\right]. \end{aligned}$$

Hence twice the whole algebraic sum (3) may be represented by one formula

$$\sum_{s=1}^{n}\sum_{r=1}^{n}\left[(\log\mu_s-\log\lambda_r)\,d\log(\lambda_r\sim\mu_s)-\log(\lambda_r\sim\mu_s)\,d(\log\mu_s-\log\lambda_r)\right.$$
$$\left.-\log\mu_s\,d\log\lambda_r+\log\lambda_r\,d\log\mu_s\right].\qquad(4)$$

Since, identically,

$$u^n-p_1u^{n-1}+\ldots\pm p_n=(u-\mu_1)(u-\mu_2)\ldots(u-\mu_n),$$

and $$mu^n-p_1u^{n-1}+\ldots\pm p_n=m(u-\lambda_1)(u-\lambda_2)\ldots(u-\lambda_n),$$

we have $$(\lambda_r-\mu_1)(\lambda_r-\mu_2)\ldots(\lambda_r-\mu_n)=\lambda_r^n-p_1\lambda_r^{n-1}+\ldots$$
$$=(1-m)\lambda_r^n,\qquad(5)$$

and $$(\mu_s-\lambda_1)(\mu_s-\lambda_2)\ldots(\mu_s-\lambda_n)=\frac{1}{m}\{m\mu_s^n-p_1\mu_s^{n-1}+\ldots\}$$
$$=\frac{m-1}{m}\mu_s^n;$$

so that $$(\lambda_1\sim\mu_s)(\lambda_2\sim\mu_s)\ldots(\lambda_n\sim\mu_s)=\frac{1-m}{m}\mu_s^n\quad(\text{since } m<1).\qquad(6)$$

Now, in (4), the terms

$$\Sigma\Sigma\{\log\mu_s\,d\log(\lambda_r\sim\mu_s)-\log(\lambda_r\sim\mu_s)\,d\log\mu_s\}$$

can be reduced by summing first with respect to r, and, using (6), and become

$$\Sigma\left\{\log\mu_s\,d\left(n\log\mu_s+\log\frac{1-m}{m}\right)-\left(n\log\mu_s+\log\frac{1-m}{m}\right)d\log\mu_s\right\},$$
$$=\log p_n\,d\log\frac{1-m}{m}-\log\frac{1-m}{m}\,d\log p_n.\qquad(7)$$

Similarly, by summing first with respect to s, and using (5), the corresponding set of terms in (4),

$$\Sigma\Sigma\{-\log\lambda_r\,d\log(\lambda_r\sim\mu_s)+\log(\lambda_r\sim\mu_s)\,d\log\lambda_r\},$$

reduce to

$$\Sigma[-\log\lambda_r\,d\{n\log\lambda_r+\log(1-m)\}+\{n\log\lambda_r+\log(1-m)\}\,d\log\lambda_r],$$
$$=-\log\frac{p_n}{m}\,d\log(1-m)+\log(1-m)\,d\log\frac{p_n}{m}.\qquad(8)$$

The remaining terms in (4) can be reduced independently in r and s, and we get

$$\Sigma\Sigma\{-\log\mu_s\,d\log\lambda_r+\log\lambda_r\,d\log\mu_s\}$$
$$=-\log p_n\,d\log\frac{p_n}{m}+\log\frac{p_n}{m}\,d\log p_n$$
$$=\log p_n\,d\log m-\log m\,d\log p_n.\qquad(9)$$

Collecting the reduced forms (7), (8), (9), we have

$$2\Sigma\Sigma\left(dL\frac{\mu_s}{\lambda_r}-dL\frac{\lambda_r}{\mu_s}\right)$$

$$\begin{aligned}
&= \log p_n\{d\log(1-m)-d\log m\}-\{\log(1-m)-\log m\}\,d\log p_n\\
&\quad -(\log p_n-\log m)\,d\log(1-m)+\log(1-m)(d\log p_n-d\log m)\\
&\quad +\log p_n\,d\log m-\log m\,d\log p_n\\
&= \log m\,d\log(1-m)-\log(1-m)\,d\log m\\
&= 2dLm.
\end{aligned}$$

Hence, integrating, we have

$$\Sigma\Sigma\left(L\frac{\mu_s}{\lambda_r}-L\frac{\lambda_r}{\mu_s}\right)=C+Lm. \tag{10}$$

To determine the constant C, we may notice that, when m approaches the value unity, the values of the λ's approach equality with the μ's, and we may suppose λ_r to be that λ which is equal to μ_r in the limit. Now μ_s/λ_r is ultimately equal to λ_s/μ_r, and, if r is not equal to s, the functions having these fractions or their reciprocals as arguments will cancel each other. Noticing from the diagram, where $OA=1/\mu_1$, that $\mu_1<\lambda_1$, $\mu_2>\lambda_2$, $\mu_3<\lambda_3$, ..., we see that, if n is even, then the left-hand side of (10) vanishes and $C=-L1$; whereas, if n is odd, the left-hand side is $L1$ and $C=0$.

We have then, for all positive integral values of n, a linear relation consisting of n^2+1 L-functions depending on n independent variables; and it is easy to see that the arguments are all real provided the curve in the diagram cuts the axis of x in n real points, and the line $y=1-m$ is taken sufficiently near to the axis of x to cut every wave of the curve.

When $n=2$, the relation is

$$L\frac{\mu_1}{\lambda_1}+L\frac{\mu_2}{\lambda_1}-L\frac{\lambda_2}{\mu_1}-L\frac{\lambda_2}{\mu_2}=-L1+Lm,$$

where $$m=\frac{\mu_1\mu_2}{\lambda_1\lambda_2}$$

and $$\frac{\lambda_1+\lambda_2}{\lambda_1\lambda_2}=\frac{\mu_1+\mu_2}{\mu_1\mu_2}.$$

Let $$\frac{\lambda_2}{\mu_2}=x,\quad m=y,$$

so that $$\frac{\mu_1}{\lambda_1}=xy;$$

then
$$Lx+Ly = L(xy)+L\frac{\mu_2}{\lambda_1}+L1-L\frac{\lambda_2}{\mu_1}$$
$$= L(xy)+L\frac{\mu_2}{\lambda_1}+L\frac{\mu_1-\lambda_2}{\mu_1}, \text{ by (1).}$$
But
$$\mu_1+\mu_2 = y(\lambda_1+\lambda_2);$$
therefore
$$\lambda_1 xy+\mu_2 = y(\lambda_1+\mu_2 x),$$
so that
$$\frac{\mu_2}{\lambda_1} = \frac{y(1-x)}{1-xy};$$
while
$$\frac{\lambda_2}{\mu_1} = \frac{1}{y}\,\frac{\mu_2}{\lambda_1} = \frac{1-x}{1-xy},$$
so that
$$1-\frac{\lambda_2}{\mu_1} = \frac{x(1-y)}{1-xy}.$$
Thus, finally, we have
$$Lx+Ly = L(xy)+L\left\{\frac{x(1-y)}{1-xy}\right\}+L\left\{\frac{y(1-x)}{1-xy}\right\}. \tag{11}$$

This formula is apparently the simplest that can be obtained in two independent variables. It is remarkable that, although (1) has been made use of in determining it, yet it is not possible to deduce (1) directly from it. We might infer, then, that formulæ included in (10) containing more than two variables may be so reduced that (11) cannot be deduced as a particular case; and, conversely, it is possible that formulæ (10) cannot be deduced by repeated application of (11). Certain facts render this very probable. For instance, by making $y = x$, we have
$$2Lx = L(x^2)+2L\left(\frac{x}{1+x}\right), \tag{12}$$
a relation which is closely connected with the obvious identity
$$2\psi x+2\psi(-x) = \psi(x^2), \tag{13}$$
where
$$\psi x = x+\frac{x^2}{2^2}+\frac{x^3}{3^2}+\ldots,$$
when we take note of the connection given in Bertrand's *Calc. Int.*, § 270, (26), between
$$\psi(-x) \quad\text{and}\quad \psi\left(\frac{x}{1+x}\right).$$
However, the equally obvious identity
$$3\psi x+3\psi(x\rho)+3\psi(x\rho^2) = \psi(x^3),$$
where
$$1+\rho+\rho^2 = 0,$$

does not appear to be made deducible in any way from (1), even when this formula is made adaptable to imaginary arguments.

2. By transforming the right-hand side of § 1, (11), by means of § 1, (1), we get

$$Lx+Ly+L(1-xy)+L\left(\frac{1-y}{1-xy}\right)+L\left(\frac{1-x}{1-xy}\right)=3L1.$$

These five arguments taken in the order

$$x, \quad y, \quad \frac{1-x}{1-xy}, \quad 1-xy, \quad \frac{1-y}{1-xy}$$

form a cyclic group in which any constituent is the same function of the two preceding it.

If a and b are the sides of a right-angled spherical triangle, these arguments may be written

$$\cos^2 a, \quad \cos^2 b, \quad \sin^2 A, \quad \sin^2 c, \quad \sin^2 B,$$

the squared cosines of Napier's circular parts.

This property shews that, if we apply § 1, (1) to the function of any consecutive three in the cyclic arrangement of the arguments, we again get § 1, (11) in another form, *e.g.*, transferring Ly, $L\left(\frac{1-x}{1-xy}\right)$, and $L(1-xy)$, we get

$$Lx+L\left(\frac{1-y}{1-xy}\right)=L\left\{\frac{x(1-y)}{1-xy}\right\}+L(xy)+L(1-y),$$

which represents § 1, (11), where y is replaced by $(1-x)/(1-xy)$.

The equation § 1, (11) may be written in another interesting form as follows.

By direct application of the formula, we get

$$L(1-x)+L\left(\frac{1}{1+m}\right)=L\left(\frac{1-x}{1+m}\right)+L\left\{\frac{m(1-x)}{m+x}\right\}+L\left(\frac{x}{m+x}\right).$$

By § 1, (1), this may be written

$$Lx+L\left(\frac{x}{x+m}\right)+L\left\{\frac{(1-x)m}{x+m}\right\}+L\left(\frac{1-x}{1+m}\right)=L1+L\left(\frac{1}{1+m}\right).$$

Now, let the function $Lx+L\left(\frac{x}{x+m}\right)$ be called Mx, where we may call x the argument and m the parameter. Then

$$L\left\{\frac{(1-x)m}{x+m}\right\}+L\left(\frac{1-x}{1+m}\right)=M\left\{\frac{(1-x)m}{x+m}\right\},$$

and
$$L1+L\left(\frac{1}{1+m}\right) = M1.$$

Hence
$$Mx+M\left\{\frac{(1-x)m}{x+m}\right\} = M1. \tag{1}$$

When $m = \infty$, this relation reduces to § 1, (1).

When $m = 1$, we have
$$f(x)+f\left(\frac{1-x}{1+x}\right) = f(1),$$
where
$$f(x) = Lx+L\left(\frac{x}{1+x}\right),$$
a result which is virtually given by Bertrand in connection with the series
$$x+\frac{x^3}{3^2}+\frac{x^5}{5^2}+\dots \quad \text{(see § 273)}.$$

Again, we have
$$Lx-L(xy)+L1-L\left\{\frac{x(1-y)}{1-xy}\right\}-L\left\{\frac{y(1-x)}{1-xy}\right\} = L1-Ly,$$

i.e.,
$$Lx-L(xy)+L\left(\frac{1-x}{1-xy}\right)-L\left(y\frac{1-x}{1-xy}\right) = L1-Ly. \tag{2}$$

Looking upon y as a parameter, and writing Yx for $Lx-Lxy$, we get
$$Yx+Y\left(\frac{1-x}{1-xy}\right) = Y1. \tag{3}$$

The equations (1) and (3) may be looked upon as a kind of generalization of the formula § 1, (2), where, if $x = \sin^2\theta$, $1-x = \cos^2\theta$, we get a type of relation of the form $F(\theta)+F(\frac{1}{2}\pi-\theta) = F(\frac{1}{2}\pi)$. In (3), if $x = \operatorname{sn}^2 u$, $y = k^2$, the relation is of the form $F(u)+F(K-u) = F(K)$. It is possible that the terms in § 1, (10) may be so grouped that by looking upon one variable as a modulus the identity may represent a relation between functions of the $(n-1)$ remaining variables, each associated with this modulus; but it is not easy to see how such a grouping may be effected.

3. The value of $L(x+yi)$ when the argument is complex will depend upon the path of integration, but will be unambiguous provided the path does not cross the axis of real quantities at any point whose abscissa >1 or <0. This value will be called the principal value of $L(x+yi)$.

We may easily see then that the formula § 1, (10) can be employed in the case of imaginary arguments, provided we make all the μ's real and positive. For, provided we make the line $y = 1-m$ lie close enough to

the axis of x, we have a case in which all the λ's are real, even if the waves of the curve do not reach $y = 1$. When m is made to diminish so that two of the intersections are lost, then two of the λ's will become imaginary and will remain imaginary up to the final value of m, viz., unity. The corresponding arguments will therefore never again cross the axis of real quantities, and the corresponding L-functions will have their principal values. The same remark applies to all imaginary arguments that may occur.

If, however, the μ's are imaginary, the same process of reasoning fails, as may be seen in the following simple example.

In the formula $$2Lz = 2z^2 + 2L\left(\frac{z}{1+z}\right),$$

let $z = \rho$, where $$\rho^2+\rho+1 = 0,$$

so that apparently $$2L\rho = L\rho^2 + 2L(-\rho^2).$$

Now, if $z = e^{\theta i}$, the value of Lz may be taken as

$$z + \frac{z^2}{2^2} + \frac{z^3}{3^2} + \ldots + \tfrac{1}{2}\log z \log(1-z),$$

the principal values of the logarithms being taken. Thus

$$L(e^{\theta i}) = \cos\theta + \frac{\cos 2\theta}{2^2} + \ldots + i\left(\sin\theta + \frac{\sin 2\theta}{2^2} + \ldots\right)$$
$$-\tfrac{1}{2}\theta i\left\{\cos\theta + \tfrac{1}{2}\cos 2\theta + \ldots + i(\sin\theta + \tfrac{1}{2}\sin 2\theta + \ldots)\right\}.$$

The real part of this is

$$\cos\theta + \frac{\cos 2\theta}{2^2} + \frac{\cos 3\theta}{3^2} + \ldots + \tfrac{1}{2}\theta\left(\sin\theta + \frac{\sin 2\theta}{2} - \ldots\right)$$
$$= \frac{\pi^2}{6} - \frac{\pi}{2}\theta + \frac{\theta^2}{4} + \tfrac{1}{2}\theta\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$$
$$= \frac{\pi^2}{6} - \frac{\pi\theta}{4} \quad \text{(provided } \theta > 0 < \pi),$$

and we get the same value for the real part of $L(e^{-\theta i})$, if $\theta > 0 < \pi$. Thus the real parts of the principal values of $L\rho$ and $L\rho^2$ are zero, while the real part of $L(-\rho^2)$ is $\frac{1}{12}\pi^2$; so that the relation cannot be true. It will be easily seen, however, that, if we make z move from (0, 0) to (0, 1) and thence along the circle $r = 1$ from $\theta = 0$ to $\theta = \frac{4}{3}\pi$, then z^2 will move across the axis of real quantities on the negative side of the origin, and the value of $L\rho^2$ will not be the same as if the point $z = \rho^2$ had been reached by making $\theta = -\frac{2}{3}\pi$. In fact it differs by $\pi i \log(1-z)$, where

$z = \rho^2$, corresponding to a circuit about 0 of $\dfrac{\log z}{1-z}$, *i.e.*, $-\dfrac{1}{2}\dfrac{1}{1-z}\displaystyle\int\frac{dz}{z}$ in the integrand, *i.e.*, $-\dfrac{\pi i}{1-z}$. The real part of $\pi i \log(1-\rho^2)$ is now $-\frac{1}{8}\pi^2$, which cancels with the real part of $2L(-\rho^2)$ in the formula.

It is seen then that even in the simplest formula depending on § 1, (11) imaginary values of the arguments may lead to what at first sight seem anomalous results. It may be observed also that in this formula, whenever the μ's are real, the λ's must be real; so that imaginary arguments always imply imaginary values of the μ's.

The actual point at which the reasoning in § 1 fails for imaginary arguments lies in taking logarithms of each side of equations § 1, (5), (6), but it is easy to see that the logarithms differ by some multiples of $2\pi i$. Where such corrections have to be made, it will be seen that the reduced form for § 1, (4) will contain further terms consisting of $2\pi i$ multiplied by differentials of logarithms; so that § 1, (10) must be corrected by the addition of terms consisting of $2\pi i$ multiplied by logarithms of known arguments, and an integral equation is thereby always obtained. General rules for determining such terms and the constant of integration would be very difficult to determine; but, on the other hand, the system of arguments of the L-functions would be simplified, as there is nothing to prevent taking complex arguments with norm greater than unity, and so in the L-function sum we may start by taking all terms positive and keeping the μ's uniformly in all the numerators of the arguments.

4. It has been shewn in § 1 that a function exists such that, for any arguments x, y which lie between 0 and 1,

$$f(x)+f(y)-f(xy) = f(u)+f(v), \tag{1}$$

where
$$u = \frac{x(1-y)}{1-xy} \quad \text{and} \quad v = \frac{y(1-x)}{1-xy}.$$

To find the differential equation which must be satisfied by any such functions we must operate on both sides by $\dfrac{\partial}{\partial x}$, $\dfrac{\partial}{\partial y}$ and $x\dfrac{\partial}{\partial x} - y\dfrac{\partial}{\partial y}$, so as to annihilate the left-hand side.

Now
$$x(1-x)\frac{\partial u}{\partial x} = \frac{1-y}{(1-xy)^2}\,x(1-x) = u(1-u);$$

therefore
$$x(1-x)\frac{\partial}{\partial x}f(u) = u(1-u)\,\frac{d}{du}f(u) = F(u) \text{ say.}$$

But
$$\frac{\partial u}{\partial y} = -\frac{x(1-x)}{(1-xy)^2};$$

therefore
$$x(1-x)\frac{\partial^2}{\partial x\,\partial y}f(u) = -\frac{x(1-x)}{(1-xy)^2}\frac{dFu}{du},$$

i.e.,
$$\frac{\partial^2}{\partial x\,\partial y}f(u) = -\frac{1}{(1-xy)^2}\frac{dFu}{du}.$$

The operation $x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}$ leaves a function of xy unchanged, while
$$\left(x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}\right)u = \frac{x}{1-xy};$$

therefore
$$\left(x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}\right)\frac{\partial^2}{\partial x\,\partial y}f(u) = -\frac{x}{(1-xy)^3}\frac{d^2}{du^2}F(u)$$
$$= -\frac{u(1-u)}{(1-x)(1-y)(1-xy)}\frac{d^2}{du^2}F(u).$$

By symmetry, the same operation on $f(v)$ gives
$$\frac{v(1-v)}{(1-x)(1-y)(1-xy)}\frac{d^2}{dv^2}F(v),$$

whence
$$u(1-u)\frac{\partial^2}{\partial u^2}u(1-u)\frac{d}{du}f(u) = v(1-v)\frac{d^2}{dv^2}v(1-v)\frac{d}{dv}f(v),$$

where u and v may be considered as independent. Hence
$$u(1-u)\frac{d^2}{du^2}u(1-u)\frac{d}{du}f(u) = \text{a constant.} \tag{2}$$

The complete solution of this equation is easily arrived at by straightforward integration and is of the form
$$A_1Lu+A_2\log u+A_3\log(1-u)+A_3;$$

so that it is possible that a more general form than Lu for $f(u)$ in (1) may satisfy that relation. Such may be shewn to be the case, provided no demand is made that the functions of the several arguments should be identical. In fact, if we write $2L(x, a)$ as denoting
$$Lx-\tfrac{1}{2}\log a\log(1-x)+\tfrac{1}{2}\log(1-a)\log x-La, \tag{3}$$

we shall have identically

$$L(x, a)+L(y, b) = L(xy, ab)+L\left(x\frac{1-y}{1-xy}, a\frac{1-b}{1-ab}\right)+L\left(y\frac{1-x}{1-xy}, b\frac{1-a}{1-ab}\right); \quad (4)$$

for the L-functions obviously cancel out, and the logarithmic terms may be easily shown to vanish. For such an identity to hold it is clear that any multiple of the logarithmic terms might have been assumed in defining $L(x, a)$, but for reasons hereafter obvious the form above given will be most important. The cancelling of these logarithmic products may easily be proved as follows, in the most general case of § 1, (10).

Let $a_1, a_2, \ldots, a_n, \beta_1, \beta_2, \ldots, \beta_n, \kappa$ be a set of quantities related in the manner given in § 1, viz.,

$$\kappa(1-\beta_1 x)(1-\beta_2 x)\ldots(1-\beta_n x) = \kappa-1+(1-a_1 x)(1-a_2 x)\ldots(1-a_n x),$$

and let us suppose that in the equation § 1, (3), viz.,

$$\pm\Sigma\Sigma\left\{\log\frac{\mu_s}{\lambda_r}\,d\log\left(1-\frac{\mu_s}{\lambda_r}\right)-\log\left(1-\frac{\mu_s}{\lambda_r}\right)d\log\frac{\mu_s}{\lambda_r}\right\},$$

we take the operation d to denote changes of the letter μ into β, λ into a, and m into κ. Then the double sum will reduce as before to $\log m\,d\log(1-m)-\log(1-m)\,d\log m$, which now denotes

$$\log m\log(1-\kappa)-\log(1-m)\log\kappa,$$

and

$$\Sigma\Sigma\pm L\left(\frac{\mu_s}{\lambda_r}, \frac{\beta_s}{a_r}\right) = L(m, \kappa), \quad (5)$$

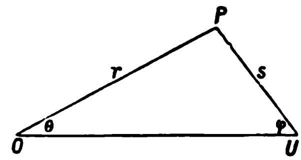
the constant C disappearing.

5. When the argument z of a L function is complex, we must define $2Lz$ as the value of the integral of

$$-\log(1-z)\,d\log z+\log z\,d\log(1-z),$$

taken along a path which starts from the origin, and where $\log(1-z)$ and $\log z$ have those values which correspond to integration along the same path. The principal value of Lz will correspond to any path connecting 0 and z for which $\log(1-z)$ and $\log z$ retain their principal values, *i.e.*, one in which z is never real unless it is a positive proper fraction.

If we take



$$z = re^{\theta i} \quad\text{and}\quad 1-z = se^{-\phi i},$$

$$\log z = \log r+\theta i \quad\text{and}\quad \log(1-z) = \log s-\phi i,$$

then the principal value will correspond to the case in which θ and ϕ both lie between π and $-\pi$, r and s being of course always positive.

We have now

$$2dLz = -(\log s - \phi i)\, d(\log r + \theta i) + (\log r + \theta i)\, d(\log s - \phi i)$$
$$= -\log s\, d\log r + \log r\, d\log s - \phi\, d\theta + \theta\, d\phi$$
$$+ i(\theta\, d\log s + \phi\, d\log r - \log s\, d\theta - \log r\, d\phi). \quad (1)$$

The real part of Lz will be written $R(\theta, \phi)$, while the coefficient of i in $2dLz$ is easily seen to be

$$\theta d\{\log\sin\theta - \log\sin(\theta+\phi)\} + \phi d\{\log\sin\phi - \log\sin(\theta+\phi)\}$$
$$-\{\log\sin\theta - \log\sin(\theta+\phi)\}\, d\theta + \{\log\sin\phi - \log\sin(\theta+\phi)\}\, d\phi$$
$$= (\theta\cot\theta - \log\sin\theta)\, d\theta + (\phi\cot\phi - \log\sin\phi)\, d\phi$$
$$-\{(\theta+\phi)\cot(\theta+\phi) - \log\sin(\theta+\phi)\}\, d(\theta+\phi).$$

Since the path of integration for Lz will always be taken as starting from 0, we see that initially $\phi = 0$; so that the coefficient of i in $L(re^{\theta i})$ is of the form

$$f(\theta) + f(\phi) - f(\theta+\phi), \quad (2)$$

where

$$f(\theta) = \tfrac{1}{2}\int_0 (\theta\cot\theta - \log\sin\theta)\, d\theta$$
$$= \int_0 \frac{\theta}{\tan\theta}\, d\theta - \tfrac{1}{2}\theta\log\sin\theta.$$

The symbol $T\theta$ will be used to denote the function $\int_0 \frac{\theta}{\tan\theta}\, d\theta$.

The formula § 1, (10) will lead, by consideration of imaginary terms, to sum-formulæ connecting T-functions in conjunction with such functions as $\theta\log\sin\theta$, but, by virtue of the extended form § 4, (5), it is not difficult to see that these latter functions will vanish identically for

$$2L(re^{\theta i}, re^{-\theta i})$$
$$= L(re^{\theta i}) - L(re^{-\theta i}) - \tfrac{1}{2}(\log r - \theta i)(\log s - \phi i) + \tfrac{1}{2}(\log s + \phi i)(\log r + \phi i)$$
$$= L(re^{\theta i}) - L(re^{-\theta i}) + i(\theta\log s + \phi\log r)$$
$$= 2i\{T\theta + T\phi - T(\theta+\phi)\}.$$

If, then, in § 4, (5) all pairs of arguments such as μ_s/λ_r and β_s/α_r, m and κ are conjugate complexes, we shall obtain a relation in which a sum of $3(n^2+1)$ T functions, involving n θ's and n ϕ's, is equated either to zero or (see the end of § 3) to a sum of logarithms. The case derived from § 1, (11) will be considered below *ab initio* in § 8, where it will be seen that the T sum will be equated to zero; and it is probable in all cases that the relation will be free from logarithmic terms.

If $r<1$, we may write

$$\begin{aligned}
d\{T\theta+T\phi-T(\theta+\phi)\} &= \theta\, d\log s+\phi\, d\log r \\
&= d(\theta\log s)-\log s\, d\theta+\phi\, d\log r \\
&= d(\theta\log s)-\tfrac{1}{2}\log(1-2r\cos\theta+r^2)\, d\theta \\
&\qquad +\frac{1}{r}\tan^{-1}\frac{r\sin\theta}{1-r\cos\theta}\, dr \\
&= d(\theta\log s)+\left(r\cos\theta+\frac{r^2}{2}\cos 2\theta+\dots\right)d\theta \\
&\qquad +\left(\sin\theta+\frac{r^2}{2}\sin 2\theta+\dots\right)dr \\
&= d(\theta\log s)+d\left(r\sin\theta+\frac{r^2}{2^2}\sin 2\theta+\frac{r^3}{3^2}\sin 3\theta+\dots\right).
\end{aligned}$$

Integrating from 0, where $s=1$ and $r=0$, we have

$$\begin{aligned}
T\theta+T\phi-T(\theta+\phi) &= r\sin\theta+\frac{r^2}{2^2}\sin 2\theta+\frac{r^3}{3^2}\sin 3\theta+\dots \\
&\qquad +\tfrac{1}{2}\theta\log(1-2r\cos\theta+r^2). \qquad (3)
\end{aligned}$$

If we treat r as constant, this right-hand side may be written in the form $\int_0 \frac{r\theta\sin\theta}{1-2r\cos\theta+r^2}\,d\theta$, but this integral will not represent the left-hand side when $r>1$. For the latter implies a path of integration from 0, and, if r is treated as a constant, we must first take the path along the axis of x as far as $(r, 0)$ before proceeding along the circle $x^2+y^2=r^2$. But, if $r>1$, this first path would pass through U and pass along the axis beyond it, which is not permissible.

However, if $r<1$, the ϕ-coordinate of $1/re^{\theta i}$ is easily seen to be $\pi-\theta-\phi$; so that the expression $T\theta+T\phi-T(\theta+\phi)$ becomes

$$T\theta+T(\pi-\theta-\phi)-T(\pi-\phi).$$

Now $\quad dT\theta+dT(\pi-\theta)=\left\{\frac{\theta}{\tan\theta}-\frac{\pi-\theta}{\tan(\pi-\theta)}\right\}d\theta=\frac{\pi}{\tan\theta}\,d\theta;$

therefore $\qquad T\theta+T(\pi-\theta)=C+\pi\log\sin\theta,$

where $\qquad C=2T(\tfrac{1}{2}\pi).$

But

$$T\tfrac{1}{2}\pi=\int_0^{\frac{1}{2}\pi}\theta\cot\theta\, d\theta=[\theta\log\sin\theta]_0^{\frac{1}{2}\pi}-\int_0^{\frac{1}{2}\pi}\log\sin\theta\, d\theta=-\int_0^{\frac{1}{2}\pi}\log\sin\theta\, d\theta,$$

which is known to be equal to $\tfrac{1}{2}\pi\log 2$.

Hence $$T\theta + T(\pi-\theta) = \pi \log(2\sin\theta), \tag{4}$$
and

$$T\theta + T(\pi-\phi-\theta) - T(\pi-\phi) = T\theta + T\phi - T(\phi+\theta) - \pi\log\frac{\sin\phi}{\sin(\phi+\theta)}$$
$$= T\theta + T\phi - T(\phi+\theta) - \pi\log r. \tag{5}$$

If, in (3), $r=1$, then $\phi = \frac{1}{2}(\pi-\theta)$.

But by differentiating it may be easily verified that

$$T\theta + T\{\tfrac{1}{2}(\pi-\theta)\} - T\{\tfrac{1}{2}(\pi+\theta)\} = 2T(\tfrac{1}{2}\theta);$$

so that, changing θ into 2θ, we have

$$T\theta = \theta\log(2\sin\theta) + \tfrac{1}{2}\left(\sin 2\theta + \frac{1}{2^2}\sin 4\theta + \frac{1}{3^2}\sin 6\theta + \dots\right). \tag{6}$$

Hence also

$$T\theta + T\phi - T(\theta+\phi) = \theta\log s + \phi\log r$$
$$+2\left\{\sin\theta\sin\phi\sin(\theta+\phi) + \frac{1}{2^2}\sin 2\theta\sin 2\phi\sin 2(\theta+\phi) + \dots\right\}. \tag{7}$$

In analogy to the above transformation, we may consider the difference of any two L functions in the form $L(\rho e^u) - L(\rho e^{-u})$, where u is positive and $\rho e^u < 1$.

If $1-\rho e^u = \sigma e^{-v}$ and $1-\rho e^{-u} = \sigma e^v$, we have $\rho = \sinh v/\sinh(u+v)$ and $\sigma = \sinh u/\sinh(u+v)$; and

$$2d[L(\rho e^u) - L(\rho e^{-u})] = -(\log\sigma - v)\,d(\log\rho + u) + (\log\rho + u)\,d(\log\sigma - v)$$
$$+(\log\sigma + v)\,d(\log\rho - u) - (\log\rho - u)\,d(\log\sigma + v)$$
$$= 2(v\,d\log\rho + u\,d\log\sigma - \log\sigma\,du - \log\rho\,dv);$$

therefore $$L(\rho e^u) - L(\rho e^{-u}) = f(u) + f(v) - f(u+v),$$

where $$f(u) = \int_0 (u\coth u - \log\sinh u)\,du = 2\int_0 \frac{u}{\tanh u}\,du - u\log\sinh u. \tag{8}$$

If in § 2, (2) we put $x = \rho e^u$, $y = e^{-2u}$, we get

$$L(\rho e^u) - L(\rho e^{-u}) = L1 - Le^{-2u} - Le^{-2v} + Le^{-2u-2v},$$

which may be easily identified with (8).

6. We may now establish the formula (*v.* Bertrand's *Calc. Int.*, § 271) connecting Lz with $L\{z/(z-1)\}$ and Lz with $L(1/z)$ where z is complex.

The four bipolar coordinates of $z/(z-1)$ corresponding to r, s, θ, ϕ are r/s, $1/s$, $\theta+\phi-\pi$, $-\pi$; therefore

$$2dL\left(\frac{z}{z-1}\right) = \log s\, d\log\frac{r}{s} - \log\frac{r}{s}\, d\log s + \phi d(\theta+\phi) - (\theta+\phi)d\phi + \pi\, d\phi$$

$$+ i\left\{-\phi\left(\frac{dr}{s}-\frac{ds}{s}\right) - (\theta+\phi-\pi)\frac{ds}{s} + \log s\, d(\theta+\phi) + \log\frac{r}{s}\, d\phi\right\}.$$

Hence $$2dLz + 2dL\left(\frac{z}{z-1}\right) = i\pi\frac{ds}{s} + \pi\, d\phi$$

by § 5, (1), and, integrating from $s = 1$, $\phi = 0$,

$$Lz + L\left(\frac{z}{z-1}\right) = \tfrac{1}{2}\pi i \log s + \tfrac{1}{2}\pi\phi. \tag{1}$$

Again, the four bipolar coordinates $r^{-1}e^{-\theta i}$ are $1/r$, s/r, $-\theta$, $\theta+\phi-\pi$; so that

$$2dL\left(\frac{1}{r}e^{\theta i}\right) = \log\frac{s}{r}\, d\log r - \log r\, d\log\frac{s}{r} \mp (\pi-\theta-\phi)d\theta - \theta d(\theta+\phi)$$

$$+ i\left\{-\theta d\log\frac{s}{r} + (\pi-\theta-\phi)\, d\log r + \log\frac{s}{r}\, d\theta + \log r\, d(\theta+\phi)\right\};$$

therefore $\quad 2dL(re^{\theta i}) + 2dL(r^{-1}e^{-\theta i}) = -\pi\, d\theta + \pi i\, d\log r$

by § 5, (1), and $\quad Lz + L(1/z) = C - \tfrac{1}{2}\pi\theta + \tfrac{1}{2}\pi i \log r$

when $z = 1$, $r = 1$, and $\theta = 0$; so that $C - 2L1 = \tfrac{1}{3}\pi^2$ and

$$Lz + L(1/z) = 2L1 - \tfrac{1}{2}\pi\theta + \tfrac{1}{2}\pi i \log r \tag{2}$$

(cf. Bertrand, *Calc. Int.*, § 272).

7. We have seen in § 2 that, if z and ω are complexes, we are not justified in assuming that

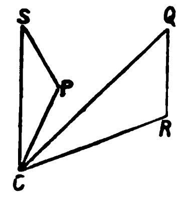
$$Lz + L\omega = L(z\omega) + L\left(z\frac{1-\omega}{1-z\omega}\right) + L\left(\omega\frac{1-z}{1-z\omega}\right),$$

but that the formula would have to be corrected by the introduction of logarithmic terms.

It is not easy to give a concise geometric meaning to the related complexes in this equation, but, by § 6, (1), we see that there can be a relation derived from the above connecting

$$Lz,\quad L\omega,\quad L\left\{\frac{z(1-\omega)}{z-1}\right\},\quad L\left\{\frac{\omega(1-z)}{\omega-1}\right\},\quad \text{and}\quad L(z\omega).$$

These complexes are connected with one another and with O, U in a manner which is worthy of notice.



If P, Q, R, S have complexes z_1, z_2, z_3, z_4, it is possible to find uniquely a point C (say, z_0) such that the triangle CPS, by rotation about C and enlargement (or diminution), may take the position CRQ; that is, there is a unique *centre of similitude* of SP and QR, which, moreover, is the centre of similitude of SQ and PR. The necessary and sufficient condition is that

$$(z_1-z_0)(z_2-z_0) = (z_3-z_0)(z_4-z_0),$$

i.e.,
$$z_0 = \frac{z_3z_4-z_1z_2}{z_3+z_4-z_1-z_2}.$$

Let $\quad z_1 = z, \quad z_2 = \omega, \quad z_3 = \dfrac{z(1-\omega)}{z-1}, \quad z_4 = \dfrac{\omega(1-z)}{\omega-1}.$

Then the vectors $\qquad OP.OQ = OR.OS;$

so that O is the centre of similitude of PS and RQ or of QS and RP. Now the vectors drawn from U are

$$1-z, \quad 1-\omega, \quad \frac{1-z\omega}{1-z}, \quad \frac{1-z\omega}{1-\omega};$$

so that $\qquad UP.UR = UQ.US,$

and U is the centre of similitude of PQ and SR or of PS and QR.

Lastly, if V is the point whose complex is $z\omega$, the vectors drawn to V from P, Q, R, S are

$$z(1-\omega), \quad \omega(1-z), \quad \frac{z(1-\omega z)}{1-z}, \quad \frac{\omega(1-\omega z)}{1-\omega};$$

so that $\qquad VP.VS = VQ.VR$

and V is the centre of similitude of PR and QS or of PQ and RS. Hence O, U, V are the three centres of similitude of pairs of sides of the quadrangle $PQRS$.

It may be observed that the *shape* of $PQRS$ is quite general.

8. In establishing a function-sum theorem for the T-function of § 5 the most convenient method will be based on the formula in § 2 where the five arguments are expressed in the cyclic form

$$z, \quad \omega, \quad \frac{1-z}{1-z\omega}, \quad 1-z\omega, \quad \frac{1-\omega}{1-z\omega},$$

after replacing x and y by z and ω.

If (θ_1, ϕ_1), (θ_2, ϕ_2), ..., (θ_5, ϕ_5) are the bipolar angular coordinates of the points represented by these five arguments, we see immediately that

$$\phi_4 = -\theta_1-\theta_2,$$

and hence, in virtue of the cyclic property, the five points are

$$(\theta_1, -\theta_3-\theta_4), \quad (\theta_2, -\theta_4-\theta_5), \quad (\theta_3, -\theta_5-\theta_1), \quad \ldots .$$

There is, however, one relation connecting the five θ's, which will necessarily be cyclic. Taking $z_1, z_2, \ldots$ as representing the five complexes and $r_1, s_1, r_2, s_2, \ldots$ as their bivectorial coordinates, we have $z_3 z_4 = 1-z$; so that $r_3 r_4 = s_1$. But in all cases

$$r = \sin\phi/\sin(\phi+\theta) \quad \text{and} \quad s = \sin\theta/\sin(\phi+\theta);$$

so that $\sin\phi_3 \sin\phi_4 \sin(\phi_1+\theta_1) = \sin(\phi_3+\theta_3)\sin(\phi_4+\theta_4)\sin\theta_1$,

i.e., by substituting for the ϕ's

$$\sin(\theta_1+\theta_5)\sin(\theta_1+\theta_2)\sin(\theta_3+\theta_4-\theta_1)$$
$$+\sin(\theta_1+\theta_5-\theta_3)\sin(\theta_1+\theta_2-\theta_4)\sin\theta_1 = 0. \quad (2)$$

This is the relation required, but does not appear here in the requisite cyclic form. To attain this end we may write $a_1, a_2, \ldots$ for $e^{2\theta_1 i}, e^{2\theta_2 i}, \ldots$ and put $z_1 = r_1 e^{\theta_1 i}$ in the form

$$\sin(\theta_3+\theta_4)e^{\theta_1 i}/\sin(\theta_3+\theta_4-\theta_1) = a_1(a_3 a_4-1)/(a_3 a_4-a_1).$$

With similar transformations for $z_2, z_3, \ldots$, we get

$$1-z_1 z_3-z_4 = -\frac{a_1 a_3 a_4}{d_1 d_3 d_4} S^*, \quad (3)$$

where $$d_1 = a_3 a_4 - a_1, \ \ldots, \ \&\text{c.},$$

and $$S = a_1 a_2 a_3 a_4 a_5 - a_1 a_2 a_3 - a_2 a_3 a_4 - a_3 a_4 a_5 - a_4 a_5 a_1 - a_5 a_1 a_2$$
$$+a_4 a_5 + a_5 a_1 + a_1 a_2 + a_2 a_3 + a_3 a_4 - 1, \quad (4)$$

i.e., $\sin(\theta_1+\theta_2+\theta_3+\theta_4+\theta_5)$

$$= \sin(\theta_1+\theta_2+\theta_3-\theta_4-\theta_5) + \text{four other terms cyclically derived.} \quad (5)$$

We may now consider the function sum relation connecting the fifteen T functions corresponding to these five complexes.

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* It is interesting to note that, if the θ's are independent, so that $S \not\equiv 0$, the solution of $a_1, a_2, \ldots$ from the equations $z_1 = a_1(a_3 a_4-1)/(a_3 a_4-a_1), \ldots$ may be effected. Writing u_1 for $1-z_1-z_3 z_4, \ldots$ and v_1 for $1-z_5-z_1-z_2+z_5 z_1 z_2, \ldots$, we get

$$u_1 = -a_1 a_3 a_4 S/d_1 d_3 d_4 \quad \text{and} \quad v_1 = a_2 a_3 a_4 a_5 S/d_5 d_1 d_2.$$

From these two results we get $a_1 u_1 v_1/v_2 v_5 =$ cyclic analogues and $d_1 u_1 v_1 u_3 u_4/v_2 v_5 =$ cyclic analogues, whence, by the relation $a_1 z_5 d_5 - a_5 d_1 = a_3 a_5 d_4$, we get $a_1 = -v_2 v_5/u_1 v_1, \ldots$, thereby obtaining the solution of the a's in terms of the z's. It is remarkable also that $(1-a_1)u_1 v_1 = u_1 v_1 + v_2 v_5 = (1-z_1)T$, where T is the cyclic expression

$$2-z_1-z_2-z_3-z_4-z_5+z_1 z_2 z_3+z_2 z_3 z_4+z_3 z_4 z_5+z_4 z_5 z_1+z_5 z_1 z_2+z_1 z_2 z_3-z_1 z_2 z_3 z_4 z_5.$$

If $T = 0$, $a_1 = 1 = a_2 = a_3 = a_4 = a_5$; so that the θ's are all zero and the z's all real.

In the differential

$$d\{T\theta_1+T\phi_1-T(\theta_1+\phi_1)+\dots\},$$
$$= d\{T\theta_1-T(\theta_3+\theta_4)+T(\theta_3+\theta_4-\theta_1)+\dots\},$$
$$= \theta_1\cot\theta_1\,d\theta_1-(\theta_3+\theta_4)\cot(\theta_3+\theta_4)\,d(\theta_3+\theta_4)$$
$$+(\theta_3+\theta_4-\theta_1)\cot(\theta_3+\theta_4-\theta_1)\,d(\theta_3+\theta_4-\theta_1)+\dots$$

the coefficient of θ_1 is

$$\cot\theta_1\,d\theta_1-\cot(\theta_1+\theta_2)\,d(\theta_1+\theta_2)-\cot(\theta_1+\theta_5)\,d(\theta_1+\theta_5)$$
$$-\cot(\theta_3+\theta_4-\theta_1)\,d(\theta_3+\theta_4-\theta_1)+\cot(\theta_1+\theta_2-\theta_4)\,d(\theta_1+\theta_2-\theta_4)$$
$$+\cot(\theta_1+\theta_5-\theta_3)\,d(\theta_1+\theta_5-\theta_3),$$
$$= d\log\frac{\sin\theta_1\sin(\theta_1+\theta_2+\theta_4)\sin(\theta_1+\theta_5-\theta_3)}{\sin(\theta_1+\theta_2)\sin(\theta_1+\theta_5)\sin(\theta_3+\theta_4-\theta_1)},$$
$$= 0 \text{ by (2).}$$

Similarly, the coefficients of θ_2, θ_3, θ_4, θ_5 are zero, and the differential of the sum vanishes. Integrating then from the origin where all the θ's are zero, we see that the sum of the fifteen T functions is zero.

This relation between the θ's, as is also (4), is cyclic, but not symmetrical. If, however, we substitute the ϕ's in the relations, they will both become symmetrical, as may be seen as follows.

Since $$\phi_1 = -\theta_3-\theta_4, \ \dots,$$

we have $\sigma = -(\theta_1+\theta_2+\theta_3+\theta_4+\theta_5) = \tfrac{1}{2}(\phi_1+\phi_2+\phi_3+\phi_4+\phi_5)$;

so that (4) becomes

$$\sin\sigma = \sin(\sigma-2\phi_1)+\sin(\sigma-2\phi_2)+\sin(\sigma-2\phi_3)+\sin(\sigma-2\phi_4)$$
$$+\sin(\sigma-2\phi_5), \qquad (6)$$

while, since

$$\theta_1 = -\sigma+\phi_2+\phi_5 \quad\text{and}\quad \theta_1+\phi_1 = -\sigma+\phi_1+\phi_2+\phi_5 = \sigma-\phi_3-\phi_4,$$

we have
$$T\phi_1-T(\sigma-\phi_2-\phi_5)-T(\sigma-\phi_3-\phi_4)$$
$$+T\phi_2-T(\sigma-\phi_3-\phi_1)-T(\sigma-\phi_4-\phi_5)$$
$$+\dots = 0, \qquad (7)$$

which is symmetric in the five angles ϕ_1, ϕ_2, ϕ_3, ϕ_4, ϕ_5.

Referring the series from § 5, (6) for $T\theta$, we see that the above

formula holds good, even if we understand by $T\theta$ the series

$$\sin 2\theta + \frac{1}{2^2}\sin 4\theta + \frac{1}{3^2}\sin 6\theta + \ldots,$$

since it is obvious that the logarithmic sum vanishes identically.

If $\phi_5 = 0$, the equation (7) becomes nugatory; for in this case

$$\sin(\sigma-\phi_1-\phi_2)\cos(\phi_1-\phi_2)+\sin(\sigma-\phi_3-\phi_4)\cos(\phi_3-\phi_4) = 0,$$

i.e., $\sin\frac{1}{2}(\phi_1+\phi_2-\phi_3-\phi_4)\sin\frac{1}{2}(\phi_1-\phi_2+\phi_3-\phi_4)\sin\frac{1}{2}(\phi_1-\phi_2-\phi_3+\phi_4) = 0$;

so that the sum of two of the remaining angles equals the sum of the other two. This makes the function sum in (7) vanish identically.

Moreover, it is impossible that four of the ϕ's should be equal and real. For, if $\phi_1 = \phi_2 = \phi_3 = \phi_4$, then

$$\sin\sigma - \sin(\sigma-2\phi_5) = 4\sin(\sigma-2\phi_1) = 4\sin\tfrac{1}{2}\phi_5,$$

$$\sin\phi_5\cos(\sigma-\phi_5) = 2\sin\tfrac{1}{2}\phi_5;$$

therefore $$\cos\tfrac{1}{2}\phi_5\cos(\sigma-\phi_5) = 1, \tag{8}$$

which is impossible, unless ϕ_5 is imaginary.

Let $\phi_4 = \phi_5$, and call each of these ϕ. Then, if $2s = \phi_1+\phi_2+\phi_3$, so that $2\sigma = 2s+2\phi$, we have

$$\sin(s+\phi) = \sin(s+\phi-2\phi_1)+\sin(s+\phi-2\phi_2)+\sin(s+\phi-2\phi_3) + 2\sin(s-\phi),$$

whence $$\tan\phi = \frac{\sin(s-2\phi_1)+\sin(s-2\phi_2)+\sin(s-2\phi_3)+\sin s}{3\cos s-\cos(s-2\phi_1)-\cos(s-2\phi_2)-\cos(s-2\phi_3)}.$$

If $\phi_1 = \phi_2 = \phi_3 = \omega$, we have

$$\tan\phi = \frac{\sin\frac{3}{2}\omega - 3\sin\frac{1}{2}\omega}{3\cos\frac{3}{2}\omega - 3\cos\frac{1}{2}\omega} = \tfrac{1}{3}\tan\tfrac{1}{2}\omega,$$

while $$3T\omega+2T\phi-3T(\phi-\tfrac{1}{2}\omega)-T(\tfrac{3}{2}\omega-\phi)-6T(\tfrac{1}{2}\omega) = 0.$$

It is possible therefore that, if the ϕ's are all real, three of them should be equal.

Again, let $\phi_2 = \phi_3 = \phi$, say; $\phi_4 = \phi_5 = \omega$; and $\phi_1 = \psi$, so that $\sigma = \psi+\phi+\omega$. Now (2) may be written

$$\sin(\sigma-\phi_1-\phi_2)\sin(\sigma-\phi_1-\phi_3)\sin(\sigma-\phi_2-\phi_3) + \sin\phi_3\sin\phi_4\sin(\sigma-\phi_3-\phi_4) = 0,$$

which now becomes

$$\sin\psi\sin(\phi-\psi)\sin(\omega-\psi)+\sin\phi\sin\omega\sin\psi = 0.$$

Rejecting $\sin\psi = 0$, we have

$$\sin(\phi-\psi)\sin(\psi-\omega) = \sin\phi\sin\omega,$$

while the T function relation becomes

$$2T\phi+2T\omega+T(2\psi)-T(\phi-\omega+\psi)-T(\omega-\phi+\psi)$$
$$-4T\psi-2T(\phi-\psi)-2T(\omega-\psi) = 0.$$

It may be observed that

$$4T\psi-T(2\psi) = \int_0 4\psi\left(\frac{1}{\tan\psi}-\frac{1}{\tan 2\psi}\right)d\psi = \int_0 \frac{4\psi\, d\psi}{\sin 2\psi}$$
$$= 4\psi\log(2\sin\psi)-2\psi\log(2\sin 2\psi)$$
$$+2\left(\sin\psi+\frac{1}{2^2}\sin 2\psi+\dots\right)$$
$$-\tfrac{1}{2}\left(\sin 2\psi+\frac{1}{2^2}\sin 4\psi+\dots\right)$$
$$= 2\psi\log\tan\psi+2\left(\sin\psi+\frac{1}{3^2}\sin 3\psi+\frac{1}{5^2}\sin 5\psi+\dots\right);$$

so that

$$\int_0 \frac{\omega}{\sin\omega}\,d\omega = \omega\log\tan\frac{\omega}{2}+2\left(\sin\frac{\omega}{2}+\frac{1}{3^2}\sin\frac{3\omega}{2}+\frac{1}{5^2}\sin\frac{5\omega}{2}+\dots\right).$$

Returning to the case in which $\phi_1 = \phi_2 = \phi_3 = \phi_4$, we have seen in (8) that

$$\cos\tfrac{1}{2}\phi_5\cos(2\phi_1-\tfrac{1}{2}\phi_5) = 1,$$

where we will suppose that $\tfrac{1}{2}\phi_5 = ui$ and $2\phi_1-\tfrac{1}{2}\phi_5 = \theta$; so that

$$\cosh u = \sec\theta.$$

The T equation becomes

$$4T\phi_1+T\phi_5-6T(\tfrac{1}{2}\phi_5)-4T(\phi_1-\tfrac{1}{2}\phi_5) = 0,$$

i.e., $$4\left[T\left(\frac{\theta+ui}{2}\right)-T\left(\frac{\theta-ui}{2}\right)\right] = -T(2ui)+6T(ui),$$

or, if we had put $\tfrac{1}{2}\phi_5 = \theta$, $2\phi_1-\tfrac{1}{2}\phi_5 = ui$, so that again $\cosh u = \sec\theta$, we should have

$$4\left[T\left(\frac{\theta+ui}{2}\right)+T\left(\frac{\theta-ui}{2}\right)\right] = -T(2\theta)+6T\theta.$$

10. It is shewn by Bertrand (*Calc. Int.*, § 271) that for certain values of the argument x the values of ψx may be determined. The notation of the present memoir considerably simplifies the results, which may be obtained as follows.

In § 1, (2) let $x = \frac{1}{2}$; then

$$L\tfrac{1}{2} = \tfrac{1}{2}L1 = \tfrac{1}{12}\pi^2.$$

In § 1, (12) let $x^2 = 2-x$, so that $x = \frac{1}{2}(\sqrt{5}-1)$; then

$$\frac{x}{1+x} = x^2 \quad \text{and} \quad 2Lx = 3L(x^2),$$

while, by § 1, (2), $$Lx + L(x^2) = L1.$$

Hence $$Lx = L\left(\frac{\sqrt{5}-1}{2}\right) = \tfrac{3}{5}L1 = \frac{\pi^2}{10},$$

while $$L(x^2) = L\left(\frac{3-\sqrt{5}}{2}\right) = \tfrac{2}{5}L1 = \frac{\pi^2}{15}.$$

Apart from these cases, it does not seem possible to obtain a special value of Lx for any real or complex argument.

ON SOMMERFELD'S DIFFRACTION PROBLEM; AND ON REFLECTION BY A PARABOLIC MIRROR

By HORACE LAMB.

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THE exact solution of a problem of diffraction is a matter of interest to many who do not feel at home in the refined mathematical theories on which Sommerfeld, in his well known paper,* has drawn with such effect. It may therefore be worth while to shew how a particular case of the problem discussed by him, viz., the case of normal incidence, may be treated by quite simple methods. Moreover, when the typical solution of the differential equation is once obtained, it can easily be varied and combined, as Sommerfeld has shewn, so as to represent the circumstances of oblique incidence.

The analysis here employed leads also to a solution of the problem of reflection by a cylindrical parabolic mirror, the incident waves, aerial or electric, being plane, with their fronts perpendicular to the axis of the parabolic section. The solution is so simple that it will be a matter of surprise if it has not been already noticed; it would seem indeed to be almost immediately indicated by the graphical representation which Sommerfeld has given of his typical function. Although the results are not altogether so representative as could be wished, owing to the special properties of the parabolic form, they may perhaps possess some degree of interest. Problems of reflection by curved surfaces have hardly been discussed hitherto, on the basis of the wave-theory, except by approximate methods. In the present case the solutions are exact, and the comparison with the results of the geometrical theory may be instructive, notwithstanding the limitation to the special form in question.

It was natural to attempt to extend the methods to the case where the reflecting surface is a paraboloid of revolution, the incident waves travelling as before in the direction of the axis. When the waves are aerial the solution is readily obtained. The case of electric waves would seem to be more difficult.

* "Mathematische Theorie der Diffraction," *Math. Ann.*, Vol. XLVII., p. 317 (1895).

Sommerfeld's Problem.

1. In the acoustical form of the problem, plane waves are supposed to npinge on a rigid semi-infinite screen. We will suppose that the screen ccupies that portion of the plane xz for which x is positive; and we will egin with the case where the incidence is normal, the primary waves eing of the type

$$\phi = e^{iky}, \tag{1}$$

le time-factor e^{ikct}, where c is the velocity of sound, being understood. he complete solution will then be of the form

$$\phi = e^{iky} + \psi, \tag{2}$$

'here ψ is the velocity-potential due to an oscillation of the screen ormal to its plane, with the amplitude and period of the incident waves, ut with exact opposition of phase, so that

$$\frac{\partial \psi}{\partial y} = -ik, \text{ for } y = 0, \ x > 0. \tag{3}$$

'he function ψ must satisfy the differential equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + k^2 \psi = 0, \tag{4}$$

r, in polar coordinates,

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} + k^2 \psi = 0, \tag{5}$$

ince the problem is virtually a two-dimensional one. Moreover, it is vident that ψ is anti-symmetrical with respect to the axis of x, and there- ore that along the negative portion of this axis we must have $\psi = 0$.

If, now, we write for a moment

$$\chi = \frac{\partial \psi}{\partial x}, \tag{6}$$

he function χ must also satisfy (4); it must vanish over the negative ortion of the axis of x, and its normal derivative $\partial \chi / \partial y$ must vanish ver the positive portion. This suggests that, as regards the region or which $y > 0$, the value of χ will be that due to a singular point f proper character at the origin, singularities at infinity being (as s easily seen) excluded by the circumstances of the physical problem. 'he required form of χ is at once indicated by our knowledge of the olutions available in the case of an incompressible fluid, when $k = 0$. Ve have then, in polar coordinates, the forms

$$r^{\frac{1}{2}} \cos \tfrac{1}{2}\theta, \qquad r^{-\frac{1}{2}} \cos \tfrac{1}{2}\theta,$$

where the range of θ is from 0 to π. The proper generalizations of these are obviously

$$J_{\frac{1}{2}}(kr)\cos\tfrac{1}{2}\theta, \qquad J_{-\frac{1}{2}}(kr)\cos\tfrac{1}{2}\theta,$$

or, omitting numerical factors,

$$\frac{\sin kr}{\sqrt{(kr)}}\cos\tfrac{1}{2}\theta, \qquad \frac{\cos kr}{\sqrt{(kr)}}\cos\tfrac{1}{2}\theta.$$

Excluding sources at infinity, and taking therefore the combination of these solutions which is appropriate to the expression of diverging waves, we may assume, at all events tentatively,

$$\frac{\partial\psi}{\partial x} = C\frac{e^{-ikr}}{\sqrt{(kr)}}\cos\tfrac{1}{2}\theta. \tag{7}$$

The integration of this partial differential equation of the first order is straightforward, and may be performed either in terms of polar coordinates, or in terms of the "parabolic" coordinates of Hankel and others. If we write

$$k(x+iy) = (\xi+i\eta)^2, \tag{8}$$

or

$$kx = \xi^2-\eta^2, \qquad ky = 2\xi\eta, \tag{9}$$

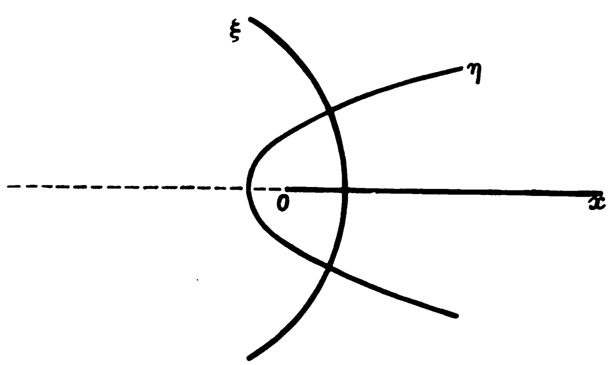
we have

$$\xi = (kr)^{\frac{1}{2}}\cos\tfrac{1}{2}\theta, \qquad \eta = (kr)^{\frac{1}{2}}\sin\tfrac{1}{2}\theta, \tag{10}$$

where (we will suppose) the range of θ is from 0 to 2π. The curves

$$\xi = \text{const.}, \qquad \eta = \text{const.}$$

form a system of confocal parabolas, the common focus being at the origin. The coordinate η is everywhere positive, and the line $\eta = 0$



represents the section of the screen. The coordinate ξ has opposite signs on the two sides of the axis of x, and the line $\xi = 0$ represents the free portion of this axis.

We easily find
$$\left.\begin{aligned}\frac{\partial\xi}{\partial x} &= \tfrac{1}{2}\frac{\xi}{r}, & \frac{\partial\eta}{\partial x} &= -\tfrac{1}{2}\frac{\eta}{r}\\ \frac{\partial\xi}{\partial y} &= \tfrac{1}{2}\frac{\eta}{r}, & \frac{\partial\eta}{\partial y} &= \ \ \tfrac{1}{2}\frac{\xi}{r}\end{aligned}\right\}, \tag{11}$$
and the equation (7) becomes
$$\xi\frac{\partial\psi}{\partial\xi} - \eta\frac{\partial\psi}{\partial\eta} = \frac{2C}{k}\xi e^{-i(\xi^2+\eta^2)}. \tag{12}$$

The subsidiary equations in Lagrange's method are then
$$\frac{d\xi}{\xi} = \frac{d\eta}{-\eta} = \frac{k\,d\psi}{2C\xi e^{-i(\xi^2+\eta^2)}}. \tag{13}$$
One integral is
$$\xi\eta = a, \tag{14}$$
where a is an arbitrary constant; and with the help of this we find
$$\begin{aligned}d\psi &= \frac{2C}{k}e^{-i(\xi^2+a^2/\xi^2)}d\xi\\ &= \frac{C}{k}\left\{e^{-i(\xi+a/\xi)^2+2ia}\,d\left(\xi+\frac{a}{\xi}\right)+e^{-i(\xi-a/\xi)^2-2ia}\,d\left(\xi-\frac{a}{\xi}\right)\right\},\end{aligned} \tag{15}$$
whence
$$\psi = \frac{C}{k}\left\{e^{2ia}\int^{\xi+a/\xi}e^{-i\zeta^2}d\zeta+e^{-2ia}\int^{\xi-a/\xi}e^{-i\zeta^2}d\zeta\right\}+b, \tag{16}$$
where b is the second arbitrary constant. The solution of (12) is therefore
$$\psi = \frac{C}{k}\left\{e^{iky}\int_0^{\xi+\eta}e^{-i\zeta^2}d\zeta+e^{-iky}\int_0^{\xi-\eta}e^{-i\zeta^2}d\zeta\right\}+F(y), \tag{17}$$
where $F(y)$ is an arbitrary function of y.

From the manner in which it has been obtained it appears that this value of ψ must satisfy
$$\frac{\partial}{\partial x}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+k^2\psi\right) = 0, \tag{18}$$
and so render the expression on the left-hand side of (4) a function of y only. Again, it is seen from (9) that when x is large, whilst y is finite, the first part of (17) reduces to a sum of constant multiples of e^{iky} and e^{-iky}. Hence, to secure that (4) shall be satisfied, it is only necessary to put
$$F(y) = Ae^{iky}+Be^{-iky}. \tag{19}$$
This may be verified independently by means of the equation
$$\frac{\partial^2\psi}{\partial\xi^2}+\frac{\partial^2\psi}{\partial\eta^2}+4(\xi^2+\eta^2)\psi = 0, \tag{20}$$
which is equivalent to (4).

It remains to determine the constants A, B, C. For $y = 0$, $x > 0$, we must have $\partial\psi/\partial y = -ik$. Making use of (11), and putting $\eta = 0$, we find

$$A - B = -1. \tag{21}$$

Again, when x is large and negative, whilst y is finite, ψ must tend to the limit 0, the primary waves (1) being alone sensible in this region. Hence, putting $\xi + \eta = \infty$, $\xi - \eta = -\infty$, and remembering that

$$\int_0^\infty \cos \zeta^2 \, d\zeta = \int_0^\infty \sin \zeta^2 \, d\zeta = \frac{\sqrt{\pi}}{2\sqrt{2}}, \tag{22}$$

we obtain $$\frac{(1-i)\sqrt{\pi}C}{2\sqrt{2k}} + A = 0, \qquad -\frac{(1-i)\sqrt{\pi}C}{2\sqrt{2k}} + B = 0. \tag{23}$$

Hence $$A = -\tfrac{1}{2}, \qquad B = \tfrac{1}{2}, \qquad C = \frac{(1+i)\,k}{\sqrt{(2\pi)}}; \tag{24}$$

and the complete solution of our problem is given by

$$\psi = \frac{1+i}{\sqrt{(2\pi)}} \left\{ e^{iky} \int_0^{\xi+\eta} e^{-i\zeta^2} d\zeta + e^{-iky} \int_0^{\xi-\eta} e^{-i\zeta^2} d\zeta \right\} - \tfrac{1}{2}(e^{iky} - e^{-iky}). \tag{25}$$

This, in conjunction with (2), is equivalent to the form which Sommerfeld's result* assumes in the case of normal incidence.

The value of ψ at a point $(x, -y)$ must be equal in magnitude and opposite in sign to that which obtains at the point (x, y). Since the expression in (25) is an odd function of ξ, it will apply to the region for which $y < 0$ without modification.

2. The types of solution of the equation (4) to which we have been led might have been obtained in a different manner. It was natural to anticipate that all parts of the field where the ratio y/x is not great would be occupied by waves whose fronts are perpendicular to the axis of y. This suggests a search for solutions of (4) of the types

$$e^{iky} u, \qquad e^{-iky} v. \tag{26}$$

Taking the first of these, we have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 2ik \frac{\partial u}{\partial y} = 0, \tag{27}$$

or, in terms of the parabolic coordinates ξ, η,

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + 4i\left(\eta \frac{\partial u}{\partial \xi} + \xi \frac{\partial u}{\partial \eta}\right) = 0. \tag{28}$$

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* Cf. Carslaw, *Proc. London Math. Soc.*, Vol. xxx., p. 134 (1898).

This is satisfied by
$$u = f(\xi+\eta) = f(\zeta), \tag{29}$$
say, provided
$$\frac{d^2f}{d\zeta^2}+2i\zeta\frac{df}{d\zeta} = 0, \tag{30}$$
i.e., provided
$$u = A+B\int_0^{\xi+\eta} e^{-i\zeta^2}\,d\zeta. \tag{31}$$
Similarly, we obtain the solution
$$u' = A'+B'\int_0^{\xi-\eta} e^{i\zeta^2}\,d\zeta. \tag{32}$$

In like manner, taking the second form in (26), we have the solutions
$$v = C+D\int_0^{\xi+\eta} e^{i\zeta^2}\,d\zeta, \tag{33}$$
$$v' = C'+D'\int_0^{\xi-\eta} e^{-i\zeta^2}\,d\zeta. \tag{34}$$

These results may be combined as above so as to satisfy all the conditions of the problem.

3. We have so far confined ourselves to the acoustical form of the question. It is known, however, that the same analysis applies to the case of electric waves polarized *perpendicularly* to the plane xy, provided the screen be regarded as perfectly conducting. In this application the symbol ϕ denotes the *magnetic* force, which is everywhere parallel to the axis of z.

When the electric waves are polarized *in* the plane xy, the surface conditions are altered. If ϕ now denote the *electric* force, which is everywhere parallel to z, the function ψ in (2) must be such that $\psi = -1$ for $y = 0$, $x > 0$, and $\partial\psi/\partial y = 0$ for $y = 0$, $x < 0$. This suggests the assumption
$$\frac{\partial\psi}{\partial x} = C\frac{e^{-ikr}}{\sqrt{(kr)}}\sin\tfrac{1}{2}\theta. \tag{35}$$

The remaining steps are analogous to those of the preceding investigation; and the final result is
$$\psi = \frac{1+i}{\sqrt{(2\pi)}}\left\{e^{iky}\int_0^{\xi+\eta} e^{-i\zeta^2}\,d\zeta - e^{-iky}\int_0^{\xi-\eta} e^{-i\zeta^2}\,d\zeta\right\} - \tfrac{1}{2}(e^{iky}+e^{-iky}). \tag{36}$$
This is easily verified.

4. It must be admitted that the procedure of § 1 becomes unsuitable when the incident waves are oblique to the screen. But we can readily

adapt the solutions of (4) there obtained so as to satisfy the altered conditions. Thus, assuming

$$\phi = e^{ik(x\sin\alpha+y\cos\alpha)}+\psi, \tag{37}$$

where the first term represents a train of incident waves coming from the direction $\theta = \frac{1}{2}\pi-\alpha$, we have the types

$$\psi_1 = Ce^{ik(x\sin\alpha+y\cos\alpha)}\int_0^{\xi+\eta'} e^{-i\zeta^2}d\zeta, \tag{38}$$

$$\psi_2 = Ce^{ik(x\sin\alpha-y\cos\alpha)}\int_0^{\xi'-\eta'} e^{-i\zeta^2}d\zeta, \tag{39}$$

provided $$k(x+iy)e^{i\alpha} = (\xi+i\eta)^2, \qquad k(x+iy)e^{-i\alpha} = (\xi'+i\eta')^2, \tag{40}$$

or $$\left.\begin{aligned} \xi &= (kr)^{\frac{1}{2}}\cos\tfrac{1}{2}(\theta+\alpha), & \eta &= (kr)^{\frac{1}{2}}\sin\tfrac{1}{2}(\theta+\alpha),\\ \xi' &= (kr)^{\frac{1}{2}}\cos\tfrac{1}{2}(\theta-\alpha), & \eta' &= (kr)^{\frac{1}{2}}\sin\tfrac{1}{2}(\theta-\alpha). \end{aligned}\right\} \tag{41}$$

If we combine these in the manner suggested by the previous investigation, we find, for $\theta=\pi$, $\psi_1+\psi_2=0$, and, for $\theta=0$, $\dfrac{\partial(\psi_1+\psi_2)}{\partial y}=0$. Hence the surface-conditions of the acoustical problem are satisfied by

$$\psi = \psi_1+\psi_2-\tfrac{1}{2}\left(e^{ik(x\sin\alpha+y\cos\alpha)}-e^{ik(x\sin\alpha-y\cos\alpha)}\right). \tag{42}$$

Moreover, when θ lies between $\frac{1}{2}\pi+\alpha$ and π, and r is infinite, the definite integrals in (38) and (39) become equal to $\frac{1}{2}(1-i)\sqrt{(\frac{1}{2}\pi)}$. Since the primary waves must alone be sensible in this region, we find

$$C = \frac{1+i}{\sqrt{(2\pi)}}. \tag{43}$$

The solution is thus completed, and will be found to agree with that given by Sommerfeld.

As before, the analysis applies also to the case of electric waves incident on a perfectly conducting screen, when the plane of polarization is perpendicular to that of incidence. For waves polarized in the plane of incidence we should find

$$\psi = \psi_1-\psi_2-\tfrac{1}{2}\left(e^{ik(x\sin\alpha+y\cos\alpha)}+e^{ik(x\sin\alpha-y\cos\alpha)}\right), \tag{44}$$

with C determined by (43) as before.

5. Perhaps the chief interest of solutions such as those above investigated lies in the opportunity which they afford of comparison with Fresnel's more empirical methods of calculation. Detailed comparisons have been made by Sommerfeld and also by Drude,* who has employed

* *Lehrbuch der Optik*, Leipzig, 1900, p. 192.

with effect the method of Cornu's spirals. It is easy to shew independently that the distribution of intensity must under the usual optical conditions* be sensibly the same as that obtained on Fresnel's hypotheses.

The value of ϕ at any point P on either side of the plane $y = 0$, in terms of the values of ϕ or of its normal derivative $\partial\phi/\partial n$, at points of this plane, is given accurately by either of the formulæ

$$\phi_P = -2\int_{-\infty}^{\infty} D_0(kR)\frac{\partial\phi}{\partial n}\,dx, \tag{45}$$

$$\phi_P = 2\int_{-\infty}^{\infty} \phi\frac{\partial}{\partial n}D_0(kR)\,dx, \tag{46}$$

where R denotes distances of P from the points $(x, 0)$ at which the value of ϕ or $\partial\phi/\partial n$ is given; and the element (δn) of the normal is supposed drawn towards the side of the axis of x on which P lies. The function D_0 represents the velocity-potential due to a "simple" line-source of unit strength, viz.,

$$D_0(kR) = \frac{1}{2\pi}\int_0^{\infty} e^{-ikR\cosh u}\,du. \tag{47}$$

For large values of R this tends to the asymptotic form

$$D_0(kR) = \frac{1}{\sqrt{(8\pi)}}\frac{e^{-i(kR+\frac{1}{4}\pi)}}{\sqrt{(kR)}}. \tag{48}$$

It may be sufficient to take the case of normal incidence, with the boundary conditions of § 1. On Fresnel's plan we should assume in the first place that over the free portion of the plane $y = 0$ the velocity-potential ϕ, or the normal velocity $\partial\phi/\partial y$, has the valu3 proper to the primary waves (1), whilst on the negative face of the screen both these functions vanish. Now it appears from (2) and (25) that for $y = 0$ we have

$$\phi_0 = 1 \quad [x < 0], \qquad \phi_0 = (1+i)\sqrt{\frac{2}{\pi}}\int_{\sqrt{(kx)}}^{\infty} e^{-i\zeta^2}\,d\zeta \quad [x > 0]. \tag{49}$$

The latter expression refers, of course, to the negative side of the screen; it fluctuates rapidly in phase, and becomes insensible when x exceeds a few wave-lengths, its asymptotic value being

$$\phi_0 = \frac{1-i}{\sqrt{(2\pi)}}\frac{e^{-ikx}}{\sqrt{(kx)}}. \tag{50}$$

For distances R which are large compared with the wave-length, and for points in the neighbourhood of the edge of the geometric shadow, the

* Cases of considerable obliquity, as regards the direction of the incident or that of the observed waves, are here excluded.

corresponding elements in the integral (46) will contribute nothing appreciable to the final result.

If we employ the alternative formula (45), we require the value of $\partial\phi/\partial n$ for $y = 0$, $x < 0$. This is found to be

$$-\left(\frac{\partial\phi}{\partial y}\right)_0 = -ik + \frac{(1+i)k}{\sqrt{(2\pi)}}\left\{2i\int_{\sqrt{(kx')}}^{\infty} e^{-i\zeta^2} d\zeta - \frac{e^{-ikx'}}{\sqrt{(kx')}}\right\}, \tag{51}$$

where x' has been written for $-x$. The asymptotic value of this is $-ik$, and the error involved in taking this as the general value is, in the circumstances supposed, insignificant.

Fresnel, of course,* did not employ the formulæ (45), (46), which are of much more recent date; but his methods give for most purposes a sufficient evaluation of the integrals, on the hypothesis (here justified) that ϕ and $\partial\phi/\partial y$ may be taken to have their simplified values in the plane $y = 0$.

Reflection by a Cylindric Parabolic Surface.

6. If in § 2 we had assumed for ϕ the forms

$$e^{ikx}u, \quad e^{-ikx}v, \tag{52}$$

we should have found, taking the first form,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 2ik\frac{\partial u}{\partial x} = 0, \tag{53}$$

or, in terms of ξ, η,

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + 4i\left(\xi\frac{\partial u}{\partial \xi} - \eta\frac{\partial u}{\partial \eta}\right) = 0. \tag{54}$$

This has the solutions

$$\left.\begin{aligned} u &= A + B\int e^{-2i\xi^2} d\xi, \\ u' &= A' + B'\int e^{2i\eta^2} d\eta. \end{aligned}\right\} \tag{55}$$

The corresponding equation in v is satisfied by

$$\left.\begin{aligned} v &= C + D\int e^{2i\xi^2} d\xi, \\ v' &= C' + D'\int e^{-2i\eta^2} d\eta. \end{aligned}\right\} \tag{56}$$

Consider, for example, the region lying to the left of the parabola

$$\eta = \eta_0, \tag{57}$$

and assume

$$\phi = \left\{1 + C\int_{\eta}^{\infty} e^{-2i\zeta^2} d\zeta\right\} e^{-ikx}. \tag{58}$$

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* Cf. Lord Rayleigh, *Phil. Mag.* (5), Vol. XLIII., p. 259 (1897). [*Sc. Papers*, Vol. IV., p. 283.]

This is symmetrical with respect to the axis of x, and, for large negative values of x, tends to the form

$$\phi = e^{-ikx}. \tag{59}$$

The condition

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{or} \quad \frac{\partial\phi}{\partial\eta} = 0 \tag{60}$$

will be satisfied at all points of the parabola (57), provided

$$2i\eta_0\left\{1+C\int_{\eta_0}^{\infty} e^{-2i\zeta^2}d\zeta\right\}-Ce^{-2i\eta_0^2} = 0. \tag{61}$$

The formula (58), with C determined by (61), represents the case of plane waves of sound (or of electric waves polarized at right angles to the plane xy) advancing from $x = -\infty$, and undergoing reflection at the convex surface of a rigid (or perfectly conducting) obstacle whose section is the parabola (57). If f denote, in the sense of geometrical optics, the focal length of the convex mirror, we have

$$kf = \eta_0^2. \tag{62}$$

When f is very great compared with the wave-length, η_0 will be large, and

$$\int_{\eta_0}^{\infty} e^{-2i\zeta^2}d\zeta = \frac{e^{-2i\eta_0^2}}{4i\eta_0}, \tag{63}$$

approximately. Hence

$$C = 4i\eta_0 e^{2i\eta_0^2}, \tag{64}$$

and the reflected waves are therefore represented by

$$\phi' = 4i\sqrt{(kf)}\,e^{-ik(x-2f)}\int_{\eta}^{\infty} e^{-2i\zeta^2}d\zeta. \tag{65}$$

For large values of η this takes the form

$$\phi' = \sqrt{(kf)}\,e^{2ikf}\frac{e^{-ikr}}{\sqrt{(kr)}\sin\frac{1}{2}\theta}. \tag{66}$$

On comparison with (48), it is seen that the waves reflected in directions nearly opposite to that of the incident waves are such as would be produced by a simple line-source of strength

$$\sqrt{(8\pi kf)}, \tag{67}$$

coincident with the axis of z.

An arrangement of virtual sources adequate for all directions is obtained if we imagine the region to which the formula (65) applies to be continued backwards through the reflecting surface. Putting $\eta = 0$, $kdy = 2\xi d\eta$, we find, for $x > 0$,

$$-\left(\frac{\partial\phi'}{\partial y}\right)_{y=0} = \pm\, 2ik\sqrt{(kf)}\,\frac{e^{-ik(x-2f)}}{\sqrt{(kx)}}, \tag{68}$$

where the alternative sign relates to the upper or lower side of the axis

of x. Taking the difference, we obtain the surface-density of the fictitious sources which would produce the system of reflected waves. These sources extend from the focus of the parabola to the right, varying in phase, and diminishing in amplitude, with increasing distance from the focus.

7. Next, consider the region to the right of the parabola $\eta = \eta_0$, in which η varies from 0 to η_0. The formula (58), with the same value (64) of C as before, will represent a radiation originating from a suitable arrangement of sources near the origin, and emerging as a uniform parallel beam in the direction of x positive, after reflection at the concave surface of the parabolic cylinder. The density of the required distribution of sources is

$$4ik\sqrt{(kf)}\,(kx)^{-\frac{1}{2}}\,e^{-ik(x-2f)}. \tag{69}$$

For large values of η the formula (58) reduces to

$$\phi = e^{-ikx} + \sqrt{(kf)}\,e^{2ikf}\frac{e^{-ikr}}{\sqrt{(kr)}\sin\frac{1}{2}\theta}, \tag{70}$$

of which the first term represents the reflected beam, and the second the direct radiation from the sources.

Again, the formula

$$\phi = \left\{1 + B\int_\eta^\infty e^{-2i\zeta^2}d\zeta\right\}e^{ikx} \tag{71}$$

will correspond to the case of a uniform parallel beam coming from the direction of $x = +\infty$, brought by reflection to a focus (in the loose sense of geometrical optics), and there absorbed by some suitable contrivance. The constant B is determined by the condition

$$-2i\eta_0\left\{1 + B\int_{\eta_0}^\infty e^{-2i\xi^2}d\xi\right\} - Be^{2i\eta_0^2} = 0, \tag{72}$$

or

$$B = -4i\eta_0 e^{-2i\eta_0^2}, \tag{73}$$

approximately, since η_0 is supposed large. The density of the sources (or rather sinks) at the "focus" is

$$-4ik\sqrt{(kf)}\,(kx)^{-\frac{1}{2}}\,e^{ik(x-2f)}. \tag{74}$$

For large values of η we have

$$\phi = e^{ikx} + \sqrt{(kf)}\,e^{-2ikf}\frac{e^{ikr}}{\sqrt{(kr)}\sin\frac{1}{2}\theta}, \tag{75}$$

of which the first term represents the incident beam, and the second the waves converging to the "focus."

It is not possible to combine the formulæ (58) and (71) so as to represent the case of an originally parallel beam reflected *through* the

real "focus," and returning finally, after a second reflection, back on its original course. It would be interesting to have a solution of such a case, but it must be recognized that the present problem is hardly suited for an elucidation of the matter. In the first place, the concentration of sources in a *plane* is peculiar to the parabolic form. Moreover, the application of our results to a *finite* mirror is subject to serious qualification. If we limit the mirror by a plane perpendicular to the axis, we cut off a whole system of rays (to use for a moment the language of geometrical optics) which after the first reflection would impinge near the vertex, and so contribute to form the central portions of the finally reflected beam. If the plane of the edge be on the nearer side of the focus, *no* reflected ray strikes the mirror a second time.

The solution obtained for the case of the convex mirror does not appear to be liable to such limitations.*

Reflection of Sound-Waves by a Paraboloid of Revolution.

9. The procedure of § 6 is readily extended to the case of a paraboloid of revolution. In the case of symmetry about the axis, the differential equation takes the form

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial\varpi^2}+\frac{1}{\varpi}\frac{\partial\phi}{\partial\varpi}+k^2\phi=0, \tag{81}$$

where ϖ denotes perpendicular distance from the axis of x, which is that of symmetry. If we write

$$kx=\xi^2-\eta^2,\quad k\varpi=2\xi\eta,\quad kr=\xi^2+\eta^2, \tag{82}$$

we find

$$\left.\begin{aligned}\frac{\partial\xi}{\partial x}&=\tfrac{1}{2}\frac{\xi}{r}, & \frac{\partial\eta}{\partial x}&=-\tfrac{1}{2}\frac{\eta}{r},\\ \frac{\partial\xi}{\partial\varpi}&=\tfrac{1}{2}\frac{\eta}{r}, & \frac{\partial\eta}{\partial\varpi}&=\tfrac{1}{2}\frac{\xi}{r};\end{aligned}\right\} \tag{83}$$

and therefore

$$\left.\begin{aligned}\frac{\partial\phi}{\partial x}&=\frac{k}{2(\xi^2+\eta^2)}\left(\xi\frac{\partial\phi}{\partial\xi}-\eta\frac{\partial\phi}{\partial\eta}\right),\\ \frac{\partial\phi}{\partial\varpi}&=\frac{k}{2(\xi^2+\eta^2)}\left(\eta\frac{\partial\phi}{\partial\xi}+\xi\frac{\partial\phi}{\partial\eta}\right).\end{aligned}\right\} \tag{84}$$

Hence, further,

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial\varpi^2}+\frac{1}{\varpi}\frac{\partial\phi}{\partial\varpi}=\frac{k^2}{4(\xi^2+\eta^2)}\left(\frac{\partial^2\phi}{\partial\xi^2}+\frac{\partial^2\phi}{\partial\eta^2}+\frac{1}{\xi}\frac{\partial\phi}{\partial\xi}+\frac{1}{\eta}\frac{\partial\phi}{\partial\eta}\right). \tag{85}$$

* [The case of electric waves polarized in the plane xy is omitted, for brevity. The formula (58) is then replaced by

$$\phi=Ce^{-ikx}\int_{\eta_0}^{\eta}e^{-2i\zeta^2}d\zeta,$$

where C is given approximately by (64).—*June*, 1906.]

If we seek for solutions of (81) of the types

$$e^{ikx}u, \qquad e^{-ikx}v, \tag{86}$$

we have, in the first place,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial \varpi^2}+\frac{1}{\varpi}\frac{\partial u}{\partial \varpi}+2ik\frac{\partial u}{\partial x}=0, \tag{87}$$

or

$$\frac{\partial^2 u}{\partial \xi^2}+\left(\frac{1}{\xi}+4i\xi\right)\frac{\partial u}{\partial \xi}+\frac{\partial^2 u}{\partial \eta^2}+\left(\frac{1}{\eta}-4i\eta\right)\frac{\partial u}{\partial \eta}=0. \tag{88}$$

This is satisfied by

$$\left.\begin{aligned} u &= A+B\int e^{-2i\xi^2}\frac{d\xi}{\xi}, \\ u' &= A'+B'\int e^{2i\eta^2}\frac{d\eta}{\eta}. \end{aligned}\right\} \tag{89}$$

Similarly, the corresponding equation in v has the particular solutions

$$\left.\begin{aligned} v &= C+D\int e^{2i\xi^2}\frac{d\xi}{\xi}, \\ v' &= C'+D'\int e^{-2i\eta^2}\frac{d\eta}{\eta}. \end{aligned}\right\} \tag{90}$$

In the case of sound-waves incident from the left on the convex surface of the paraboloid

$$\eta=\eta_0, \tag{91}$$

we assume

$$\phi=\left\{1+C\int_\eta^\infty e^{-2i\zeta^2}\frac{d\zeta}{\zeta}\right\}e^{-ikx}, \tag{92}$$

where C is to be determined by the condition that

$$\frac{\partial \phi}{\partial \eta}=0, \quad \text{for} \quad \eta=\eta_0. \tag{93}$$

This gives

$$2i\eta_0\left\{1+C\int_\eta^\infty e^{-2i\zeta^2}\frac{d\zeta}{\zeta}\right\}-C\frac{e^{-2i\eta_0^2}}{\eta_0}=0. \tag{94}$$

For large values of η, we have

$$\int_\eta^\infty e^{-2i\zeta^2}\frac{d\zeta}{\zeta}=\frac{e^{-2i\eta^2}}{4i\eta^2}, \tag{95}$$

approximately. Hence, if the focal length f is large compared with the wave-length,

$$C=4i\eta_0^2e^{2i\eta_0^2}=4ikfe^{2ikf}, \tag{96}$$

and the formula (92) becomes

$$\phi=\left\{1+4ikfe^{2ikf}\int_\eta^\infty e^{-2i\zeta^2}\frac{d\zeta}{\zeta}\right\}e^{-ikx}. \tag{97}$$

For large values of η,

$$\phi = e^{-ikx} + fe^{2ikf} \frac{e^{-ikr}}{r \sin^2 \frac{1}{2}\theta}. \tag{98}$$

Hence the waves reflected at small obliquities will appear to come from a virtual source of strength $4\pi f$ at the origin.

From (92) and (83), we find

$$\frac{\partial \phi}{\partial \varpi} = -\tfrac{1}{2} C \frac{\xi}{\eta} \frac{e^{-ikr}}{r} = -\tfrac{1}{2} C \frac{r+x}{\varpi} \frac{e^{-ikr}}{r}. \tag{99}$$

Hence, if we suppose the function which represents the reflected waves to be continued behind the reflecting surface, we find that these waves may be regarded as due to a continuous distribution of point-sources along the axis of x, to the right of the origin, with line-density

$$2\pi C e^{-ikf} \quad \text{or} \quad 8i\pi k f e^{-ik(x-2f)}, \tag{100}$$

if we substitute the approximate value of C from (96).

The amplitude of these sources is the same at all points of the axis of x to the right of the origin, but the phase runs along continually with the wave-velocity appropriate to the medium. The reader who is interested in tracing physical continuities will observe that the uniformity of amplitude is connected with the fact that the equipotential surfaces of a semi-infinite uniform line of matter are confocal paraboloids.

The case of a concave parabolic reflector might be treated in a similar manner.

INVESTIGATIONS ON SERIES OF ZONAL HARMONICS

By T. J. I'A. BROMWICH.

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A NUMBER of results have been established by Abel, Appell, Picard, Cesàro, Lasker, Pringsheim, Le Roy, Lindelöf, and Hardy* on the behaviour of a function $\Sigma a_n x^n$ in the neighbourhood of the point $x = 1$, according to the mode of convergence or divergence of the series Σa_n.

In the following paper some results analogous to those just mentioned are obtained for the behaviour of the series of zonal harmonics $\Sigma a_n r^n P_n(\cos\theta)$ in the neighbourhood of the point $(r = 1,\ \theta = 0)$. It is proved, for example (§ 7), that, if $\lim_{n=\infty} a_n n^{1-p} = g$, where p is positive, then

$$\lim_{\rho=0} [\rho^p \Sigma a_n r^n P_n(\cos\theta)] = \Gamma(p)\, g\, P_{p-1}(\mu_1),$$

where $$\rho^2 = 1-2r\cos\theta+r^2, \qquad \rho\mu_1 = 1-r\cos\theta.$$

But, if $p = 0$ and $a_n = 1/(n+\gamma)$, we have (§ 3)

$$\lim_{\rho=0} [\Sigma a_n r^n P_n(\cos\theta) - \log\{2/\rho(1+\mu_1)\}] = \int_0^1 \frac{t^{\gamma-1}-1}{1-t}\,dt.$$

On the other hand (§ 1), if Σa_n *converges*,

$$\lim_{\rho=0} \Sigma a_n r^n P_n(\cos\theta) = \Sigma a_n.$$

It is, perhaps, noteworthy that the use of the asymptotic formula for $P_n(\cos\theta)$ leads to erroneous results with regard to the behaviour of $\Sigma a_n r^n P_n(\cos\theta)$; see § 4 below.

One point of difference from the power series is to be noted: in the power series we can deduce the behaviour of the series at any other point of the circle $|x| = 1$ from the behaviour at $x = 1$. To be more exact, if $|t| = 1$ and $\Sigma a_n t^n$ diverges, the behaviour of $\Sigma a_n x^n$ near $x = t$ is easily determined by writing the series as $\Sigma a_n t^n (x/t)^n$; such a method *cannot* be

* Abel, *Œuvres*, t. I., pp. 223, 618; t. II., p. 203 (Sylow-Lie's edition). Appell, *Comptes Rendus*, t. LXXXVII., 1878, p. 689. Picard, *Traité d'Analyse*, t. II., 1893, p. 73. Cesàro, *Rend. d. R. Acc. d. Napoli* (2), 1893, p. 187. Lasker, *Phil. Trans.*, Vol. 196, A, 1901, p. 431. Pringsheim, *Acta Mathematica*, Bd. XXVIII., 1904, p. 1. Le Roy, *Bulletin des Sci. Math.*, t. XXIV., 1900, p. 245; *Annales de Toulouse* (2), t. II., 1900. Lindelöf, *Acta Soc. Sci. Fennicæ*, t. XXXI., 1902, No. 3. G. H. Hardy, *Proc. London Math. Soc.* (2), Vol. 3, 1905, p. 381.

applied to series of zonals. I have found some results concerning the behaviour of $\Sigma a_n r^n P_n(\cos\theta)$ near $r=1$, $\theta=a$, when $\Sigma a_n \cos na$ and $\Sigma a_n \sin na$ are divergent; but the present paper deals only with the point $(r=1, \theta=0)$, as there are some difficulties still to be settled in the other case.

1. *Extension of Abel's Theorem on Power Series.*

In the first place, if the series Σa_n is absolutely convergent, it is evident from the familiar result

$$|P_n(\cos\theta)| \leqslant 1$$

that the terms of the series

$$\Sigma a_n r^n P_n(\cos\theta) = \Sigma a_n X_n$$

are not greater, in absolute value, than those of the convergent series $\Sigma |a_n|$, provided that $r \leqslant 1$. Thus the series $\Sigma a_n X_n$ converges uniformly at all points (r, θ) within or on the unit circle $r=1$.

But, if Σa_n converges, although not absolutely, another method must be adopted. By Abel's transformation we can write

$$\sum_{n=\mu+1}^{\nu} a_n X_n = \sum_{\mu+1}^{\nu-1} (A_n - A_\mu)(X_n - X_{n+1}) + (A_\nu - A_\mu) X_\nu,$$

where $\qquad A_n = a_0 + a_1 + a_2 + \ldots + a_n \quad \text{or} \quad a_n = A_n - A_{n-1}.$

Thus $\qquad \left|\sum_{\mu+1}^{\nu} a_n X_n\right| \leqslant \sum_{\mu+1}^{\nu-1} |A_n - A_\mu|\, |X_n - X_{n+1}| + |A_\nu - A_\mu|\, |X_\nu|,$

and, by § 2, we have $\qquad |X_n - X_{n+1}| \leqslant \rho r^n,$

where $\rho^2 = 1 - 2r\cos\theta + r^2$. Also $|X_\nu| \leqslant r^\nu$; and, in virtue of the convergence of Σa_n, it is possible to fix μ so that

$$|A_n - A_\mu| < \sigma, \quad \text{if} \quad n > \mu.$$

Combining these results, we see that

$$\left|\sum_{\mu+1}^{\nu} a_n X_n\right| < \rho\sigma \sum_{\mu+1}^{\nu-1} r^n + \sigma r^\nu;$$

that is, $\qquad \left|\sum_{\mu+1}^{\nu} a_n X_n\right| < \sigma\left[\frac{\rho r^{\mu+1}}{1-r} - \frac{r^\nu(r+\rho-1)}{1-r}\right].$

Now $\rho > 1-r$; so that $\left|\sum_{\mu+1}^{\nu} a_n X_n\right| < \frac{\rho\sigma r^{\mu+1}}{1-r} < \frac{\rho\sigma}{1-r}.$

Further, Pringsheim* has proved that

$$\frac{\rho}{1-r} \leqslant \frac{2}{\cos a}$$

* *Münchener Sitzungsberichte*, Bd. XXXI., 1901, p. 514; *Acta Mathematica*, Bd. XXVIII., 1904, p. 4.

if the point (r, θ) is restricted to lie within, or on the boundary of, the area $OABC$ (Fig. 1) bounded by the circle of radius $\frac{1}{2}$ (touching the unit circle at 1) and two straight lines making angles α with the radius through 1.

Thus, finally, within $OABC$ we have

$$\left|\sum_{\mu+1}^{\nu} a_n X_n\right| < 2\sigma/\cos\alpha;$$

which proves that the series $\Sigma a_n X_n$ converges uniformly at all points within (or on the boundary of) the area $OABC$.

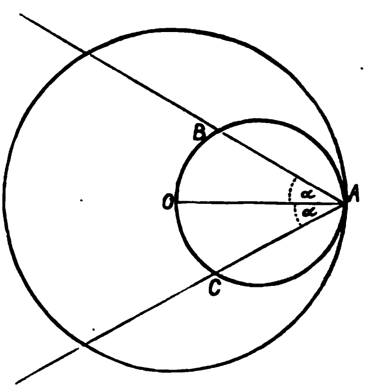

FIG. 1.

2. *Lemma on Zonal Harmonics.*

We have assumed in § 1 that

$$|X_n - X_{n+1}| \leqslant \rho r^n,$$

and we now proceed to prove this inequality.

It is known that, if $\mu = \cos\theta$,

$$P_n(\cos\theta) = \frac{1}{2\pi i}\int_C \frac{t^n\,dt}{(1-2\mu t+t^2)^{\frac{1}{2}}}$$

where the path of integration C is a closed curve enclosing the two points $t = e^{i\theta}$, $t = e^{-i\theta}$ in the plane of the complex variable t.

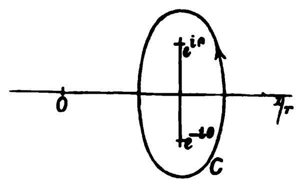

FIG. 2.

Thus
$$X_n - X_{n+1} = \frac{r^n}{2\pi i}\int_C \frac{t^n(1-rt)\,dt}{(1-2\mu t+t^2)^{\frac{1}{2}}};$$

and now take as the path C an ellipse of semi-axes a, b, whose foci coincide with the points $e^{i\theta}$, $e^{-i\theta}$, so that $a^2 - b^2 = \sin^2\theta$.

If ϕ is the eccentric angle on the ellipse, we can write in the integral

$$t = \cos\theta + b\sin\phi - ai\cos\phi.$$

It is then easily seen that

$$\left(\frac{dt}{d\phi}\right)^2 = (b\cos\phi + ai\sin\phi)^2 = (b^2-a^2) - (t-\cos\theta)^2 = -(1-2\mu t+t^2);$$

so that
$$\frac{|dt|}{|(1-2\mu t+t^2)^{\frac{1}{2}}|} = d\phi.$$

Hence
$$|X_n - X_{n+1}| \leqslant \frac{r^n}{2\pi}\int_0^{2\pi} |t|^n\,|1-rt|\,d\phi.$$

Now

$$|t|^2 = (\cos\theta + b\sin\phi)^2 + a^2\cos^2\phi = (\cos\theta + b\sin\phi)^2 + (b^2 + \sin^2\theta)\cos^2\phi\,;$$

so that $$|t|^2 = 1 + 2b\cos\theta\cos\phi + b^2 - \sin^2\theta\sin^2\phi,$$

and therefore $$|t| \leqslant 1 + b.$$

Again

$$\begin{aligned}|1 - rt|^2 &= (1 - r\cos\theta - br\sin\phi)^2 + a^2r^2\cos^2\phi \\ &= 1 - 2r\cos\theta + r^2 - 2br\sin\phi(1 - r\cos\theta) + b^2r^2 - r^2\sin^2\theta\sin^2\phi\,;\end{aligned}$$

so that $$|1 - rt|^2 \leqslant \rho^2 + 2br\rho + b^2r^2$$

and $$|1 - rt| \leqslant \rho + br$$

where $$\rho^2 = 1 - 2r\cos\theta + r^2.$$

Hence $$|X_n - X_{n+1}| \leqslant \frac{r^n}{2\pi}\int_0^{2\pi}(1+b)^n(\rho + br)\,d\phi$$

or $$|X_n - X_{n+1}| \leqslant [r(1+b)]^n(\rho + br).$$

Now b may be supposed as small as we please, and so the last inequality implies $$|X_n - X_{n+1}| \leqslant \rho r^n.$$

3. *Some special Series of Zonal Harmonics.*

(A) Consider the special series of zonal harmonics

$$F(r, \theta) = 1 + \sum_1^\infty (p + n - 1)_n r^n P_n(\cos\theta),$$

where $$(p + n - 1)_n = \frac{p(p+1)\ldots(p+n-1)}{n!} = \frac{\Gamma(p+n)}{\Gamma(p)\Gamma(n+1)} \quad (p > 0).$$

Now $$P_n(\cos\theta) = \frac{1}{2\pi i}\int_C \frac{t^n\,dt}{(1 - 2\mu t + t^2)^{\frac{1}{2}}},$$

where the contour C is any closed regular curve in the plane of the complex variable t, which encloses the two points $t = e^{i\theta}$, $t = e^{-i\theta}$. Thus

$$F(r, \theta) = \frac{1}{2\pi i}\int_C \frac{(1 - rt)^{-p}\,dt}{(1 - 2\mu t + t^2)^{\frac{1}{2}}}$$

because $$(1 - rt)^{-p} = 1 + \sum_1^\infty (p + n - 1)_n r^n t^n,$$

and this series converges uniformly on the path C, provided that the curve does not pass through the point $t = 1/r$.

Write now $$1-rt = \rho u$$
where $\rho^2 = 1-2r\cos\theta+r^2$.

We have then
$$r^2(1-2\mu t+t^2)$$
$$= r^2-2\mu r(1-\rho u)+(1-\rho u)^2$$
$$= \rho^2(1-2\mu_1 u+u^2)$$

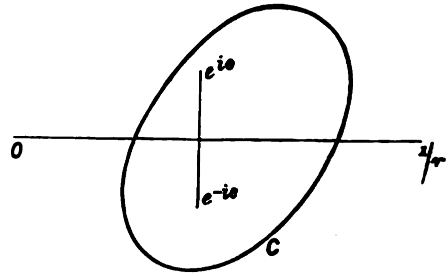

FIG. 3.

where $$\rho\mu_1 = 1-\mu r.$$

It is easy to give a geometrical interpretation to ρ and μ_1; mark the points P, (r, θ), and A, $(1, 0)$ in a diagram. Then $AP = \rho$ and $\mu_1 = \cos\theta_1$

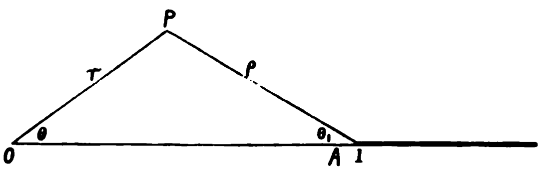

FIG. 4.

if $\theta_1 = OAP$. When the substitution for t is made in the integral $F(r, \theta)$ we find
$$F(r, \theta) = \frac{\rho^{-p}}{2\pi i}\int_{C'} \frac{u^{-p}du}{(1-2\mu_1 u+u^2)^{\frac{1}{2}}}$$
where the path of integration C' is now a closed regular curve in the plane of the complex variable u which encloses the two points* $u = e^{i\theta_1}$, $u = e^{-i\theta_1}$. Hence
$$F(r, \theta) = \rho^{-p}P_{p-1}(\cos\theta_1),$$
and this expression gives the value of the series so long as $r < 1$; but analytically the expression $\rho^{-p}P_{p-1}(\cos\theta_1)$ has a meaning for all values of (r, θ) except those for which $\theta = 0$ and $r > 1$ (making $\cos\theta_1 = -1$). Thus $\rho^{-1}P_{p-1}(\cos\theta_1)$ gives an analytical continuation of $F(r, \theta)$ over the whole plane if we exclude by a cut the part of the line $\theta = 0$ which lies beyond the point $r = 1$. Of course, this analytical continuation is not to be confused with the potential function
$$V = r^{-1}+\sum_1^{\infty}(p+n-1)_n r^{-(n+1)}P_n(\cos\theta),$$
which is appropriate for $r > 1$; a similar investigation will show that V can be put in the form
$$\rho^{-p}r^{p-1}P_{p-1}(\cos OPA),$$
which is plainly continuous with $F(r, \theta)$ at $r = 1$.

* Because $t = e^{i\theta}$ gives $\rho u = 1-re^{i\theta} = 1-r\cos\theta-ir\sin\theta$ or $\rho u = \rho\cos\theta_1-i\rho\sin\theta_1$; *i.e.*, $u = e^{-i\theta_1}$.

(B) Take next the series

$$\Phi(r,\theta) = \frac{1}{\gamma^q} + \sum_1^\infty \frac{r^n P_n(\cos\theta)}{(\gamma+n)^q} \qquad (q<1).$$

If we follow the method given by G. H. Hardy* for the associated power series, we shall write

$$\frac{\Gamma(q)}{(\gamma+n)^q} = \int_0^1 \left[\log\left(\frac{1}{t}\right)\right]^{q-1} t^{\gamma-1+n}\,dt,$$

which can then be transformed into an integral along a closed regular curve in the t-plane, starting and ending at the origin, and enclosing $t=1$. In this manner we have

$$\frac{1}{(\gamma+n)^q} = \frac{\Gamma(1-q)}{2\pi i}\int_C (\log t)^{q-1}\, t^{\gamma-1+n}\,dt,$$

where

$$(\log t)^{q-1} = \exp\left[(q-1)\log(\log t)\right],$$

$$t^{\gamma-1} = \exp\left[(\gamma-1)\log t\right],$$

and $\log t$, $\log(\log t)$ are real at the point A (see Fig. 5).

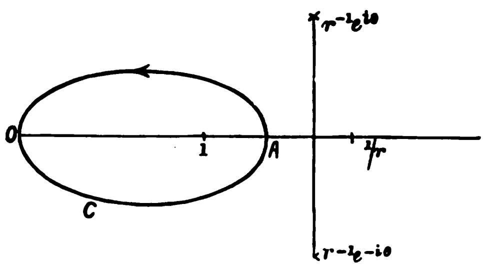

FIG. 5.

Hence
$$\Phi(r,\theta) = \frac{\Gamma(1-q)}{2\pi i}\int_C \frac{(\log t)^{q-1}\, t^{\gamma-1}}{(1-2\mu r t+r^2t^2)^{\frac{1}{2}}}\,dt,$$

where the square root is taken so as to be positive at the point A; and the path C is supposed to be drawn so as *not* to enclose the points $t=r^{-1}e^{i\theta}$, $t=r^{-1}e^{-i\theta}$ at which the denominator is zero. That the series $\Sigma(rt)^n P_n(\cos\theta)$ converges uniformly at all points of C is evident if A lies between 1 and $1/r$.

If we write t instead of rt, we obtain (see Fig. 6)

$$\Phi(r,\theta) = \frac{\Gamma(1-q)}{2\pi i r^\gamma}\int_{C_1} \frac{[\log(t/r)]^{q-1}\, t^{\gamma-1}\,dt}{(1-2\mu t+t^2)^{\frac{1}{2}}}.$$

If cuts are made from 0 to r and from $e^{-i\theta}$ to $e^{i\theta}$ in the plane of the complex variable t, it will be evident that the subject of integration is

* *Proc. London Math. Soc.*, Ser. 2, Vol. 3, 1905, p. 381.

made one-valued; and, further, that

$$\int_{C_1} = \int_{C_3} - \int_{C_2},$$

the paths being those indicated in Fig. 6.

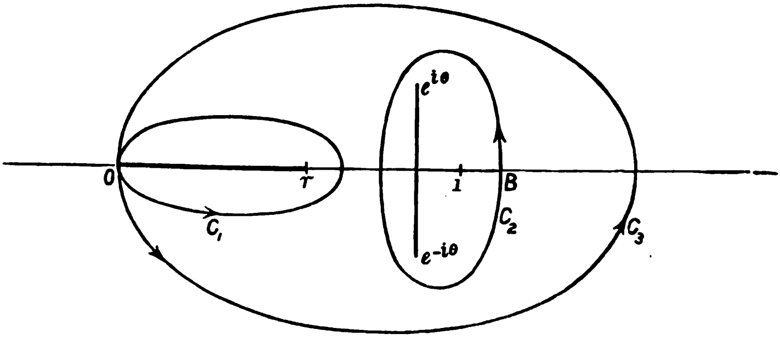

FIG. 6.

Now, when r tends to 1 and θ to 0, the integral along C_3 approaches a definite finite limit

$$L = \frac{\Gamma(1-q)}{2\pi i} \int_{C_3} \frac{(\log t)^{q-1} t^{\gamma-1}}{1-t}\, dt.$$

Thus we may now write

$$\Phi(r, \theta) = \frac{\Gamma(1-q)}{2\pi i\, r^{\gamma}} \int_{C_2} \frac{[\log (t/r)]^{q-1} t^{\gamma-1}}{(1-2\mu t+t^2)^{\frac{1}{2}}}\, dt + L + \epsilon,$$

where ϵ tends to zero as (r, θ) approaches 1; *and now the square root is supposed to be positive at the point B.* (Fig. 6.)

To evaluate the remaining integral, write $t = r+\rho v$, where

$$\rho^2 = 1-2r\cos\theta+r^2;$$

and then we obtain

$$\Phi(r, \theta) = \frac{\Gamma(1-q)}{2\pi i r} \int \frac{[\log (1+\rho v/r)]^{q-1} (1+\rho v/r)^{\gamma-1}\, dv}{(1-2\mu_2 v+v^2)^{\frac{1}{2}}} + L + \epsilon,$$

where $\rho\mu_2 = \mu - r$; or $\mu_2 = \cos(\theta+\theta_1)$ in the notation of (A) above. The path of integration is now a regular closed curve surrounding

$$v = e^{i\theta_2}, \quad e^{-i\theta_2}$$

and excluding the origin. Now, since our object is to determine the behaviour of $\Phi(r, \theta)$, when (r, θ) is near to 1, so that ρ is small, we can easily assign a value of ρ_0 such that $|\rho v/r|$ is less than 1 at all points of the path of integration provided that $\rho < \rho_0$; and then

$$\log(1+\rho v/r) = \rho v/r - \tfrac{1}{2}(\rho v/r)^2 + \tfrac{1}{3}(\rho v/r)^3 - \ldots,$$

$$(1+\rho v/r)^{\gamma-1} = 1 + (\gamma-1)(\rho v/r) + \tfrac{1}{2}(\gamma-1)(\gamma-2)(\rho v/r)^2 + \ldots.$$

Thus

$$\Phi(r,\theta) = \frac{\Gamma(1-q)}{2\pi i r}\int\left(\frac{\rho v}{r}\right)^{q-1}\left[1+\tfrac{1}{2}(2\gamma-q-1)\frac{\rho v}{r}+\dots\right](1-2\mu_2 v+v^2)^{-\frac{1}{2}}dv$$
$$+L+\epsilon$$
$$=\Gamma(1-q)\left[\frac{\rho^{q-1}}{r^q}P_{q-1}(\mu_2)+\tfrac{1}{2}(2\gamma-q-1)\frac{\rho^q}{r^{q+1}}P_q(\mu_2)+\dots\right]+L+\epsilon.$$

Consequently, when ρ is small, the most important term in the value of $\Phi(r,\theta)$ is

$$\Gamma(1-q)\rho^{q-1}r^{-q}P_{q-1}(\mu_2),$$

because $q<1$.

It may be noted that, if $q=1-p$,

$$\lim_{\rho=0}\ \Phi(r,\theta)/F(r,\theta) = \Gamma(1-q) = \Gamma(p)$$

because

$$\lim_{\theta=0}\mu_2 = \lim_{\theta=0}\mu_1.$$

The case $q=1$ can be deduced from the expression just found for $\Phi(r,\theta)$ by passing to the limit and using the identity

$$(1-q)\Gamma(1-q) = \Gamma(2-q).$$

But it is easier to proceed directly; for we have

$$\frac{1}{\gamma+n} = \int_0^1 t^{n+\gamma-1}dt,$$

so that

$$\frac{1}{\gamma}+\sum_1^\infty\frac{r^nP_n(\cos\theta)}{\gamma+n} = \int_0^1\frac{t^{\gamma-1}dt}{(1-2\mu rt+r^2t^2)^{\frac{1}{2}}},$$

the convergence being uniform. Now we have

$$\int_0^1(1-2\mu rt+r^2t^2)^{-\frac{1}{2}}dt = \frac{1}{r}\log\frac{1+\mu}{\rho(1+\mu_2)},$$

so that

$$\int_0^1\frac{t^{\gamma-1}dt}{(1-2\mu rt+r^2t^2)^{\frac{1}{2}}} = \frac{1}{r}\log\frac{1+\mu}{\rho(1+\mu_2)}+\int_0^1\frac{(t^{\gamma-1}-1)dt}{(1-2\mu rt+r^2t^2)^{\frac{1}{2}}}.$$

But

$$\int_0^1\frac{(t^{\gamma-1}-1)dt}{(1-2\mu rt+r^2t^2)^{\frac{1}{2}}} = \left(\frac{1}{\gamma}-1\right)+\sum_1^\infty\frac{(1-\gamma)r^n}{(n+\gamma)(n+1)}P_n(\cos\theta),$$

and in this series the coefficients form an absolutely convergent series. Thus, by § 1,

$$\lim_{\rho=0}\int_0^1\frac{(t^{\gamma-1}-1)dt}{(1-2\mu rt+r^2t^2)^{\frac{1}{2}}} = \left(\frac{1}{\gamma}-1\right)+\sum_1^\infty\frac{(1-\gamma)}{(n+\gamma)(n+1)} = \int_0^1\frac{t^{\gamma-1}-1}{1-t}dt,$$

where the transformation of the series into the integral is justifiable by the aid of a test due to G. H. Hardy.* It is also possible to justify the step by direct reasoning.

* *Messenger of Mathematics*, December, 1905.

Consequently

$$\lim_{\rho=0}\left[\left\{\frac{1}{\gamma}+\sum_1^\infty \frac{r^n P_n(\cos\theta)}{\gamma+n}\right\}-\log\frac{2}{\rho(1+\mu_2)}\right]=\int_0^1 \frac{t^{\gamma-1}-1}{1-t}\,dt.$$

4. In § 3 we have determined the behaviour near (1, 0) of certain series of the type

$$\Sigma d_n r^n P_n(\cos\theta) = \Sigma d_n X_n,$$

where Σd_n is divergent, though $\Sigma d_n r^n$ converges if $r<1$. From the results there established it is clear that the behaviour of the series $\Sigma d_n X_n$ near (1, 0) is analogous to that of the associated power series $\Sigma d_n x^n$ near $x=1$; and we are led to expect the existence of theorems for series $\Sigma d_n X_n$ of a similar character to the theorems established for $\Sigma d_n x^n$ by Pringsheim and others (see the papers quoted on p. 204 above).

It is, of course, obvious that when $\theta=0$ the series $\Sigma d_n X_n$ reduces to the power series $\Sigma d_n r^n$; and, consequently, any result established for $\Sigma d_n X_n$ ought to reduce (for $\theta=0$) to a known result for the power series $\Sigma d_n r^n$. This remark gives some verification of the conclusions already obtained in § 3; thus, in (A), $\Sigma d_n r^n = (1-r)^{-p}$ and $\rho = 1-r$, $\theta_1=0$ (when $\theta=0$); so that $\rho^{-p}P_{p-1}(\cos\theta_1) = (1-r)^{-p}$.

It might be thought at first sight that, if Σd_n is a divergent series, the behaviour of the series

$$\Sigma d_n r^n P_n(\cos\theta)$$

near the point ($r=1$, $\theta=0$) could be determined by making use of the familiar asymptotic formula

$$P_n(\cos\theta)=\left(\frac{2}{n\pi\sin\theta}\right)^{\frac{1}{2}}\sin\left[(n+\tfrac{1}{2})\theta+\tfrac{1}{4}\pi\right].$$

For it might be urged that, although this formula is not correct for small values of n and θ, yet it gives a good approximation (even when θ is small), provided that ($n\sin\theta$) is large; thus, since the divergence of $\Sigma d_n r^n P_n(\cos\theta)$ is due to the terms of high order, one might anticipate that the asymptotic formula would lead to correct results. But, as a matter of fact, this anticipation is quite incorrect.

For instance, take the series $\Sigma r^n P_n(\cos\theta)$; if we replace P_n by the asymptotic formula, we obtain

$$1+(\pi\sin\theta)^{-\frac{1}{2}}\left[(\cos\tfrac{1}{2}\theta+\sin\tfrac{1}{2}\theta)\sum_1^\infty n^{-\frac{1}{2}}r^n\cos n\theta \right.$$
$$\left. +(\cos\tfrac{1}{2}\theta-\sin\tfrac{1}{2}\theta)\sum_1^\infty n^{-\frac{1}{2}}r^n\sin n\theta\right].$$

Now, according to Pringsheim [*Acta Mathematica*, t. XXVIII., p. 11, (10)],

the value of $\sum_1^{\infty} n^{-\frac{1}{2}} r^n e^{in\theta}$ is given by the approximation

$$\Gamma(\tfrac{1}{2})/(1-re^{i\theta})^{\frac{1}{2}} = \pi^{\frac{1}{2}}\rho^{-\frac{1}{2}}e^{\frac{1}{2}i\theta_1}$$

(using the notation of § 3). Hence the asymptotic formula combined with Pringsheim's would give for the approximation to $\Sigma r^n P_n(\cos\theta)$

$$\rho^{-1}(\cos\tfrac{1}{2}\theta_1+\sin\tfrac{1}{2}\theta_1)(\sin\theta_1)^{-\frac{1}{2}}$$

by writing $$\sin\theta = \rho\sin\theta_1/r.$$

But the actual value of $\Sigma r^n P_n(\cos\theta)$ is ρ^{-1}, no matter what may be the value of θ_1; and this contradicts the result obtained from the asymptotic formula for $P_n(\cos\theta)$. The correct value ρ^{-1} may be found from either (A) or (B) of § 3.

5. *Uniform Divergence.*

Pringsheim has proved* that, when x is free to take complex values near the point $x = 1$, the equation $\lim_{x=1}|\Sigma d_n x^n| = \infty$ is by no means a consequence of the divergence of Σd_n. That the same is true for $\Sigma d_n X_n$ is evident by considering § 3 (A). For example, if $p = 3$, $P_2(\mu_1) = 0$ if $\cos\theta_1 = 3^{-\frac{1}{2}}$; and so, for this special value of θ_1, $\Sigma d_n X_n$ is zero for any value of ρ; thus, $\lim_{\rho=0}|\Sigma d_n X_n|$ is not always ∞.

Following Pringsheim, we shall say that:—*The series of zonals* $\Sigma d_n X_n$ *diverges uniformly at* (1, 0), *provided that*

$$\lim_{\rho=0}|\Sigma d_n X_n|/|\Sigma d_n r^n| = a > 0,$$

within a properly chosen area of the type indicated in Fig. 1 (*p.* 206).

6. *Comparison of the Divergence of Two Series.*

The fundamental theorem is:—

If $\Sigma d_n r^n$ *is convergent for* $r < 1$, *but* Σd_n *is divergent* (*where* $d_n > 0$), *then, if* $\Sigma d_n X_n$ *is uniformly divergent at* (1, 0), *the series* $\Sigma a_n X_n$ *converges for* $r < 1$ *and diverges uniformly at* (1, 0), *provided that*

$$\lim_{\nu=\infty}\left(\sum_0^{\nu} a_n \Big/ \sum_0^{\nu} d_n\right) = g.$$

Further, $$\lim_{\rho=0}(\Sigma a_n X_n/\Sigma d_n X_n) = g.$$

For, write $D_\nu = \sum_0^{\nu} d_n$, $A_\nu = \sum_0^{\nu} a_n$; then it is evident† that $\Sigma D_n r^n$ converges if $r < 1$; and consequently $\Sigma A_n r^n$ is also convergent, and therefore $\Sigma a_n r^n$ converges. Thus $\Sigma a_n X_n$ converges for $r < 1$.

* *L.c.*, p. 5. Pringsheim's example is $e^{1/(1-x)^2}$, which gives $\lim_{x=1}|e^{1/(1-x)^2}| = 1$ if $\theta_1 = \frac{1}{4}\pi$, or 0 if $\theta_1 > \frac{1}{4}\pi$.

† Because $\Sigma D_n r^n = (1-r)^{-1}\Sigma d_n r^n$.

Now $$\sum_0^\infty a_n X_n = \sum_0^{\nu-1} A_n(X_n - X_{n+1}) + \sum_\nu^\infty A_n(X_n - X_{n+1}),$$

so that $$\left|\sum_0^\infty a_n X_n\right| \leqslant \left|\sum_0^{\nu-1} A_n(X_n - X_{n+1})\right| + \left|\sum_\nu^\infty A_n(X_n - X_{n+1})\right|.$$

Assume now for the present that $g = 0$, so that by proper choice of ν we can make *all* the quotients

$$A_\nu/D_\nu, \quad A_{\nu+1}/D_{\nu+1}, \quad A_{\nu+2}/D_{\nu+2}, \quad \dots \text{ to } \infty$$

less than an arbitrary positive fraction σ; thus

$$\left|\sum_\nu^\infty A_n(X_n - X_{n+1})\right| < \sigma \sum_\nu^\infty D_n \,|X_n - X_{n+1}|.$$

But, by § 2, $$|X_n - X_{n+1}| \leqslant \rho r^n,$$

and so $$\left|\sum_\nu^\infty A_n(X_n - X_{n+1})\right| < \rho\sigma \sum_\nu^\infty D_n r^n < \rho\sigma \sum_0^\infty D_n r^n.$$

Now $$\sum_0^\infty D_n r^n = (1-r)^{-1} \sum_0^\infty d_n r^n \leqslant a^{-1}(1-r)^{-1} \left|\sum_0^\infty d_n X_n\right|$$

in virtue of the uniform divergence of $\Sigma d_n X_n$, provided that the point (r, θ) lies within a suitably chosen area round the point $(1, 0)$. Hence

$$\left|\sum_\nu^\infty A_n(X_n - X_{n+1})\right| < \rho(1-r)^{-1}\sigma a^{-1} \left|\sum_0^\infty d_n X_n\right|;$$

or, using Pringsheim's inequality,* we find

$$\left|\sum_\nu^\infty A_n(X_n - X_{n+1})\right| < 2\sigma(a \cos a)^{-1} \left|\sum_0^\infty d_n X_n\right|.$$

Thus we have

$$\left|\sum_0^\infty a_n X_n\right| < \left|\sum_0^{\nu-1} A_n(X_n - X_{n+1})\right| + 2\sigma(a\cos a)^{-1} \left|\sum_0^\infty d_n X_n\right|$$

or $$\left|\frac{\sum_0^\infty a_n X_n}{\sum_0^\infty d_n X_n}\right| < \frac{\left|\sum_0^{\nu-1} A_n(X_n - X_{n+1})\right|}{\left|\sum_0^\infty d_n X_n\right|} + \frac{2\sigma}{a\cos a}.$$

But, from § 2, $$\left|\sum_0^{\nu-1} A_n(X_n - X_{n+1})\right| \leqslant \rho \sum_0^{\nu-1} |A_n| r^n,$$

and $$\left|\sum_0^\infty d_n X_n\right| \geqslant a \sum_0^\infty d_n r^n = a(1-r)^{-1} \sum_0^\infty D_n r^n,$$

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* See § 1 above, p. 205 (foot).

so that
$$\frac{\left|\sum_0^{\nu-1} A_n(X_n-X_{n+1})\right|}{\left|\sum_0^{\infty} d_n X_n\right|} \leqslant \frac{\rho}{a(1-r)} \frac{\sum_0^{\nu-1}|A_n|r^n}{\sum_0^{\infty} D_n r^n}.$$

Hence
$$\left|\frac{\sum_0^{\infty} a_n X_n}{\sum_0^{\infty} d_n X_n}\right| < \frac{2}{a\cos a}\left\{\frac{\sum_0^{\nu-1} |A_n| r^n}{\sum_0^{\infty} D_n r^n}+\sigma\right\},$$

so that
$$\lim_{\rho=0}\left|\frac{\sum_0^{\infty} a_n X_n}{\sum_0^{\infty} d_n X_n}\right| \leqslant \frac{2\sigma}{a\cos a},$$

since*
$$\lim_{\rho=0}\sum_0^{\infty} D_n r^n = \lim_{r=1}\sum_0^{\infty} D_n r^n = \infty.$$

But σ can be made arbitrarily small by proper choice of ν, so that
$$\lim_{\rho=0}\left(\sum_0^{\infty} a_n X_n \Big/ \sum_0^{\infty} d_n X_n\right) = 0.$$

Return now to the case $\lim_{\nu=\infty} A_\nu/D_\nu = g$; we have then
$$\lim_{\nu=\infty}(A_\nu - gD_\nu)/D_\nu = 0,$$
so that, in virtue of what has just been proved,
$$\lim_{\rho=0}\left(\sum_0^{\infty}(a_n-gd_n)X_n \Big/ \sum_0^{\infty} d_n X_n\right) = 0.$$

Consequently
$$\lim_{\rho=0}\left(\sum_0^{\infty} a_n X_n \Big/ \sum_0^{\infty} d_n X_n\right) = g,$$
and from this the uniform divergence of $\Sigma a_n X_n$ is at once evident.

If it happens that $\lim_{n=\infty}(a_n/d_n) = g$, it is known that $\lim_{\nu=\infty}(A_\nu/D_\nu) = g$ by the Cauchy-Stolz theorem (see p. 219); consequently the same conclusions apply.

7. If now we take, as the series of comparison, the series $F(r, \theta)$ discussed in § 3 (A), we have at once the following conclusions from § 6:—

If† $\sum_0^{\nu} a_n \simeq (p+\nu)_\nu g$, *or if* $a_n \simeq (p+n-1)_n g$, *then*
$$\lim_{\rho=0}\rho^p \sum_0^{\infty} a_n X_n = g P_{p-1}(\mu_1).$$

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* This theorem is due to Abel, but was published posthumously (*Œuvres*, t. II., p. 203).

† The equation $A_\nu \simeq B_\nu g$ implies $\lim_{\nu=\infty}(A_\nu/B_\nu) = g$. It should be noted that
$$\sum_0^{\nu}(p+n-1)_n = (p+\nu)_\nu.$$

A direct application of § 6 would require the exclusion [from the domain of (r, θ)] of every line through A on which $P_{p-1}(\mu_1) = 0$; for we have seen that $F(r, \theta)$ is zero corresponding to these values of μ_1. Consequently the series $F(r, \theta)$ ceases to diverge uniformly on these lines. But the theorem at the foot of p. 215 is still true even for these values of μ_1; for, just as in § 6, we can prove that

$$\left|\sum_0^\infty (a_n - gd_n) X_n\right| < \rho \sum_0^{\nu-1} |A_n - gD_n| r^n + \rho\sigma \sum_0^\infty D_n r^n.$$

But here $\sum_0^\infty D_n r^n = (1-r)^{-(p+1)}$; so that

$$\rho^p \left|\sum_0^\infty (a_n - gd_n) X_n\right| < \rho^{p+1} \sum_0^{\nu-1} |A_n - gD_n| r^n + \rho^{p+1}(1-r)^{-(p+1)}\sigma.$$

Now, by Pringsheim's inequality, $\rho(1-r)^{-1} \leqslant 2/\cos a$; so that

$$\rho^p \left|\sum_0^\infty (a_n - gd_n) X_n\right| < \rho^{p+1} \sum_0^{\nu-1} |A_n - gD_n| r^n + \sigma (2/\cos a)^{p+1}.$$

Hence
$$\lim_{\rho=0} \rho^p \left|\sum_0^\infty (a_n - gd_n) X_n\right| \leqslant \sigma (2/\cos a)^{p+1},$$

and so
$$\lim_{\rho=0} \rho^p \sum_0^\infty a_n X_n = g \lim_{\rho=0} \rho^p \sum_0^\infty d_n X_n = gP_{p-1}(\mu_1).$$

Since $\lim_{n=\infty} (p+n-1)_n n^{1-p} = \Gamma(p)$, the theorem obtained above may be put in the slightly more convenient form:—

If $\sum_0^\nu a_n \simeq g\, n^p$, *then* $\lim_{\rho=0} \left(\rho^p \sum_0^\infty a_n X_n\right) = \Gamma(p+1).g.P_{p-1}(\mu_1)$; *or, if* $a_n \simeq g\, n^{p-1}$, *then* $\lim_{\rho=0} \left(\rho^p \sum_0^\infty a_n X_n\right) = \Gamma(p).g.P_{p-1}(\mu_1)$.

It is at once evident that the second of these results is in agreement with § 3 (B).

8. We pass now to the consideration of series which diverge more slowly than those already discussed. For this purpose we need certain properties given by Pringsheim* with reference to a continuous function $\lambda(x)$ which steadily increases to infinity with x, but more slowly† than any power of x; for integral values $x = n$, it is convenient to write

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* *Acta Mathematica*, Bd. XXVIII., 1904, pp. 12–14.

† For instance, $\log x$, $\log|\log x|$, ...

$\lambda(n) = \lambda_n$. Then the two chief properties are

$$\text{(i.)} \quad \lim_{n=\infty} \frac{n}{\lambda_n}(\lambda_n - \lambda_{n-1}) = 0,$$

$$\text{(ii.)} \quad \lim_{x=\infty} \lambda(kx)/\lambda(x) = 1.$$

The first of these is more conveniently written

$$\text{(iii.)} \quad \lim_{x=\infty} x\lambda'(x)/\lambda(x) = 0,$$

in case the differential coefficient $\lambda'(x)$ exists.

Consider now the series

$$F_\lambda = \lambda_0 + \sum_1^\infty (\lambda_n - \lambda_{n-1}) X_n :$$

we shall show that
$$\lim_{\rho=0} \frac{F_\lambda}{\lambda(1/\rho)} = 1.$$

The proof is exactly similar to Pringsheim's for the corresponding theorem on power series.*

For any integral value of ν, we have

$$\lambda_\nu = \lambda_0 + \sum_1^\nu (\lambda_n - \lambda_{n-1}),$$

so that
$$F_\lambda - \lambda_\nu = \sum_1^\nu (\lambda_n - \lambda_{n-1})(X_n - 1) + \sum_{\nu+1}^\infty (\lambda_n - \lambda_{n-1}) X_n$$

and
$$| F_\lambda - \lambda_\nu | \leqslant \sum_1^\nu (\lambda_n - \lambda_{n-1}) \, | X_n - 1 | + \sum_{\nu+1}^\infty (\lambda_n - \lambda_{n-1}) \, | X_n | .$$

Now
$$| X_n - 1 | \leqslant (1-r) R + |\theta| \Theta,$$

where R, Θ are the maximum values of $\left|\frac{\partial X_n}{\partial r}\right|$ and $\left|\frac{\partial X_n}{\partial \theta}\right|$ in the neighbourhood of the point (1, 0).

But
$$\frac{\partial X_n}{\partial r} = nr^{n-1} P_n(\cos\theta), \qquad \frac{\partial X_n}{\partial \theta} = r^n \frac{dP_n}{d\theta},$$

so that $R = n$; as for Θ, we can easily prove that† $\left|\frac{dP_n}{d\theta}\right| < n$, and so

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* *L.c.*, pp. 16–18.

† In fact, $P_n(\cos\theta) = a_0 \cos n\theta + a_1 \cos(n-2)\theta + \ldots$ to $\frac{1}{2}(n+1)$ or $\frac{1}{2}(n+2)$ terms, where $a_0, a_1, \ldots$ are all positive and $\Sigma a = 1$. Thus

$$\frac{dP_n}{d\theta} = -na_0 \sin n\theta - (n-2) a_1 \sin(n-2)\theta - \ldots ;$$

or
$$\left|\frac{dP_n}{d\theta}\right| < n \Sigma a.$$

$\Theta < n$. Thus

$$|X_n - 1| < n[(1-r) + |\theta|].$$

Further, we have $(1-r)^2 \leqslant 1 - 2r\cos\theta + r^2 \leqslant \rho^2$

and $$|\theta| \leqslant |\tan\theta| < \rho/(1-\rho).$$

Hence $$|X_n - 1| < n\rho(2-\rho)/(1-\rho) < 3n\rho,$$

provided that $\rho < \frac{1}{2}$. Thus, if we write

$$n(\lambda_n - \lambda_{n-1})/\lambda_n = \mu_n,$$

we find $$|F_\lambda - \lambda_\nu| < 3\rho \sum_1^\nu \mu_n \lambda_n + \sum_{\nu+1}^\infty (\lambda_n - \lambda_{n-1}) r^n,$$

because $$|X_n| \leqslant r^n.$$

Now, since $\lim \mu_n = 0$ (in virtue of the first of Pringsheim's properties), we can choose N so that

$$0 < \mu_n < \sigma, \quad \text{if} \quad n \geqslant N,$$

however small the positive number σ may be. Thus, if

$$n \geqslant \nu > N,$$

we can write $$\lambda_n - \lambda_{n-1} = \lambda_n \mu_n / n < \sigma\lambda_n / n.$$

Now, although λ_n tends to ∞ with n, yet it does so more slowly than n, and therefore (at least after a certain stage) λ_n/n must steadily decrease as n increases.* Hence $\lambda_n/n < \lambda_\nu/\nu$ and so

$$\lambda_n - \lambda_{n-1} < \sigma\lambda_\nu/\nu, \quad \text{if} \quad n > \nu > N,$$

thus $$\sum_{\nu+1}^\infty (\lambda_n - \lambda_{n-1}) r^n < (\sigma\lambda_\nu/\nu) \sum_{\nu+1}^\infty r^n < \frac{\sigma\lambda_\nu}{\nu(1-r)}.$$

Making use of this, we have

$$|F_\lambda - \lambda_\nu| < 3\rho \sum_1^\nu \mu_n \lambda_n + \frac{\sigma\lambda_\nu}{\nu(1-r)}$$

or $$\left|\frac{F_\lambda}{\lambda_\nu} - 1\right| < 3\nu\rho \frac{\sum_1^\nu \mu_n \lambda_n}{\nu\lambda_\nu} + \frac{\sigma}{\nu(1-r)} < 3\nu\rho \frac{\sum_1^\nu \mu_n \lambda_n}{\nu\lambda_\nu} + \frac{2\sigma}{\nu\rho\cos\alpha},$$

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* In fact, $$\frac{\lambda_n}{n} - \frac{\lambda_{n-1}}{n-1} = \frac{n(\lambda_n - \lambda_{n-1}) - \lambda_n}{n(n-1)} = \frac{\lambda_n(\mu_n - 1)}{n(n-1)},$$

which, after a certain stage, must be constantly negative, because μ_n tends to zero.

in virtue of Pringsheim's inequality,

$$\frac{\rho}{1-r} \leqslant \frac{2}{\cos a}.$$

Now, by a theorem of Cauchy's, generalized by Stolz,

$$\lim_{\nu=\infty} \frac{a_\nu}{b_\nu} = \lim_{\nu=\infty} \frac{a_\nu - a_{\nu-1}}{b_\nu - b_{\nu-1}},$$

provided that the second limit exists and that b_ν steadily increases with ν; thus we have

$$\lim_{\nu=\infty} \frac{\sum_1^\nu \mu_n \lambda_n}{\nu\lambda_\nu} = \lim_{\nu=\infty} \frac{\mu_\nu \lambda_\nu}{\nu\lambda_\nu - (\nu-1)\lambda_{\nu-1}} = \lim_{\nu=\infty} \frac{\mu_\nu \lambda_\nu}{\mu_\nu \lambda_\nu + \lambda_{\nu-1}} = 0,$$

because $$\lim_{\nu=\infty} \mu_\nu = 0 \quad \text{and} \quad \lim_{\nu=\infty} \lambda_\nu / \lambda_{\nu-1} = 1.$$

Hence, ϵ being arbitrarily small and positive, by a proper adjustment of N, we can make both $\frac{1}{\nu\lambda_\nu} \sum_1^\nu \mu_n \lambda_n < \frac{1}{6}\epsilon$, and $\sigma \sec a < \frac{1}{8}\epsilon$ simply by taking $\nu > N$.

Suppose now that ν and ρ are connected by the condition that ν is the integral part of $1/\rho$; thus $\nu > N$, if $\rho < 1/(N+1)$. Then

$$\left| \frac{F_\lambda}{\lambda_\nu} - 1 \right| < \frac{3}{\nu\lambda_\nu} \sum_1^\nu \mu_n \lambda_n + 4\sigma \sec a < \epsilon, \qquad \text{if} \quad \rho < 1/(N+1).$$

Hence $$\lim_{\rho=0} (F_\lambda / \lambda_\nu) = 1.$$

But, since $\nu\rho$ lies between 1 and $\nu/(\nu+1)$, we have, by the second of Pringsheim's results quoted on p. 217,

$$\lim_{\rho=0} \frac{\lambda(1/\rho)}{\lambda_\nu} = 1,$$

so that $$\lim_{\rho=0} \frac{F_\lambda}{\lambda(1/\rho)} = 1.$$

In particular, if $\theta = 0$, we get

$$\lim_{r=1} \frac{\Phi_\lambda}{\lambda[1/(1-r)]} = 1,$$

where $$\Phi_\lambda = \lambda_0 + \sum_1^\infty (\lambda_n - \lambda_{n-1}) r^n.$$

Now, since $$1 \leqslant \frac{\rho}{1-r} \leqslant \frac{2}{\cos a}$$

within the area $OABC$ (Fig. 1, p. 206), we have

$$\lim_{\rho=0} \frac{\lambda(1/\rho)}{\lambda[1/(1-r)]} = 1,$$

in virtue of Pringsheim's second property for $\lambda(x)$. Hence

$$\lim_{\rho=0} (F_\lambda/\Phi_\lambda) = 1;$$

and therefore *the series F_λ diverges uniformly at* (1, 0); it is therefore possible to apply § 6 and deduce the theorem :—

If $\sum_0^\nu a_n \simeq g\lambda_\nu$, *or if* $a_n \simeq g(\lambda_n - \lambda_{n-1})$, *or again if* $a_n \simeq g\lambda'(n)$, *then*

$$\lim_{\rho=0} [\lambda(1/\rho)]^{-1} \sum_0^\infty a_n X_n = g.$$

An example of this theorem is given by the series discussed in § 3 (B), $\Sigma X_n/(\gamma+n)$, for which we can take $\lambda(n) = \log n$ and $g = 1$.

9. Although it has not proved difficult to modify (see § 8) Pringsheim's discussion of the series

$$\lambda_0 + \sum_1^\infty (\lambda_n - \lambda_{n-1}) x^n,$$

so as to obtain results for the associated series of zonal harmonics, yet it does not seem to be possible to obtain from this series information as to the two series

$$\sum_0^\infty \lambda_n X_n \quad \text{and} \quad \sum_1^\infty \lambda_n^{-1} X_n$$

by the method which Pringsheim adopts for the corresponding power series.

However, we have, just as in § 3 (A),

$$\sum_0^\infty \lambda_n X_n = \frac{1}{2\pi i} \int_C \frac{\Sigma \lambda_n (rt)^n}{(1-2\mu t+t^2)^{\frac{1}{2}}} dt,$$

where the path of integration is that given in Fig. 3 (see p. 208). And Pringsheim's results* show that

$$\sum_0^\infty \lambda_n (rt)^n = \frac{1}{1-rt} \lambda\left(\frac{1}{|1-rt|}\right)(1+\xi),$$

where $|\xi|$ can be made arbitrarily small by taking $|1-rt|$ small enough, provided that the phase of $(1-rt)$ is less, numerically, than some fixed value a ($< \frac{1}{2}\pi$). Now, if we take the path C sufficiently close to the cut, this condition will be satisfied at all points of C, provided that

$$1 \geqslant \mu_1 > k > \cos a > 0,$$

where, as in § 3 (A), $\rho\mu_1 = 1 - \rho\mu$.

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* *L.c.*, pp. 19–21.

Thus we find, by writing $\rho v = 1 - rt$,

$$\sum_0^\infty \lambda_n X_n = \frac{1}{2\pi i}\int_{C'} \frac{(1+\xi)\,dv}{\rho v \lambda\left(\dfrac{1}{\rho\,|v|}\right)(1-2\mu_1 v+v^2)^{\frac{1}{2}}},$$

where $|\xi| < \frac{1}{2}\epsilon$, if $\rho < \rho_0$.

Now the path C' can be taken so close to the cut in the v-plane that at all points on C' we have

$$\cos a < |v| < l,$$

if l is an arbitrary number slightly greater than 1. Thus at all points of C' we have

$$\lim_{\rho=0} \lambda\left(\frac{1}{\rho\,|v|}\right)\Big/\lambda\left(\frac{1}{\rho}\right) = 1,$$

in virtue of the second of Pringsheim's properties as to $\lambda(x)$; and therefore we can find ρ_1 such that

$$\frac{1+\xi}{\lambda\left(\dfrac{1}{\rho\,|v|}\right)} = \frac{1+\xi'}{\lambda\left(\dfrac{1}{\rho}\right)},$$

where $|\xi'| < \epsilon$, if $\rho < \rho_1$. It follows that

$$\lim_{\rho=0}\left[\rho\lambda\left(\frac{1}{\rho}\right)\left(\sum_0^\infty \lambda_n X_n\right)\right] = \frac{1}{2\pi i}\int_{C'}\frac{dv}{v(1-2\mu_1 v+v^2)^{\frac{1}{2}}},$$

because the subject of integration is continuous at all points of C'. The last integral is $P_0(\mu_1) = 1$, so that

$$\lim_{\rho=0}\left[\rho\lambda\left(\frac{1}{\rho}\right)\left(\sum_0^\infty \lambda_n X_n\right)\right] = 1.$$

In like manner we can show from Pringsheim's results as to $\Sigma n^{p-1}\lambda_n^{\pm 1}x^n$ that

$$\lim_{\rho=0}\rho^p\left[\lambda\left(\frac{1}{\rho}\right)\right]^{\mp 1}\left(\sum_0^\infty n^{p-1}\lambda_n^{\pm 1}X_n\right) = \Gamma(p)\,.\,P_{p-1}(\mu_1) \quad (p>0).$$

From this, by an obvious modification of the discussion in § 7, we deduce that

If $a_n \simeq g n^{p-1}\lambda_n^a$ (*where* $p>0$, $a = -1$, 0, or $+1$), *then*

$$\lim_{\rho=0}\rho^p\left[\lambda\left(\frac{1}{\rho}\right)\right]^{-a}\left(\sum_0^\infty a_n X_n\right) = \Gamma(p)\,.\,g\,.\,P_{p-1}(\mu_1);$$

while, if $a_n \simeq g(\lambda_n - \lambda_{n-1})$, *or if* $a_n \simeq g\lambda'(n)$, *then*

$$\lim_{\rho=0}\left[\lambda\left(\frac{1}{\rho}\right)\right]^{-1}\left(\sum_0^\infty a_n X_n\right) = g.$$

We can obtain an independent treatment of the series $\sum_{2}^{\infty} \log n X_n$, by differentiating the series $\sum_{1}^{\infty} n^p X_n$ with respect to p; now the leading term (in the second series) is

$$\frac{\Gamma(1+p)}{\rho^{1+p}} r^p P_p(\mu_2),$$

by § 3 (B). Thus, if we differentiate* with respect to p, we get, as the leading term in $\Sigma \log n X_n$,

$$\frac{1}{\rho}\left[\Gamma'(1)-\log\rho+\log\tfrac{1}{2}(1+\mu_2)\right];$$

and consequently $\lim_{\rho=0} \rho\,[\log(1/\rho)]^{-1}\left(\sum_{2}^{\infty}\log n X_n\right) = 1,$

in agreement with the theorem at the foot of the last page (with $a = 1$, $p = 1$).

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* Since $$P_p(\mu_2) = 1-\frac{p(p+1)}{(1\,!)^2}\frac{1-\mu_2}{2}+\frac{(p-1)p(p+1)(p+2)}{(2\,!)^2}\left(\frac{1-\mu_2}{2}\right)^2 - \frac{(p-2)(p-1)p(p+1)(p+2)(p+3)}{(3\,!)^2}\left(\frac{1-\mu_2}{2}\right)^3+\dots,$$

we have $$\left(\frac{\partial P_p}{\partial p}\right)_{p=0} = -\left[\frac{1-\mu_2}{2}+\tfrac{1}{2}\left(\frac{1-\mu_2}{2}\right)^2+\tfrac{1}{3}\left(\frac{1-\mu_2}{2}\right)^3+\dots\right]$$
$$= \log\left(1-\frac{1-\mu_2}{2}\right) = \log\frac{1+\mu_2}{2}.$$

THE CANONICAL FORMS OF THE TERNARY SEXTIC AND QUATERNARY QUARTIC

By A. C. DIXON.

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LET Q be a homogeneous sextic in x, y, z; it is known* that there exist linear expressions $L_1, L_2, \ldots$ and constants $A_1, A_2, \ldots$ such that Q is identically equal to

$$\sum_1^{10} A_r L_r^6,$$

and, further, that L_{10} may be arbitrarily chosen. I propose to shew how the reduction is to be actually carried out, and to find the number of ways in which it is possible.

Since a cubic can be described through nine points, there must be a contracubic, U, such that

$$\Delta^3 U L_r^n = 0 \quad (r = 1, 2, \ldots, 9),$$

and therefore
$$\Delta^3 U Q = A_{10} \Delta^3 U L_{10}^6,$$

a simple multiple of L_{10}^3. The ratios of the ten coefficients in U are uniquely determined by the fact that $\Delta^3 UQ$ is to be a constant multiple of L_{10}^3, and the value of A_{10} is thus also uniquely determined.

Hence the problem is to reduce the known sextic $Q - A_{10}L_{10}^6$, which is destroyed by the known cubic operator $\Delta^3 U$, to the form

$$\sum_1^9 A_r L_r^6.$$

Now, if we take 18 arbitrary tangents $L'_1, \ldots, L'_{18}$ to the contracubic U, we can express $Q - A_{10}L_{10}^6$ in the form

$$\sum_1^{18} A'_r L'^6_r.$$

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* See, for instance, Palatini (*Rom. Acc. Lincei, Rendiconti*, Vol. XII.). For the notation see a paper on the the ternary quintic and septimic by Dr. Stuart and the writer (*ante*, p. 160).

With 19 arbitrary tangents or with 18 that touch a contrasextic there is a syzygy of the form
$$\Sigma A_r L_r^6 = 0.$$

We shall discuss this syzygy and deduce a method for reducing
$$\sum_1^{18} A'_r L'^6_r$$
to an expression of the same form containing only nine terms.

Since L_r is a tangent to a given contracubic, we may, by a suitable linear transformation, put
$$L_r = (x - y\wp u_r - \wp' u_r) = \theta . \sigma(u_r - a)\sigma(u_r - b)\sigma(u_r + a + b)/\sigma^3 u_r,$$
where a, b, θ depend on x, y.

Now let $v_1, \ldots, v_{18}$ be any 18 arguments whose sum is 0 and let
$$u_1 + u_2 + \ldots + u_{19} = w;$$
the function
$$\sigma(u-w)\prod_1^{18}\sigma(u-v_r)\Big/\prod_1^{19}\sigma(u-u_r)$$
is doubly periodic in u and its poles are $u_1, u_2, \ldots, u_{19}$. Hence the sum of the residues
$$\frac{\sigma(u_1-w)\prod_1^{18}\sigma(u_1-v_r)}{\prod_2^{19}\sigma(u_1-u_r)} + \ldots = 0.$$

Putting $v_1 = v_2 = \ldots = v_6 = a, \quad v_7 = v_8 = \ldots = v_{12} = b,$
$$v_{13} = \ldots = v_{18} = -a-b,$$
we have here the syzygy
$$\sum_1^{19} A_r L_r^6 = 0,$$
and it appears that
$$A_r = \sigma(u_r - w)(\sigma u_r)^{18}\Big/\prod_{\substack{s=1,2,\ldots,19\\ s\neq r}}\sigma(u_r - u_s).$$

The next thing is to find $u_1, u_2 \ldots, u_9$ when $u_{10}, \ldots, u_{19}$ and the ratios $A_{10} : A_{11} : \ldots : A_{19}$ are given. Let $u_1 + u_2 + \ldots + u_9 = \alpha$, $u_{10} + \ldots + u_{19} = \beta$, and consider the function
$$F(u) \equiv \sigma^2(u-\beta)\prod_1^9\sigma(u-u_r)\Big/\sigma(u-\alpha-\beta)\prod_{10}^{19}\sigma(u-u_r).$$
$F(u)$ is doubly periodic and its residue at the pole u_r $(r = 10, \ldots, 19)$ is
$$B_r = \frac{1}{A_r}(\sigma u_r)^{18}\sigma^2(u_r - \beta)\Big/\prod_{\substack{s=10,\ldots,19\\ s\neq r}}\sigma^2(u_r - u_s).$$

has one other pole, $\alpha+\beta$, and thus

$$F(u) = \sum_{r=10}^{19} B_r\{\zeta(u-u_r)-\zeta(u-\alpha-\beta)\}+B,$$

here B is constant.

We also have $F(\beta)=0$, $F'(\beta)=0$. From the second of these

$$\sum_{10}^{19} B_r\{\wp(\beta-u_r)-\wp\alpha\} = 0,$$

hich gives $\wp\alpha$ uniquely, since β and the ratios $B_{10}:B_{11}:\ldots:B_{19}$ are iown. There are two distinct values of α; taking either of them, we id B from the condition

$$F(\beta) = 0,$$

id then $u_1, u_2, \ldots, u_9$ are the zeros of $F(u)$ other than β.

Hence it follows that the sum of ten terms of the form AL^6 can be duced to nine in two ways. The two sets of nine must be tangents the same contrasextic; this follows either from the syzygy * or from e fact that the two values of α are equal and opposite.

The sum of 18 terms can be reduced successively to 17, 16, ..., and stly to 9. Hence the original problem is solved and the number of lutions is two.

The reduction from 18 to 9 may be accomplished directly as follows:—

If $n>18$, there will be $n-18$, say m, syzygies

$$\sum_1^n A_r L_r^6 = 0.$$

) exhibit them, take the doubly periodic function

$$\prod_1^m \sigma(u-w_r)\prod_1^{18}\sigma(u-v_r)\Big/\prod_1^n\sigma(u-u_r),$$

iere $\Sigma v_r = 0$ as before, $\Sigma w_r = \Sigma u_r$. The sum of the residues is again ro, and we deduce, as before, that

$$A_r = \prod_{s=1}^m \sigma(u_r-w_s)(\sigma u_r)^{18}\Big/\prod_{\substack{s=1,2,\ldots,n\\ s\neq r}}\sigma(u_r-u_s).$$

ere $w_1, w_2, \ldots, w_{m-1}$ are arbitrary.

To make the reduction we must take $n=27$, suppose $u_{10}, u_{11}, \ldots, u_n$ d $A_{10}:A_{11}:\ldots:A_n$ given, and find the values of $u_1, u_2, \ldots, u_9$, , $w_2, \ldots, w_m$.

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* It could be seen from the beginning that not more than two solutions were possible; for, ve had $\sum_1^9 AL^6 = \sum_1^9 A'L'^6 = \sum_1^9 A''L''^6$, the lines L would be residual to L' and also to L'', and the s L' to L''. Hence each set of nine would be residual to itself and would not be of full ierality.

Consider the function

$$F(u) \equiv \prod_1^{2m} \sigma(u-\beta_r) \prod_1^{9} \sigma(u-u_r) \Big/ \prod_1^{m} \sigma(u-w_r) \prod_{10}^{n} \sigma(u-u_r),$$

where $\beta_1, \beta_2, \ldots, \beta_{2m}$ are any fixed arguments whose sum is $2(u_{10}+\ldots+u_n)$. We have

$$F(u) = \sum_{10}^{n} B_r \zeta(u-u_r) + \sum_1^m C_r \zeta(u-w_r) + C,$$

where $$A_r B_r = (\sigma u_r)^{18} \prod_{s=1}^{2m} \sigma(u_r-\beta_s) \Big/ \prod_{\substack{s=10,\ldots,n \\ s\neq r}} \sigma^2(u_r-u_s),$$

so that the ratios of $B_{10}, B_{11}, \ldots, B_n$ are known, while $C, C_1, C_2, \ldots$ are to be determined.

Let $$\sum_{10}^{n} B_r \{\zeta(u-u_r)-\zeta u\} = G(u),$$

$$\sum_1^m C_r \{\zeta(u-w_r)-\zeta u\} + C = H(u),$$

so that $$G(u)+H(u) = F(u).$$

Then $G(u)$ is a known function, and $H(u)$ is a doubly periodic function, of which we know the following facts:—

(1) $H(\beta_r) = -G(\beta_r) \quad (r = 1, 2, \ldots, 2m)$;

(2) $H(0)$ is infinite, the residue being the known quantity ΣB_r $(= -\Sigma C_r)$;

(3) $H(u)$ has $m+1$ poles in all.

Hence we may put

$$H(u) = \frac{M+M_0 \wp u + M_1 \wp' u + \ldots + M_m \wp^{(m)} u}{N+N_0 \wp u + N_1 \wp' u + \ldots + N_m \wp^{(m)} u};$$

from (2) we have $$N_m = 0$$

and $$(m+1) M_m + \Sigma B_r N_{m-1} = 0;$$

from (1) we have $2m$ other equations linear in the $2m+4$ unknowns $M, N, \ldots$. Thus all the coefficients are determined in terms of two of them, say M, N and $H(u) = \{MP_1(u)+NP_2(u)\}/\{MQ_1(u)+NQ_2(u)\}$, where P_1, P_2, Q_1, Q_2 are known functions with poles at 0 of orders $m+2$, $m+2$, $m+1$, $m+1$. To satisfy (3) we must make the numerator and denominator of $H(u)$ have a common zero; this will be a zero of

$$P_1 Q_2 - P_2 Q_1.$$

Now $P_1 Q_2 - P_2 Q_1$ has the $2m$ known zeros $\beta_1, \beta_2, \ldots, \beta_{2m}$, and its pole 0 is only of order $2m+2$. Hence it has two other zeros, say γ_1, γ_2.

Taking either of these, γ, we can complete the determination of $H(u)$. Then $u_1, u_2, \ldots, u_9$ are fixed as the unknown roots of the equation

$$G(u)+H(u) = 0,$$

and when they are determined the problem is solved: $w_1, w_2, \ldots, w_m$ are the zeros, other than γ, of the denominator of $H(u)$. The two values of Σw are $-\gamma_1$, $-\gamma_2$, whose sum is $\Sigma\beta$ or $2\sum_{10}^{n} u_r$. Hence, as before, the two values of $\sum_1^9 u_r$ are equal and opposite.

The use of elliptic functions in the discussion is convenient, but not essential. As a matter of actual working, the process of reduction is algebraic.

The Quaternary Quartic.

Let Q be a homogeneous quartic in x, y, z, w. The problem is to reduce it to the form $\sum_1^{10} A_r L_r^4$, where L_r is linear (see Reye, *Crelle's Journal*, Vol. LXXVIII., pp. 123–9). Here, again, L_{10} may be chosen at will, and the quadratic operator $\Delta^2 U$ (notation as in the ternary case) which destroys L_r^n $(r = 1, 2, \ldots, 9)$ is determined uniquely by the fact that it reduces Q to a multiple of L_{10}^2; A_{10} is also determined. Then L_9 can be chosen at will among the tangent planes to $U = 0$; the quadratic operator $\Delta^2 V$ which reduces Q to a multiple of L_9^2 can be found, as also A_9, and again uniquely.

We now have a known quartic $Q-A_9L_9^4-A_{10}L_{10}^4$ destroyed by two known quadratic operators $\Delta^2 U$, $\Delta^2 V$. Such a quartic can be put in the form $\sum_1^{16} A_r' L_r'^4$ where $L_1', \ldots, L_{16}'$ are arbitrary common tangent planes to $U = 0$, $V = 0$. There would be a syzygy $\Sigma AL^4 = 0$ with 17 such planes, or with 16 if they all touched a contraquartic surface.

Here we have $L_r \propto (z-x\wp u_r-y\wp' u_r-\wp'' u_r)$ after a suitable linear transformation. The syzygy may be discussed as before, and we again conclude that Reye's reduction is possible in two ways.

ON PERPETUANTS AND CONTRA-PERPETUANTS

By E. B. ELLIOTT.

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[SINCE the reading of this paper the desirability of a bibliographical preface to it has been urged upon me. The following are salient writings on the subject, and they present a number of different methods of investigation, none of which can be regarded as superseded by those who would appreciate aright the problem of the enumeration and specification of perpetuant systems.

1. Sylvester. "On Subinvariants, *i.e.*, Semi-Invariants to Binary Quantics of an Unlimited Order": *Amer. Journ.*, v. (1882), pp. 79–136.
2. Cayley. "A Memoir on Seminvariants": *Amer. Journ.*, VII. (1885), pp. 1–25.
3. MacMahon. "On Perpetuants": *Amer. Journ.*, VII. (1885), pp. 26–46.
4. ——— "A Second Paper on Perpetuants": *Amer. Journ.*, VII. (1885), pp. 259–263.
5. Hammond. "On Perpetuants, with Applications to the Theory of Binary Quantics": *Amer. Journ.*, VIII. (1886), pp. 105–126.
6. Stroh. "Ueber die symbolische Darstellung der Grundsyzyganten einer binären Form sechster Ordnung und eine Erweiterung der Symbolik von Clebsch": *Mathematische Annalen*, XXXVI. (1890).
7. Cayley. "On Symmetric Functions and Seminvariants": *Amer. Journ.*, XV. (1893), pp. 1–75.
8. MacMahon. "The Perpetuant Invariants of Binary Quantics": *Proc. Lond. Math. Soc.*, XXVI. (1895), pp. 262–284.
9. ——— "Seminvariants of Binary Quantics, the order of each Quantic being Infinite": *Camb. Phil. Soc. Trans.*, XIX. (1901), pp. 234–248.
10. Grace. "Types of Perpetuants": *Proc. Lond. Math. Soc.*, XXXV. (1902), pp. 107–111.
11. ——— "On Perpetuants": *Proc. Lond. Math. Soc.*, XXXV. (1902–3), pp. 319–331.
12. Wood. "On the Irreducibility of Perpetuant Types": *Proc. Lond. Math. Soc.*, Ser. 2, Vol. I. (1904), pp. 480–484.
13. ——— "On the Unique Expression of a Quantic of any Order in any Number of Variables, with an Application to the Theory of Binary Perpetuants": *Proc. Lond. Math. Soc.*, Ser. 2, Vol. 2 (1904), pp. 70–87.
14. ——— "Perpetuant Syzygies of Degree Four": *Proc. Lond. Math. Soc.*, Ser. 2, Vol. 2 (1904), pp. 144–149.
15. Young and Wood. "Perpetuant Syzygies": *Proc. Lond. Math. Soc.*, Ser. 2, Vol. 2 (1904), pp. 221–265.
16. Elliott. "An Integration Theorem as to Rational Integral Functions, with the Bearing on the Theory of Forms": *Quart. Journ.*, XXXVI. (1904), pp. 124–139.
17. Wood. "On the Reducibility of Covariants of Binary Quantics of Infinite Order": *Proc. Lond. Math. Soc.*, Ser. 2, Vol. 2 (1904), pp. 352–370.
18. ——— "On the Reducibility of Covariants of Binary Quantics of Infinite Order," Part II.: *Proc. Lond. Math. Soc.*, Ser. 2, Vol. 3 (1905), pp. 316–333.
19. ——— "Alternative Expressions for Perpetuant Type Forms": *Proc. Lond. Math. Soc.*, Ser. 2, Vol. 3 (1905), pp. 334–344.
20. Young. "On Relations between Perpetuants": *Camb. Phil. Soc. Trans.*, Vol. XX. (1905).

In 1 the term *perpetuant* is introduced, the idea being that, in the consideration of seminvariants which are irreducible in terms of other seminvariants of extents greater as well as not greater than their own, there is involved, not the whole of the old severe question as to what seminvariants are leaders of irreducible covariants of an assigned quantic or system of quantics, but the perpetually relevant simpler and self-contained question as to what seminvariants always lead irreducible covariants of quantics with orders as great as we please, to which they can appertain. These perpetuants are enumerated up to degree 4, after which progress is blocked by difficulties as to syzygies. In 2 the degrees 5 and 6 are dealt with, without complete success. In 3 the difficulties of degree 5 were conquered; and in 4 MacMahon devised a method which enabled him to state with some confidence that for every value of n the correct numerical generating function is that written down in § 1 below. In 5 the accuracy of this for $n = 6$ and 7 was established.

In 6 Stroh introduced a new and powerful method, and, on certain very natural but unproved assumptions, justified MacMahon's presumption in all its generality. His new symbolism, given as purely umbral, is effectively the same as that of the differential operators θ used below. The beginnings of the use of the operators θ themselves are, I believe, to be found in a paper by Kempe (*Proc. Lond. Math. Soc.*, Vol. xxiv., 1893, p. 102) which has no direct reference to perpetuant systems. In 7 Cayley associates, examines, and elucidates the methods of MacMahon and Stroh in great detail. In 8 and 9 MacMahon adopts the theory of Stroh, brings to bear on its conclusions his own discovery of the one-to-one correspondence between seminvariants and non-unitary symmetric functions, and succeeds in exhibiting in a notation of partitions all the individuals of the complete system of perpetuants with which he is concerned, in 8 for the case of a given degree in one set, and in 10 for given partial degrees in different sets. His investigations thus reach their climax.

No. 10, strangely enough, marks the first determined effort to bring the orthodox Clebschian symbolism to bear on the subject. It deals with perpetuants of n unit degrees in n sets. In 11 there is at the outset an introduction of the idea, hardly thought worth noticing hitherto, of alternative complete systems, and then a passage to perpetuants of degree n in one set, and to those of assigned partial degrees in different sets. The numerical generating functions of MacMahon's 4 and 9 are arrived at from conclusions which effectively afford real generating functions in umbral symbolism. In 12 a logical defect in the earlier theory is remedied, so far as perpetuants of n unit degrees in n sets are concerned, by a proof that all forms retained as no doubt irreducible are so in reality.

(It is believed that the paper here prefaced has the same efficacy for any given degrees, when taken in connection with 16, and with the fact that, if one system obtained as complete is non-redundant, so also must be any alternative complete system with the same number of individuals.) The remaining papers of the list, except those dealing with syzygies, are to a great extent concerned with the possible vast choice between alternative complete perpetuant systems. They depend largely on the notice in 16 and 17 of the critical importance of the question whether a given rational integral function is or is not annihilated by a product of simple linear operators, denoted in 16, and in the paper now to follow, by ΠD.]

I. *Perpetuants.*

1. A theorem proved by Mr. Grace* may be stated as follows, when translated from the symbolical notation ordinarily used in the invariant theory of binary forms into the notation of differential operators, which I prefer and have employed † in exhibiting systems of perpetuant types. A complete system of perpetuants of degree n in a single set of coefficients $a_0, a_1, a_2, \ldots$—*i.e.*, a system of seminvariants of degree n in the set, with the property that no linear function of them is a linear function of products of seminvariants of lower degrees, but that all seminvariants of degree n can be linearly expressed in terms of such products and of individuals included in the system—is exactly specified by the system of operating products which are terms in the expansion, in products of powers of the δ's, of

$$\frac{\delta_1^{2^{n-2}}\delta_2^{2^{n-3}}\ldots\delta_{n-2}^2\delta_{n-1}}{(1-\delta_1^2)(1-\delta_1^2\delta_2)(1-\delta_1^2\delta_2\delta_3)\ldots(1-\delta_1^2\delta_2\delta_3\ldots\delta_{n-1})}.$$

Here, for $r = 1, 2, 3, \ldots, n-1$,

$$\delta_r \equiv \theta_r - \theta_{r+1},$$

in which, for $r = 1, 2, 3, \ldots, n$,

$$\theta_r \equiv a_1^{(r)}\frac{\partial}{\partial a_0^{(r)}} + a_2^{(r)}\frac{\partial}{\partial a_1^{(r)}} + a_3^{(r)}\frac{\partial}{\partial a_2^{(r)}} + \cdots,$$

and the explicit form of the seminvariant specified by any function of the δ's is what we arrive at by operating with the function on the product

$$a_0^{(1)}a_0^{(2)}\ldots a_0^{(n)},$$

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* Paper 11.

† Paper 16. For independent and almost simultaneous investigations covering much the same ground in ordinary notation, cf. P. W. Wood, *Proc. Lond. Math. Soc.* (Papers 17, 19).

and eventually, in the type seminvariant arrived at, replacing all the n sets of letters $a_0^{(r)}$, $a_1^{(r)}$, $a_2^{(r)}$, ..., for $r = 1, 2, \ldots, n$, by the single set $a_0, a_1, a_2, \ldots$.

This accords with the fact that the number of weight w in a complete system of perpetuants of degree n, in one set of coefficients, is the coefficient of x^w in the expansion of

$$x^{2^{n-1}-1}/(1-x^2)(1-x^3)\ldots(1-x^n).$$

Mr. Grace also obtained a complete system of perpetuants of given partial degrees in two sets of coefficients, the number of those of weight w in a complete system of partial degrees m, n being, in accordance with a previous result of MacMahon's,

$$x^{2^{m+n-1}-1}/(1-x^2)(1-x^3)\ldots(1-x^m)(1-x)(1-x^2)\ldots(1-x^n).$$

His method is to examine all symbolic products which remain in a linearly independent system, selected from a formally full system by systematic application of the fundamental identity which, in my notation, is

$$(\theta_2-\theta_3)+(\theta_3-\theta_1)+(\theta_1-\theta_2) = 0,$$

and further reduce their effective number by consideration of identities which must hold among them in consequence of interchanges in them of symbols which eventually refer to the same set of coefficients being lawful.

It occurs to me that a more symmetrical and less searchingly analytic method, based on the use of symmetric functions, ought to suffice for the discovery of complete systems equivalent in their aggregates to those at which Mr. Grace has arrived, and to have suggestive value.

2. A little recapitulation from my paper in the *Quarterly Journal* (*loc. cit.*) is needed for clearness. I showed that, if R is a function of the differences of $\theta_1, \theta_2, \ldots, \theta_n$ of $2^{n-1}-1$ dimensions, which is not annihilated by the product, ΠD_θ, of the $2^{n-1}-1$ distinct linear operators that are parts of

$$\frac{\partial}{\partial\theta_1}+\frac{\partial}{\partial\theta_2}+\frac{\partial}{\partial\theta_3}+\ldots+\frac{\partial}{\partial\theta_n}$$

and do not contain $\dfrac{\partial}{\partial\theta_1}$, then the system of operators included in

$R\times$all linearly independent functions of differences of $\theta_1, \theta_2, \ldots, \theta_n$

exactly specifies a complete system of perpetuants of the first degree in each of n different sets of coefficients, *i.e.*, a system of perpetuant types for degree n.

For R may, for instance, be taken either of Grace's two functions

$$(\theta_1-\theta_2)^{2^{n-2}}(\theta_1-\theta_3)^{2^{n-3}} \dots (\theta_1-\theta_{n-1})^2(\theta_1-\theta_n),$$

$$(\theta_1-\theta_2)^{2^{n-2}}(\theta_2-\theta_3)^{2^{n-3}} \dots (\theta_{n-2}-\theta_{n-1})^2(\theta_{n-1}-\theta_n);$$

but some prominence was given to the lawfulness of taking

$$R \equiv (\theta_1+\theta_2+\dots+\theta_n-n\theta_1)^{2^{n-1}-1}.$$

This last has advantages, with symmetricization in view, but it will be noticed that it has not the advantage of introducing the individual θ's to comparatively low degrees, which attaches to Grace's second form: that form is better, and probably the best possible with n general, when it is desired to exhibit a complete perpetuant system without going on to any unnecessary *extent* in any series $a_0, a_1, a_2, \dots$ of coefficients.

Now take $\phi_r \equiv \theta_1+\theta_2+\dots+\theta_n-n\theta_r \quad (r = 1, 1, 3, \dots, n).$

Instead of $\phi_1^{2^{n-1}-1}$, we may take for R any $\phi_r^{2^{n-1}-1}$ or, again,

$$\phi_1^{2^{n-1}-1}+\phi_2^{2^{n-1}-1}+\dots+\phi_n^{2^{n-1}-1}.$$

In fact, every $\Pi D_\theta \phi_r^{2^{n-1}-1}$ is the same non-vanishing constant as $\Pi D_\theta \phi_1^{2^{n-1}-1}$. For, if in ΠD_θ as formed above we replace each factor involving $\frac{\partial}{\partial\theta_r}$ by *minus* the complementary part of

$$\frac{\partial}{\partial\theta_1}+\frac{\partial}{\partial\theta_2}+\dots+\frac{\partial}{\partial\theta_r},$$

which annihilates every function of differences, we produce $(-1)^{2^{n-2}}$ times the product of those parts of the sum which do not involve $\frac{\partial}{\partial\theta_r}$; and we take $n > 2$.

We notice that every difference of two θ's is one n-th of the difference of the two corresponding ϕ's, so that functions of differences of θ's are functions of differences of ϕ's, and so are functions of ϕ's. Moreover, every ϕ is a sum of differences of θ's; and the sum of all the ϕ's vanishes, so that one of them may be replaced by *minus* the sum of all the others. Thus linearly independent functions of $n-1$ of the ϕ's, linearly independent functions of differences of the ϕ's, and linearly independent functions of differences of the θ's are equally numerous. It follows that

$$(\text{any form of } R)/(1-\phi_1)(1-\phi_2) \dots (1-\phi_{n-1}),$$

equally with

$$(\text{any form of } R)/[1-(\theta_1-\theta_2)][1-(\theta_1-\theta_3)] \dots [1-(\theta_1-\theta_n)]$$

and $(\text{any form of } R)/[1-(\theta_1-\theta_2)][1-(\theta_2-\theta_3)] \dots [1-(\theta_{n-1}-\theta_n)],$

specifies by the terms in its expansion a complete system of perpetuants of n unit degrees in n sets.

All alike give the correct generating function $\dfrac{x^{2^{n-1}-1}}{(1-x)^{n-1}}$ for numbers of perpetuant types of the various possible weights, of which $2^{n-1}-1$ is the least.

3. Every reducible seminvariant in n different sets either remains reducible or vanishes when the sets are identified. Thus every perpetuant of degree n in one set is the result of identification of sets in a perpetuant of unit degrees in n sets. A complete system of perpetuants of degree n in one set cannot then contain more, for any weight, than persist—*i.e.*, do not vanish after identification of the sets—in either of the complete systems above specified.

We will now see that

$$\frac{\Sigma\phi^{2^{n-1}-1}}{(1-p_2)(1-p_3)\,\ldots\,(1-p_n)},$$

in which p_r denotes the sum of the products r together of $\phi_1, \phi_2, \ldots, \phi_n$, and in which the numerator may be replaced by any other of the forms of R, exactly includes the aggregate of perpetuants which persist. When we have also seen that the persistents remain linearly independent after the identifications, and that no linear function of them loses its property of irreducibility, we shall have established that the aggregate constitutes a complete system of perpetuants of degree n in one set of coefficients.

Now, first, if $F(\phi_1, \phi_2, \ldots, \phi_n)\Sigma\phi^{2^{n-1}-1}$, or, more generally,

$$F(\phi_1, \phi_2, \ldots, \phi_n)\Sigma R,$$

specifies one of the perpetuants of unit degrees in n sets, the result of identifying in this perpetuant the sets of coefficients referred to by $\theta_1, \theta_2, \ldots, \theta_n$ is the same as the result similarly obtained from the perpetuant specified by any rearrangement of the θ's in the same operator; and any rearrangement of the θ's means the same rearrangement of the ϕ's. Consequently $F(\phi_{r_1}, \phi_{r_2}, \ldots, \phi_{r_n})\Sigma\phi^{2^{n-1}-1}$, where $r_1, r_2, \ldots, r_n$ are $1, 2, \ldots, n$ in any order, leads to the same result after the identifications; and so does $\dfrac{1}{n!}\Sigma F(\phi_1, \phi_2, \ldots, \phi_n)\Sigma\phi^{2^{n-1}-1}$, which is symmetric. It follows that $S(\phi_1, \phi_2, \ldots, \phi_n)\Sigma\phi^{2^{n-1}-1}$, where S is the most general symmetric function of dimensions $w-2^{n-1}+1$ in its n arguments, includes enough of the complete system of perpetuants of n unit degrees to provide all those of weight w which persist in the case of one degree n.

But $S(\phi_1, \phi_2, \ldots, \phi_n)$ can be rationally and integrally expressed in

terms of the elementary symmetric functions

$$\Sigma\phi = p_1 = 0,\quad \Sigma\phi_1\phi_2 = p_2,\quad \Sigma\phi_1\phi_2\phi_3 = p_3,\quad \ldots,\quad \phi_1\phi_2\ldots\phi_n = p_n.$$

Consequently $\qquad G(p_2, p_3, \ldots, p_n)\Sigma\phi^{2^{n-1}-1},$

where G has full generality as a rational integral function of its arguments of weight $w-2^{n-1}+1$ in suffixes, includes all perpetuants of weight w which persist.

No particular G is such that the above, looked at as a function of θ's, vanishes. For $p_1 = 0$ is the only relation connecting the n functions ϕ of the θ's. Moreover, no particular G is such that the result of operating with the above on the product of n leading coefficients, and then identifying the sets of coefficients, vanishes. For every $G\Sigma\phi^{2^{n-1}-1}$ is a non-vanishing symmetric function of the θ's, and every non-vanishing symmetric function leads to a non-vanishing result after the operation and the identifications: if, for instance, the symmetric function contains a part $k\Sigma\theta_1^{l_1}\theta_2^{l_2}\ldots\theta_n^{l_n}$, as it must for some values of the l's and k, the final result contains a term $n!\,ka_{l_1}a_{l_2}\ldots a_{l_n}$, with no other against which it can cancel. Thus the full number of perpetuants in n sets of which the operators have the above symmetric form produces the same exact number of linearly independent seminvariants, including all perpetuants, of degree n in one set.

Lastly, the full number are irreducible after the identifications, as well as beforehand. For suppose, if possible, that a particular

$$g(p_2, p_3, \ldots, p_n)\Sigma\phi^{2^{n-1}-1}$$

produces a reducible seminvariant γ of degree n in the eventual single set of coefficients. From γ can be derived by $n-1$ polarizations a seminvariant γ' of unit degrees in n sets, and this is still reducible. γ' reproduces γ upon identification of sets, by Euler's theorem as to homogeneous functions. It is not specified by $g(p_2, p_3, \ldots, p_n)\Sigma\phi^{2^{n-1}-1}$, as it is reducible. But it presents itself as symmetric in the n sets, and, even if it did not, could be symmetricized by taking a mean, and still reproduce γ. There is then a symmetric $\sigma(\phi_1, \phi_2, \ldots, \phi_n)$ distinct from

$$g(p_2, p_3, \ldots, p_n)\Sigma\phi^{2^{n-1}-1}$$

such that the difference

$$g(p_2, p_3, \ldots, p_n)\Sigma\phi^{2^{n-1}-1}-\sigma(\phi_1, \phi_2, \ldots, \phi_n)$$

produces a vanishing result of operation on $a_0^{(1)}a_0^{(2)}\ldots a_0^{(n)}$ when the sets

are identified. But it has just been seen that no non-vanishing symmetric function can produce a vanishing result. The supposition made is then untenable.

There is then neither incompleteness nor redundancy in the entire system of operators given by expanding

$$\frac{\Sigma\phi^{2^{n-1}-1}}{(1-p_2)(1-p_3)\dots(1-p_n)}$$

for the specification of a complete system of perpetuants of degree n in one set.

The numerator, the specifier of a single perpetuant of the minimum possible weight $2^{n-1}-1$, may, of course, be replaced by any other R of § 2, symmetricized or not. The correct numerical generating function (of § 1) is afforded.

4. In precisely the same way, if we identify, not all the n sets of coefficients, but only $n-1$ of them, namely those referred to by $\theta_2, \theta_3, \dots, \theta_r$, and if we take the elementary symmetric functions

$$q_1 = -\phi_1, \qquad q_2, q_3, \dots, q_{n-1}$$

of $\phi_2, \phi_3, \dots, \phi_n$, we obtain, as specifying a complete system of perpetuants of partial degrees 1, $n-1$ in two sets, the terms of the expansion of

$$\frac{\phi_1^{2^{n-1}-1}}{(1-\phi_1)(1-q_2)(1-q_3)\dots(1-q_{n-1})}.$$

Here the numerator is already symmetrical in $\theta_2, \theta_3, \dots, \theta_n$. It may, of course, be replaced by any more convenient R. The correct numerical generating function according to § 1 immediately results.

It is easy to obtain a complete system for the general case of any assigned partial degrees in assigned sets of coefficients. Take

$$n = \nu+\nu'+\nu''+\dots,$$

and identify the sets referred to by $\theta_1, \theta_2, \dots, \theta_\nu$ with a set $a_0, a_1, a_2, \dots$, those referred to by $\theta_{\nu+1}, \theta_{\nu+2}, \dots, \theta_{\nu+\nu'}$ with a set $a'_0, a'_1, a'_2, \dots$, those referred to by $\theta_{\nu+\nu'+1}, \theta_{\nu+\nu'+2}, \dots, \theta_{\nu+\nu'+\nu''}$ with $a''_0, a''_1, a''_2, \dots$, and so on. Take, as before,

$$\phi_r = \Sigma_n\theta - n\theta_r = \Sigma_n\theta - (\nu+\nu'+\nu''+\dots)\theta_r,$$

for $r = 1, 2, 3, \dots, n$. Denote by Σ_ν symmetric summation in the first set of θ's, *i.e.*, of ϕ's; by $\Sigma_{\nu'}$ symmetric summation in the second set; and so on. Also let $p_1, p_2, \dots, p_\nu$; $p'_1, p'_2, \dots, p'_{\nu'}$; $p''_1, p''_2, \dots, p''_{\nu''}$; ... be

the elementary symmetric functions of the various sets of ϕ's separately. The one relation connecting all the ϕ's is

$$p_1+p_1'+p_1''+\ldots = 0.$$

We observe that every function of the ϕ's which is symmetric in the first ν of them, in the second ν', in the third ν'', &c., may be linearly expressed in terms of products of the form

$$\Sigma_\nu \phi_1^{l_1}\phi_2^{l_2}\ldots\phi_\nu^{l_\nu}.\Sigma_{\nu'}\phi_{\nu+1}^{l_{\nu+1}}\ldots\phi_{\nu+\nu'}^{l_{\nu+\nu'}}.\Sigma_{\nu''}\phi_{\nu+\nu'+1}^{l_{\nu+\nu'+1}}\ldots\phi_{\nu+\nu'+\nu''}^{l_{\nu+\nu'+\nu''}}\ldots,$$

so that it is a rational integral function of $p_1, p_2, \ldots, p_\nu$; $p_1', p_2', \ldots, p_{\nu'}'$; $p_1'', p_2'', \ldots, p_{\nu''}''$; ...; and that, conversely, any rational integral function of these has the various symmetries described. A single one of $p_1, p_1', p_1'', \ldots$ may be omitted without loss of generality, in virtue of $\Sigma p_1 = 0$. Otherwise there is complete linear independence of all products of p's from the various sets. The details of the reasoning of § 3 need not be repeated; but, exactly as there, we find, for a complete system of perpetuants of partial degrees $\nu, \nu', \nu'', \ldots$ in different sets of coefficients, the system obtained by identification of the first ν, the second ν', &c., sets of coefficients in the results obtained from the product of

$$n = \nu+\nu'+\nu''+\ldots$$

leading coefficients by operation on it with the terms of the expansion of

$$\frac{\Sigma\phi^{2^{n-1}-1}}{(1-p_2)(1-p_3)\ldots(1-p_\nu).(1-p_1')(1-p_2')\ldots(1-p_{\nu'}').(1-p_1'')(1-p_2'')\ldots(1-p_{\nu''}'')\ldots},$$

and this gives for the numerical generating function which determines, by the coefficient of x^w in its expansion, the number of those perpetuants of the system which have any possible weight w,

$$\frac{x^{2^{n-1}-1}}{(1-x^2)(1-x^3)\ldots(1-x^\nu).(1-x)(1-x^2)\ldots(1-x^{\nu'}).(1-x)(1-x^2)\ldots(1-x^{\nu''})\ldots}.$$

The form of this is that indicated by the continuation of Mr. Grace's series of theorems. It has also been arrived at by MacMahon (paper 9).

There is, of course, no reason to believe the particular symmetric form of generating function of operators to specify a complete system, which it has been found easy to arrive at above, to be in itself preferable to others which can be formed in great variety. Operators of any one system provide seminvariants which differ from linear functions of seminvariants provided by operators of any other only by reducibles

II. *The Dual Theory.*

5. So far as I know, no attention has been paid to the question whether anything of interest can be obtained by correlating with the theory of perpetuants Hermite's doctrine of reciprocity between degree and extent in systems of seminvariants. My representation appears to be the one best fitted for use in examining the duality; and I proceed to write down a few of the more obvious results of such an examination, confessing in advance that I am disappointed at finding them to be *prima facie* of but small importance, more useful probably in throwing reflected light on perpetuants than intrinsically.

The duality is that which arises from the formal identity of the adequately symmetricized operators in

$$\theta_1,\ \theta_2,\ \ldots,\ \theta_n,$$

which produce seminvariants of total degree n from a product of n leading coefficients, and the expressions in terms of

$$a_1,\ a_2,\ \ldots,\ a_n,$$

the n roots of quantics with n for the sum of their orders, for seminvariants appertaining to those quantics.

For instance, there is one-to-one correspondence between seminvariants of n unit degrees in distinct sets of coefficients and invariants of n distinct forms of unit order; and, again, between seminvariants of degree n in one set of coefficients and expressions in terms of the roots for seminvariants of one n-ic.

Two corresponding seminvariants have the same weight; for a multiplication by an $a_r - a_s$ and an operation with a $\theta_r - \theta_s$ equally increase weight by unity. We are going, as in the last statement, to exclude the case of weight zero, and not always to distinguish between S, the seminvariant of lowest degree provided by an expression in the roots, and the result of multiplying S by any powers of the leading coefficients in the quantics with which it is associated.

Consider a seminvariant, defined by a θ-operator, and let its partial degrees in the different sets of coefficients which it involves be ν, ν', ν'', ... We may consider that the sets referred to by the first ν, the second ν', &c., of the θ's have been identified. The lowest possible order of a quantic with coefficients belonging to the first set to which the seminvariant can appertain is the index of the highest power to which any one of θ_1, θ_2, ..., θ_ν occurs in the operator, when this is symmetricized in θ_1, θ_2, ..., θ_ν, and in the other sets of θ's separately; and similarly as to lowest orders of quantics to whose coefficients the other sets of θ's

respectively refer. The partially symmetricized operator leads, when we replace in it every θ by the corresponding a, to the corresponding root-expression for a seminvariant of a ν-ic, a ν'-ic, a ν''-ic, &c. The lowest degree in the coefficients of the ν-ic of a seminvariant provided by this root-expression is the index of the highest power to which each of $a_1, a_2, \ldots, a_\nu$ enters, *i.e.*, is the same as the lowest order of the quantic to which the degree ν could refer in the case of the corresponding θ-seminvariant; and similarly as to lowest degrees in coefficients of the ν'-ic, the ν''-ic, &c. Other seminvariants afforded by the same root-expression are only results of multiplying the one of lowest partial degrees by powers of the leading coefficients. Those seminvariants of the ν-ic, the ν'-ic, &c., whose partial degrees are $\mu, \mu', \ldots$, and cannot be depressed, correspond to those of partial degrees $\nu, \nu', \ldots$ which belong to a μ-ic, a μ'-ic, &c., but not to lower quantics in the same sets respectively; and hence arises a simple classification which gives Hermite's principle in its generality: that there is a one-to-one correspondence between all seminvariants of partial degrees $\nu, \nu', \ldots$ appertaining to a μ-ic, a μ'-ic, &c., and all seminvariants of partial degrees $\mu, \mu', \ldots$ appertaining to a ν-ic, a ν'-ic, &c., conjugate weights being always equal.

But, in the case of perpetuants of assigned partial degrees $\nu, \nu', \nu'', \ldots$, the idea of orders of quantics referred to is irrelevant. These orders are only taken large enough to provide all the coefficients which occur in the full explicit expressions of the perpetuants in question. For the class of seminvariants of quantics of assigned orders $\nu, \nu', \nu'', \ldots$ which correspond, in the above scheme of duality, to perpetuants of degrees $\nu, \nu', \nu'', \ldots$, I venture to use the name *contra-perpetuants* of those quantics. In respect of contra-perpetuants the idea of degrees in the coefficients of the ν-ic, &c. is irrelevant. The degrees must be made great enough, by powers of the leading coefficients applied as factors to their root-expressions, to make them integral in the coefficients, and need not, but may, be greater. No harm can arise if we take unity for the leading coefficient in each of the ν-ic, the ν'-ic, &c. The necessary least degrees of any contra-perpetuant of which we have obtained the explicit expression in terms of coefficients, subject to such an apparent particularization, will be put in evidence as what are necessary to secure homogeneity in each set of coefficients, retaining integralness.

The prerogative of a contra-perpetuant of a quantic or quantics, among other seminvariants of the quantic or quantics, cannot then have reference to irreducibility in degree. Its reference must be essentially to the quantic or quantics to which the contra-perpetuant as such appertains, and be concerned with their orders.

6. Let us first consider the prerogative, and the specification, of those seminvariants of n linear forms

$$a^{(1)}(x-a_1 y),\quad a^{(2)}(x-a_2 y),\quad \ldots,\quad a^{(n)}(x-a_n y)$$

which are contra-perpetuants of the system of forms.

Seminvariants of the forms, omitting mere products of leading coefficients, are the various homogeneous functions of the differences of $a_1, a_2, \ldots, a_n$, each made integral in the coefficients by any adequate product of powers of $a^{(1)}, a^{(2)}, \ldots, a^{(n)}$ as a factor. The contra-perpetuants of the forms are given in this way (cf. § 2) by those homogeneous functions of the differences which are not annihilated by ΠD_a, which is the product of the $2^{n-1}-1$ linear operators that are parts of the sum

$$\frac{\partial}{\partial a_1}+\frac{\partial}{\partial a_2}+\ldots+\frac{\partial}{\partial a_n}$$

and do not involve $\dfrac{\partial}{\partial a_1}$ (or any other one term we like to choose). The prerogative of these functions of differences is that no one of them can be written as a sum of parts each of which is a product of two factors (both functions of differences, or one perhaps constant) such that one involves some, but not all, of $a_1, a_2, \ldots, a_n$, and the other does not involve any of those letters which are present in the first. In other words, the prerogative of the contra-perpetuants is that, however multiplied by products of leading coefficients of the n linear forms, they cannot be written as sums of parts each of which is a product of a seminvariant of some (but not all) of the n forms into a seminvariant of the remainder.

A complete system of contra-perpetuants of the n linear forms is given, but for leading-coefficient factors which have lower, but not upper, limits of degree, by a complete system of linearly independent functions of the differences of $a_1, a_2, \ldots, a_n$ such that every one of them has the prerogative above explained, and that every function of differences which has the prerogative, but is not a linear function of them, can be expressed as a linear function of them and of other functions of differences not possessing the prerogative.

By § 2 there is great variety of ways in which a complete system of contra-perpetuant functions of the differences can be written down, one of the best consisting of the products of the β's which occur in the expansion of

$$\beta_1^{2^{n-1}-1}/(1-\beta_1)(1-\beta_2)\ldots(1-\beta_{n-1}),$$

where $$\beta_r \equiv a_1+a_2+\ldots+a_n-na_r \quad (r=1, 2, 3, \ldots, n).$$

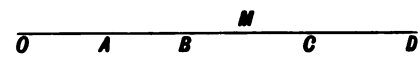
The common factor given in the numerator of this generating function

may be replaced by any R_a of dimensions $2^{n-1}-1$ which is not annihilated by ΠD_a, and in particular by

$$(a_1-a_2)^{2^{n-2}}(a_1-a_3)^{2^{n-3}}\dots(a_1-a_{n-1})^2(a_1-a_n),$$

or by $$(a_1-a_2)^{2^{n-2}}(a_2-a_3)^{2^{n-3}}\dots(a_{n-2}-a_{n-1})^2(a_{n-1}-a_n).$$

It is interesting to take distances from an origin along a line measured numerically by $a_1, a_2, \dots, a_n$, and express our conclusions as facts in the geometry of ranges. For instance, taking $n=4$, we obtain the following:—



(1) Every expression of dimensions less than 7 ($=2^3-1$) in the distances from one another of four collinear points A, B, C, D can be linearly separated into parts in which only two of the points are mentioned, parts in which only three are, and parts which are products of powers of AB and CD, of AC and BD, and of AD and BC.

(2) Every expression of dimensions 7 can be linearly separated into parts as in (1) and a multiple of a single term, which may, for instance, be taken as either $AB^4.AC^2.AD$ or $AB^4.BC^2.CD$ or MA^7, where M is the centre of mean position of A, B, C, D, *i.e.*, the centroid of equal masses at those points: but neither MA^7 nor $AB^4.AC^2.AD$ nor $AB^4.BC^2.CD$ can be separated as in (1).

(3) Every expression of dimensions exceeding 7 can be separated into parts as in (1) and parts of either we like to choose of the forms $MA^{r+7}.MB^s.MC^t$, $AB^{r+4}.AC^{s+2}.AD^{t+1}$, $AB^{r+4}.BC^{s+2}.CD^{t+1}$, with r, s, t zero and positive; but no linear function of terms of either of these last three forms can be separated as in (1).

7. Consider now contra-perpetuants of one n-ic. Their prerogative among seminvariants appertaining to the n-ic is that none of them either is, or can be multiplied by any power of the leading coefficient so as to be made, linearly expressible in terms of products of seminvariants of quantics of lower orders which are complementary factors of the n-ic.

For, as in § 6, the essential fact is that the root-expression for the contra-perpetuant be not annihilated by ΠD_a. This prevents it from being a linear function of products of functions of the differences of complementary sets of the roots a. Now it is symmetric in the roots, and a function of the differences of $a_1, a_2, \dots, a_n$ which is symmetric must, if it be a sum of terms like

(function of differences of $a_1, a_2, \dots, a_\nu$)

$\times$ (function of differences of $a_{\nu+1}, a_{\nu+2}, \dots, a_n$),

be a sum of terms like

$$\frac{1}{\nu!}\,\Sigma(\text{function of differences of } a_1, a_2, \ldots, a_\nu)$$

$$\times\frac{1}{(n-\nu)!}\,\Sigma(\text{function of differences of } a_{\nu+1}, a_{\nu+2}, \ldots, a_n),$$

which is the product of the root-expressions for two seminvariants of complementary factors of the n-ic. Accordingly the prerogative of contra-perpetuants is as stated above.

To specify a complete system of contra-perpetuants of the n-ic we can employ § 3. For instance, we can at once assert that, if $a_1, a_2, \ldots, a_n$ are the roots of the n-ic, and $\beta_1, \beta_2, \ldots, \beta_n$ are written down as in § 6, the root-expressions for a complete system are afforded by the terms in the expansion of

$$\Sigma\beta^{2^{n-1}-1}/(1-p_2')(1-p_3')\ldots(1-p_n'),$$

where $p_1'\,(=0)$, $p_2', p_3', \ldots, p_n'$ are the elementary symmetric functions of $\beta_1, \beta_2, \ldots, \beta_n$.

Accordingly we have the following conclusions.

(1) Every seminvariant of weight less than $2^{n-1}-1$ appertaining to an n-ic can, at any rate when multiplied by a power of the leading coefficient, be linearly expressed in terms of seminvariants of quantics obtained by omitting factors from the n-ic and products of seminvariants of quantics of lower orders than n which are complementary factors of the n-ic.

(2) Every seminvariant of weight $2^{n-1}-1$ appertaining to an n-ic which cannot be expressed as described in (1) can, at any rate when multiplied by a power of the leading coefficient, be given the form of a linear function as in (1) together with a multiple of a single contra-perpetuant seminvariant, which may, for instance, be taken as the one specified by

$$\Sigma\beta^{2^{n-1}-1}, \quad\text{or}\quad \Sigma(a_1-a_2)^{2^{n-2}}(a_1-a_3)^{2^{n-3}}\ldots(a_1-a_{n-1})^2(a_1-a_n),$$

or $$\Sigma(a_1-a_2)^{2^{n-2}}(a_2-a_3)^{2^{n-3}}\ldots(a_{n-2}-a_{n-1})^2(a_{n-1}-a_n);$$

but this single contra-perpetuant cannot be expressed as in (1).

(3) For a weight exceeding $2^{n-1}-1$ there are as many contra-perpetuants in a complete system, linearly independent when made of one degree by powers of the leading coefficient a as factors, *i.e.*, as many seminvariants, of the weight w and appertaining to the n-ic, but not considered as distinct when they differ only by powers of a as factors, which cannot be expressed as in (1), as there are seminvariants, a factors dis-

regarded, appertaining to the n-ic and of weight $w-2^{n-1}+1$; and in terms of these and sums as in (1) all seminvariants of the weight for the n-ic can, at any rate when raised in degree by a factors, be linearly expressed. Such a system of contra-perpetuants has been exhibited above; and others will be noticed presently.

8. To lessen verbiage take, as it was noticed in § 5 that we may,

$$(1, a_1, a_2, \ldots, a_n)(x, y)^n$$

for our n-ic. Notice that, if we deprive it of its second term by putting

$$\frac{x}{y}=\frac{x'}{y}-a_1=\frac{x'}{y}+\frac{1}{n}(a_1+a_2+\ldots+a_n),$$

we give it a form $\qquad (1, 0, a_2', \ldots, a_n')(x', y)^n,$

of which the n roots are $-1/n$ times $\beta_1, \beta_2, \ldots, \beta_n$ respectively.

Thus $a_2', a_3', \ldots, a_n'$ are multiples of the $p_2', p_3', \ldots, p_n'$ of the generating function in § 7, and may be put in place of them. Also they are the results of putting 1 for a_0 in the unreduced protomorph seminvariants

$$A_2=a_0a_2-a_1^2, \qquad A_3=a_0^2a_3-3a_0a_1a_2+2a_1^3,$$
$$A_4=a_0^3a_4-4a_0^2a_1a_3+6a_0a_1^2a_2-3a_1^4, \quad \ldots,$$
$$A_n=\frac{1}{a_0}(a_n, a_{n-1}, \ldots, a_0)(a_0, -a_1)^n.$$

Consequently, if C denote a contra-perpetuant of least weight $2^{n-1}-1$, either that specified by $\Sigma\beta^{2^{n-1}-1}$, or that by any other proper R_a, a complete system of contra-perpetuants is comprised in the terms of the expansion of

$$C/(1-A_2)(1-A_3)\ldots(1-A_n).$$

The system here presented is one the members of which are of unnecessarily high degrees, and, though the notion of degree is irrelevant to that of a complete system, it would seem best to exhibit a system with members of as low degrees as convenient. Now we only need a generating function of which the expansion shall be

$$C\times \text{all seminvariants of } (1, a_1, a_2, \ldots, a_n)(x, y)^n,$$

the replacing of a_0 by 1 being effective in removing all distinction between seminvariants which only differ by a_0-factors. The right minimum degree of a seminvariant when made homogeneous by the reintroduction of a_0 is that of the highest terms in $a_1, a_2, \ldots, a_n$ present in its expression when $a_0=1$.

We depress degrees in a complete system, for the various weights, by expressing the factor "all seminvariants of $(1, a_1, a_2, \ldots, a_n)(x, y)^n$" in terms of a most reduced system of protomorphs

$$Q_2,\ C_3,\ Q_4,\ C_5,\ Q_6,\ \ldots,\ S_n,$$

which are alternately quadratic and cubic, so that S_n means Q_n or C_n according as n is even or odd, instead of in terms of $A_2, A_3, \ldots, A_n$. Thus the expansion of

$$C/(1-Q_2)(1-C_3)(1-Q_4)(1-C_5)\ldots(1-S_n)$$

comprises the members of a depressed complete system of contra-perpetuants, and this, except in so far as the common factor C is concerned, is the most depressed which can be expected to be attainable in general. It is not the most depressed which exists, but to obtain a perfectly depressed system the probably inaccessible knowledge of a real generating function for seminvariants of the n-ic in which all irreducibles should be represented would be needed.

Notice that the duals of

$$\Sigma\delta_1^2,\ \Sigma\delta_1^2\delta_2,\ \Sigma\delta_1^2\delta_2\delta_3,\ \Sigma\delta_1^2\delta_2\delta_3\delta_4,\ \ldots,$$

where the terms summed are what occur in Grace's denominator of § 1, and the summations are symmetrical in $\theta_1, \theta_2, \ldots, \theta_n$, are the root-expressions for a set of quadratic and cubic protomorphs $Q_2, C_3, Q_4, C_5, \ldots$. We may expect then that a symmetrical perpetuant system closely allied to Grace's unsymmetrical one will have the advantage, among other equally complete and non-redundant systems, of being reduced in extent, in the set of coefficients referred to, to the minimum which there is much hope of arriving at without knowledge of a real generating function for seminvariants of an n-ic in which all irreducibles occur.

But the best numerator, or common factor, R or C for the purpose has yet to be exhibited, for the general n. Resuming the language of the dual theory of contra-perpetuants, the C of lowest degree for the weight $2^{n-1}-1$, which is at present before us, is the

$$\Sigma(a_1-a_2)^{2^{n-2}}(a_2-a_3)^{2^{n-3}}\ldots(a_{n-2}-a_{n-1})^2(a_{n-1}-a_n)$$

provided by Grace's numerator. This cannot be replaced by one of lower degree when $n=4$; but already for $n=5$ its degree 12 is unnecessarily high. It is pointed out to me* that Mr. P. W. Wood has examined all

* Since the reading of this paper.

products of the form

$$(a_1-a_2)^{\mu_1}(a_2-a_3)^{\mu_2}(a_3-a_4)^{\mu_3}(a_4-a_5)^{\mu_4},$$

to ascertain which of them are not annihilated by ΠD_a, for $n=5$. From his list (*Proceedings*, Ser. 2, Vol. 3, p. 318) it is at once clear that there is a choice of suitable contra-perpetuants C of degree 8 only. For instance, we may have

$$C=\Sigma(a_1-a_2)^4(a_2-a_3)^4(a_3-a_4)^4(a_4-a_5)^3.$$

9. Let us exemplify the results of the last two articles by consideration of the case $n=4$, *i.e.*, of contra-perpetuants of the quartic

$$(a, b, c, d, e)(x, y)^4 \equiv a(x-\alpha y)(x-\beta y)(x-\gamma y)(x-\delta y).$$

The degree of $\quad \Sigma(\alpha-\beta)^4(\beta-\gamma)^2(\gamma-\delta)$

is 5, not 6 as would at first appear, for β^6 has a vanishing coefficient in the sum. The expression for the symmetric function is readily seen to be a numerical multiple of

$$\frac{1}{a^5}(a^2d-3abc+2b^3)(ae-4bd+3c^2).$$

In fact the numerator here is the only seminvariant of weight 7 and degree 5 for the quartic. [No seminvariant of weight 7 and degree less than 5 exists.]

Thus a complete system of contra-perpetuants of the quartic is comprised in all products of the form

$$(ac-b^2)^r(a^2d-3abc+2b^3)^{s+1}(ae-4bd+3c^2)^{t+1},$$

with r, s, t zero and positive, *i.e.*, in the terms of the expansion of

$$C_3Q_4/(1-Q_2)(1-C_3)(1-Q_4);$$

and this system consists of members depressed in degree as far as is possible by the methods of general applicability which have been explained.

But for the quartic there is a known real generating function of seminvariants, expressed entirely in irreducibles, namely,

$$(1+C_3)/(1-a)(1-Q_2)(1-Q_4)(1-J),$$

where J is the invariant $ace+2bcd-ad^2-b^2e-c^3$, and Q_4 is the invariant usually called I. Accordingly, as we ignore superfluous powers of a, we are able for the quartic to write down a system of contra-perpetuants, given by the expansion in terms of Q_2, C_3, I, J of

$$IC_3(1+C_3)/(1-Q_2)(1-I)(1-J),$$

the members of which are all of absolutely indepressible degrees. Notice that the numerical generating function, which tells us how many there are in the system for each weight, is given by this real generating function to be

$$\frac{x^7(1+x^8)}{(1-x^2)(1-x^4)(1-x^6)} = \frac{x^7}{(1-x^2)(1-x^8)(1-x^4)},$$

which is what we know ought to be the case.

For a cubic the equally perfect complete system of contra-perpetuants, arrived at in like manner, consists of the terms of the expansion of

$$C_3(1+C_3)/(1-Q_2)(1-\Delta),$$

where Δ is the discriminant.

Reverting to the dual theory of perpetuants of (§ 1), we deduce that, for degrees 4 and 3, complete systems of perpetuants, rendered perfect in that all their members go to no unnecessary extent in the set of coefficients, are given by the expansions respectively of

$$\Sigma\delta_1^2\delta_2\delta_3\,\Sigma\delta_1^2\delta_2(1+\Sigma\delta_1^2\delta_2)/(1-\Sigma\delta_1^2)(1-\Sigma\delta_1^2\delta_2\delta_3)(1-\Sigma\delta_1^2\delta_2\delta_3^2\delta_4),$$

where δ_4 means $\delta_1+\delta_2+\delta_3$, *i.e.*, $\theta_1-\theta_4$, and

$$\Sigma\delta_1^2\delta_2(1+\Sigma\delta_1^2\delta_2)/(1-\Sigma\delta_1^2)(1-\Delta'),$$

where Δ' means the product of the squares of differences of θ_1, θ_2, θ_3, θ_4.

Lastly, from § 7 with $n=4$, we get the following examples of theorems in the geometry of ranges of points:—

(1) Every rational integral expression of dimensions w less than 7 in the mutual distances of four collinear points A, B, C, D, which is symmetrical in its reference to the points, can be written as a sum of zero or numerical multiples of terms of the forms ΣAB^w, $\Sigma AB^r.AC^{w-r}$, $\Sigma AB^r.CD^{w-r}$.

(2) $\Sigma AB^4.AC^2.AD$, $\Sigma AB^4.BC^2.CD$, ΣMA^7, where M is the centre of mean position of A, B, C, D, cannot be expressed as in (1). Every other symmetric expression of dimensions 7 which cannot is the sum of a multiple of either of these three we like and an expression which can.

(3) If $w>7$, no expression of dimensions w such as described which has either ΣMA^7 or $\Sigma AB^4.AC^2.AD$ or $\Sigma AB^4.BC^2.CD$ or

$$\Sigma AB^2.BC\,\Sigma AB^2.BC.CD$$

for a factor can be expressed as in (1); but every other symmetric function can be expressed as the sum of a part as in (1), and a part with either of the above symmetric functions we please as a factor.

10. It would be easy to give the dual interpretation to the general expansions in differential operators, written down in § 4, for complete systems of perpetuants of any partial degrees we like in different sets; but the statement would be only tedious. It is not unlikely, however, that a close examination of the dual conclusions would decide, for instance, whether a system of perpetuants closely allied to that exhibited by Grace for two partial degrees, and naturally led on to in general by his processes, has the same superiority, in respect of lowness of extents in the various sets of coefficients, over other equally complete and non-redundant systems of which the formation is all that is practicable except for particular and very low degrees, as attaches to a system like his for the case of a single set with the degree general.

SOME THEOREMS CONNECTED WITH ABEL'S THEOREM ON THE CONTINUITY OF POWER SERIES

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1. It will probably make the object of this paper more easily intelligible if, at the risk of repeating a certain number of well known facts, I preface it with a brief historical *résumé*.

In his famous memoir on the Binomial Series Abel proved that, *if a series* Σa_n *is convergent, the series* $\Sigma a_n x^n$ *is convergent for all positive values of* x *less than unity, and represents a function* $f(x)$ *which is continuous for all such values of* x, *unity included.**

An alternative proof of Abel's theorem was given later by Dirichlet.†

Stated in the language of the modern theory of functions, Abel's theorem runs: "If a power series in x converges to the sum s at a point P on its circle of convergence, and $f(x)$ is the function represented by the series within the circle, then $f(x)$ tends to the limit s when x tends to P along a radius vector from the origin."

This theorem has proved the starting point for a considerable number of later researches. Stolz was the first to prove that the result still holds if x tends to P along any path which lies entirely within the circle of convergence.‡ At a later date Pringsheim returned to the subject in a very instructive memoir,§ in which he shows that Abel's proof suffices to prove not only the continuity of $f(x)$, but also the *uniform convergence* of the series $\Sigma a_n x^n$ throughout the interval (0, 1). Of this the continuity of $f(x)$ for $x = 1$ is a corollary; but Abel had really proved more than mere continuity, and Pringsheim justly remarks that Dirichlet's proof is inferior to Abel's in that it obscures this fundamental point.

This is not the only direction in which Abel's theorem has been generalised. The property of the special function x^n, upon which Abel's

* *Crelle*, Bd. I.; *Œuvres*, T. I., p. 223.

† *Liouville*, Sér. 2, T. VII.; *Werke*, Bd. II., p. 305.

‡ *Zeitschr. f. Math.*, Bd. XX., p. 370, and Bd. XXIX., p. 127. This statement is somewhat loose; see § 4.

§ *Münchener Sitzungsberichte*, 1897, p. 343.

proof was based, was simply that expressed by the inequality

$$x^n \geqslant x^{n+1} \quad (0 \leqslant x \leqslant 1),$$

and it was at once suggested that similar theorems must hold for more general classes of series of the type $\Sigma a_n f_n(x)$. And, in fact, Dirichlet and Dedekind* arrived at the following results, which for the sake of brevity I state on the hypothesis that the functions $f_n(x)$ are real functions of x defined for the interval $0 \leqslant x \leqslant 1$.

(*a*) If $$f_n(x) \geqslant f_{n+1}(x) \geqslant 0 \quad (0 \leqslant x \leqslant 1),$$
and Σa_n is convergent, then $\Sigma a_n f_n(x)$ is convergent and, if every f_n is continuous, the sum of the series is a continuous function of x.

(*b*) If Σa_n oscillates between finite limits of indetermination,

$$f_n(x) \geqslant f_{n+1}(x), \quad \text{and} \quad \lim f_n = 0,$$

then $\Sigma a_n f_n(x)$ is convergent; and, if every f_n is continuous, the sum of the series is a continuous function of x.

Dirichlet and Dedekind were concerned mainly with applications of these theorems to Dirichlet's series, and pass somewhat lightly over the general properties of series which are involved in them. Their exposition is also obscured to some extent by the fact that they do not utilize the notion of *uniform convergence*. I have therefore discussed the question further in § 2, and have stated a few theorems which summarize the conclusions which can be drawn from the discussion. I cannot claim any particular originality for these theorems, but, so far as I know, they have not, in the form in which I state them, been included in any published work. They would naturally suggest themselves to any one who undertook a careful analysis of the various theorems stated in this section, and Prof. Bromwich informs me that he has himself included Theorem I. *a* in a tract on the theory of series which will ultimately form one of the *Cambridge Tracts in Mathematics and Mathematical Physics*.

I have also included in §§ 3, 4 some applications of these theorems which do not appear to have been noticed hitherto, and in § 5 I have discussed a passage in Kronecker's *Vorlesungen über Integrale* which is concerned with the subject, but appears to contain serious errors.

There is yet another form of generalisation of Abel's theorem which has occupied the attention of mathematicians. It may happen that the series $\Sigma a_n x^n$ is divergent at a point on the circle of convergence, but is capable of "summation" by one or other of the methods furnished by

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* *Vorlesungen über Zahlentheorie*, §§ 100 and 143–4.

the theory of divergent series, Cesàro's method of mean values, or Borel's method of exponential summation; or one of the various generalisations of either method. And it results from the combined researches of a number of writers that, *if* Σa_n *has the sum* s *when summed according to any of these methods, then* $f(x)$ *tends to the limit* s *when* x *tends to the point in question on the circle of convergence by any path subject to certain restrictions.* In the latter part of the paper I have occupied myself with series summable by Cesàro's method. The theorem for such series which corresponds to Abel's original theorem was first proved by Frobenius,* and states that, if

$$s_n = a_0 + a_1 + \ldots + a_n,$$

and
$$\lim_{n=\infty} \frac{s_0+s_1+\ldots+s_n}{n+1} = s,$$

then
$$\lim_{x=1} f(x) = s.$$

I have attempted to prove a general theorem which shall stand to this theorem in the same relation as Theorem I. to Abel's theorem. This theorem (Theorem II.) is the principal result of the paper: it will be found in § 6.

Finally, I have illustrated some of the most obvious applications of this general theorem, and I have indicated some further questions which are naturally suggested, but which I cannot profess to have completely solved.

I may remark that I was led to this investigation by considering various problems concerning the limits approached by the q-series of elliptic functions, when q tends to a point on the unit circle, and a number of my illustrations are furnished by q-series. But I have not in this paper attempted to treat any such particular class of problems systematically.

2. THEOREM I. *a.—If* $f_0(x)$, $f_1(x)$, $f_2(x)$, ... *is a series of real finite positive functions*† *such that*

(1) $$f_n(x) \geqslant f_{n+1}(x) \quad (0 \leqslant x \leqslant 1),$$

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* *Crelle*, Bd. LXXXIX., p. 262.

† A finite function (*fonction bornée*) is a function whose absolute value is, throughout the interval of variation of the independent variable, less than a constant K. It would obviously be enough to assert that $|f_0| < K$.

and Σa_n is any convergent series, then the series $\Sigma a_n f_n(x)$ is uniformly convergent throughout the interval (0, 1).

For

$$(1) \quad \sum_{\nu=m}^{n} a_\nu f_\nu = \sum_{\nu=m}^{n-1} (a_m+a_{m+1}+\ldots+a_\nu)(f_\nu-f_{\nu+1})+(a_m+a_{m+1}+\ldots+a_n) f_n.$$

Choose m_0 so that, for $\nu \geqslant m \geqslant m_0$,

$$| a_m+a_{m+1}+\ldots+a_\nu | < \epsilon.$$

Then
$$\left| \sum_{\nu=m}^{n} a_\nu f_\nu \right| < \epsilon f_m < \epsilon M,$$

where M is the maximum of $f_0(x)$ in the range (0, 1). The theorem is therefore proved.

COROLLARY.—*If the functions $f_n(x)$ are continuous, the series $\Sigma a_n f_n(x)$ represents a function of x continuous throughout the interval $0 \leqslant x \leqslant 1$.*

THEOREM I. a 1.—*If the restriction that f_n is real and positive is removed, and the condition* (1) *is replaced by the condition that*

$$(1a) \qquad \sum_{m}^{n} | f_\nu(x)-f_{\nu+1}(x) | < K,$$

*where K is a constant, then the series $\Sigma a_n f_n$ is still uniformly convergent.**

We first observe that the existence of such a constant K involves that of a constant L, such that $|f_n(x)| < L$, for all values of x and n. For

$$|f_n(x)| \leqslant |f_0(x)| + \sum_{0}^{n-1} |f_\nu(x)-f_{\nu+1}(x)| < M+K.$$

Hence
$$\left| \sum_{m}^{n} a_\nu f_\nu \right| < \epsilon \left\{ \sum_{m}^{n-1} |f_\nu-f_{\nu+1}| + |f_n| \right\} < \epsilon(M+2K),$$

and the result follows as before.

COROLLARY.—*If the functions f_n are continuous, the sum of the series is continuous.*

An obvious generalisation is—

THEOREM I. a 2. — *The conclusions of the preceding theorems and corollaries still hold if the terms of the series Σa_n are functions of x, provided the series is uniformly convergent, and (in the corollaries) the functions a_n are continuous.*

* We may suppose either that f_n is a complex function of a real variable, or a function of a complex variable; in the latter case the interval (0, 1) must be replaced by a region.

These theorems all arise from the Theorem (a) of Dirichlet-Dedekind. It is with this rather than with Theorem (b) that I am concerned in this paper; but the latter also raises interesting questions.

THEOREM I. b.—*If the functions $f_n(x)$ satisfy, in addition to the conditions of I., the condition $\lim_{n=\infty} f_n(x) = 0$, and if Σa_n oscillates between finite limits of indetermination,* then the series $\Sigma a_n f_n$ is uniformly convergent.*

In the first place there is a number K such that

$$|a_m + a_{m+1} + \ldots + a_\nu| < K$$

for all values of m and ν. In the second place $f_n(x)$ is a function of x which never increases as n increases, and whose limit zero is a continuous function of x. The convergence of $f_n(x)$ to its limit is therefore *uniform*,† and we can choose m_0 so that, for $m \geqslant m_0$, and for all values of x,

$$|f_m(x)| < \epsilon.$$

The theorem now follows immediately from (1).

COROLLARY.—*If the functions f_n are continuous, the sum of the series $\Sigma a_n f_n(x)$ is a continuous function of x.*

THEOREM I. b 1.—*If the restriction that the functions $f_n(x)$ are real and positive is removed, and the conditions to which they are subject are replaced by the condition that the series $\Sigma |f_n(x) - f_{n+1}(x)|$ is convergent, the series $\Sigma a_n f_n$ is convergent.*

THEOREM I. b 2.—*If in addition the functions f_n are continuous and either of the equivalent conditions* (i.) *that the series $\Sigma |f_n - f_{n+1}|$ is uniformly convergent, or* (ii.) *that its sum represents a continuous function of x, is satisfied, the series $\Sigma a_n f_n$ will be uniformly convergent and continuous.*

THEOREM I. b 3.—*The preceding conclusions are not affected if the a_n's are functions of x, provided a constant K exists such that*

$$|a_0 + a_1 + \ldots + a_n| < K$$

for all values of n and x, and (if the continuity of the series is asserted) the functions a_n are continuous.

These theorems follow at once by trifling modifications of the preceding arguments. It will be seen that the series of theorems I. b, b 1, b 2, b 3 runs almost, though not exactly, parallel to the series I. a, a 1, a 2.

* *I.e.*, $|a_0 + a_1 + \ldots + a_n| < K$.

† Dini, *Grundlagen*, pp. 148, 149. The corollary is substantially Dedekind's theorem: his proof is less simple, owing to the fact that he does not employ the notion of uniform convergence.

3. Of the preceding theorems those of which the applications are most interesting are I. a and its extension I. a 1.

Since $f_n \geqslant f_{n+1}$, f_n tends to a limit for $n = \infty$ for all values of x; but in general it will not tend *uniformly* to this limit, and the limit will not be a continuous function of x. In the most important applications such a non-uniformity or discontinuity occurs at one or other end of the interval (0, 1), and the interest of the theorem lies in its application to establish the continuity of the series $\Sigma a_n f_n$ at this end. Thus

(i.) If
$$f_n(x) = x^n, \quad f_n \geqslant f_{n+1},$$
$$\lim f_n = 0 \ (0 \leqslant x < 1), \quad \lim f_n = 1 \ (x = 1),$$
and we obtain Pringsheim's form of Abel's theorem.

(ii.) If
$$f_n(x) = n^{-x}, \quad f_n \geqslant f_{n+1},$$
$$\lim f_n = 0 \ (0 < x \leqslant 1), \quad \lim f_n = 1 \ (x = 0),$$
and we deduce that the Dirichlet's series
$$\frac{a_1}{1^x} + \frac{a_2}{2^x} + \frac{a_3}{3^x} + \dots$$
is uniformly convergent throughout (0, 1), and so continuous for $x = 0$, which is one of the Dirichlet-Dedekind theorems.

(iii.) If (denoting the independent variable now by q) we take
$$f_n(q) = \frac{q^n}{1+q^n},$$
so that
$$f_n - f_{n+1} = \frac{q^n(1-q)}{(1+q^n)(1+q^{n+1})} \geqslant 0,$$
and
$$\lim_{n=\infty} f_n = 0 \ (q < 1), \quad = \frac{1}{2} \ (q = 1),$$
and we deduce that, if Σa_n is convergent,
$$\lim_{q=1} \Sigma \frac{a_n q^n}{1+q^n} = \tfrac{1}{2}\Sigma a_n,$$
numerous applications of this result [and the similar results for $\Sigma a_n q^n/(1+q^{2n})$, ...] may be made in the theory of elliptic functions. For instance, from
$$\log k = \log 4\sqrt{q} + 4\Sigma \frac{(-q)^n}{n(1+q^n)} \, *$$
we deduce
$$\lim_{q=1} \log k = 2\log 2 - 4\left(\tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{6} - \dots\right) = 0,$$
as may be verified independently.

(iv.) Let us next consider the series
$$\Sigma \frac{na_n q^n(1-q)}{1-q^n} = \Sigma \frac{na_n q^n}{1+q+q^2+\dots+q^{n-1}}.$$
Here
$$f_n(q) = \frac{nq^n}{1+q+q^2+\dots+q^{n-1}},$$
$$f_n(q) - f_{n+1}(q) = \frac{(1-q)^2 q^n}{(1-q^n)(1-q^{n+1})}(n-1-q-\dots-q^{n-1}) \geqslant 0.$$

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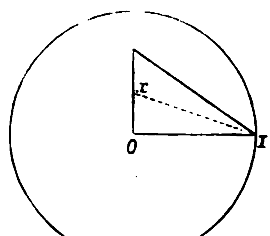
* Jacobi, *Fundamenta Nova*, p. 103.

We deduce that
$$\lim_{q=1}(1-q)\,\Sigma\frac{na_nq^n}{1-q^n} = \Sigma a_n,$$
provided only the latter series is convergent. This result has been proved (by a special method depending upon integrals) by Franel.* Similar results may, of course, be proved for such series as
$$\Sigma\frac{2na_nq^n}{1-q^{2n}},\quad \Sigma\frac{(2n+1)\,a_n\,q^{2n+1}}{1-q^{4n+2}},\quad \ldots.$$
For instance, from
$$-\log k' = 8\sum_0^\infty \frac{q^{2n+1}}{(2n+1)(1-q^{4n+2})}\dagger$$
we deduce
$$-\log_{q=1} k' \sim \frac{\pi^2}{2\,(1-q)},$$
and from
$$\frac{2\omega}{\pi}\sqrt{(\wp u-e_3)} = \operatorname{cosec}\frac{u\pi}{2\omega}+4\Sigma\frac{q^{2n+1}}{1-q^{2n+1}}\sin\left\{(2n+1)\frac{u\pi}{2\omega}\right\}\ddagger$$
we deduce
$$\frac{2\omega}{\pi}\sqrt{(\wp u-e_3)} \sim \frac{4}{1-q}\,\Sigma\,\frac{\sin\left\{(2n+1)\frac{u\pi}{2\omega}\right\}}{2n+1} = \pm\frac{\pi}{1-q},$$
according to the value of u. In the last equation we must suppose that ω is constant and that ω' varies in such a way that q tends to 1 along the real axis.

In an interesting note recently published in the *Messenger of Mathematics*,§ Prof. Bromwich establishes the asymptotic equality
$$f(\theta) = \sum_1^\infty \frac{(-)^{n-1}}{\sinh n\theta} \sim \frac{\log 2}{\theta}$$
for $\theta = 0$. This result follows immediately from what precedes if we write q for e^{θ}. I shall refer later on to Prof. Bromwich's further results.

4. I shall now consider some examples of the use of Theorem I. a 1.

(i.) Suppose that $f_n(x) = x^n$, and that the region of variation of x is a triangle formed by joining 0 and 1 to any point inside the unit circle.



It is easily verified that a constant K (depending only on the triangle) can be found such that for all points within or on the boundary of the triangle
$$\frac{|1-x|}{1-|x|} < K.$$
Hence, if $|x| = r$,
$$\sum_m^n |f_\nu(x)-f_{\nu+1}(x)| = \sum_m^n r^\nu|1-x| < K\sum_m^n r^\nu(1-r) < K,$$

* *Math. Annalen*, Bd. LII.

† *Fundamenta Nova*, l.c.

‡ Halphen, *Fonctions Elliptiques*, t. I., p. 431.

§ "Some Contributions to the Theory of Two Electrified Spheres," *Messenger*, Vol. XXXV., p. 1.

and the conditions of the theorem are satisfied. We thus obtain Pringsheim's generalisation of Abel's theorem.*

(ii.) The theorem may be applied to q-series such as those previously considered when q moves (let us say) along a radius vector to a *rational point* on the unit circle, *i.e.*, a point $e^{\pi ib/a}$, where a and b are integers. Take, *e.g.*, the series for $\log k$ considered above,† and suppose that $q = re^{\pi ib/a}$, where b is even and a odd, and that r tends to unity along the radius vector $(0, 1)$. Then none of the terms of the series become infinite in the limit; also

$$\sum_1^\infty \frac{(-q)^n}{n(1+q^n)} = \sum_{s=1}^{a} \sum_{m=0}^{\infty} \frac{(-q)^{ma+s}}{(ma+s)(1+q^{ma+s})} = \sum_{s=1}^{a} (-)^s r^s e^{s\pi ib/a} F_s(r^a),$$

where
$$F_s(\rho) = \sum_0^\infty \frac{(-\rho)^m}{(ma+s)(1+\rho^{m+s/a} e^{s\pi ib/a})}.$$

This last series satisfies the criteria of I. a 1 for uniform convergence throughout the interval $(0, 1)$ of values of ρ. For, if $a_m = (-)^m/(ma+s)$, Σa_m is convergent. Also, if

$$f_m(\rho) = \frac{\rho^m}{1+\rho^{m+s/a} e^{s\pi ib/a}},$$

$$f_m(\rho) - f_{m+1}(\rho) = \frac{\rho^m(1-\rho)}{(1+A\rho^{m+s/a})(1+A\rho^{m+1+s/a})},$$

where $A = e^{s\pi ib/a}$. Now

$$|1+A\rho^{m+s/a}| = \surd\{1+\rho^{2(m+s/a)}+2\rho^{m+s/a}\cos(s\pi b/a)\}.$$

If $\cos(s\pi b/a) > 0$, this is greater than unity; if $\cos(s\pi b/a) < 0$, it has a minimum when $\rho^{m+s/a} = -\cos(s\pi b/a)$, this minimum being $|\sin(s\pi b/a)|$. And in any case

$$|f_m(\rho) - f_{m+1}(\rho)| < K\rho^m(1-\rho),$$

from which it follows at once that the conditions of I. a 1 are satisfied.

Hence the original series for $\log k$ converges uniformly when $q = re^{\pi ib/a}$, $0 \leqslant r \leqslant 1$. For $r = 1$ it assumes the form

$$2\log 2 + \frac{\pi ib}{2a} + 2\sum_1^\infty \frac{(-)^n}{n}\left(1 + i\tan\frac{n\pi b}{a}\right) = \frac{\pi ib}{a} + 2i\sum_1^\infty \frac{1}{n}\tan\frac{n\pi b}{a},$$

and this is therefore the value to which $\log k$ tends as r approaches unity. The series on the right may be summed in finite terms.‡

5. In a passage in his *Vorlesungen über Integrale*, which has doubtless puzzled many readers besides myself, Kronecker apparently essays to prove a theorem designed to be a generalisation of Abel's theorem somewhat on the lines of Theorem I. a, except that there is no mention of uniform convergence. The whole passage is obscure; but the suggested

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* *Münchener Sitzungsberichte*, *l.c.*

† § 3, iii.

‡ See H. J. S. Smith, "On some Discontinuous Series considered by Riemann" (*Messenger*, Vol. XI., pp. 1-11; *Collected Math. Papers*, Vol. II., p. 312); Dedekind's Note in Riemann's *Werke*, pp. 427-447; G. H. Hardy, "Note on the Limiting Values of the Elliptic Modular Functions," *Quarterly Journal*, Vol. XXXIV., pp. 76-86.

theorem seems to be as follows:—* "If

(i.) Σa_n is a convergent series,

(ii.) the functions $f_n(x)$ are positive and continuous throughout (a, A),

(iii.) $f_n(x) \geqslant f_{n+1}(x)$,

(iv.) $\lim_{x=A} f_m(x) = \lim_{x=A} f_n(x)$, for all values of m and n,

then $\Sigma a_n f_n(x)$ will be convergent and continuous for $x = A$."

My criticisms on the passage are in brief (i.) that the conditions are redundant, the fourth of them being quite unnecessary and having nothing to do with the essence of the matter; and (ii.) that the proof is altogether unsound. The unsoundness of the proof appears to have arisen from a mistaken idea of the importance of condition (iv.). Kronecker argues as follows. Starting from Abel's partial summation lemma, the origin of all these theorems, viz.,

$$c_0 f_0 + \sum_1^n (c_\nu - c_{\nu-1}) f_{\nu-1} = \sum_1^n c_\nu (f_{\nu-1} - f_\nu) + c_n f_n,$$

and putting
$$c_\nu = -(a_\nu + a_{\nu+1} + \ldots),$$
he deduces

$$-f_0 \sum_0^\infty a_\nu + \sum_1^n a_{\nu-1} f_{\nu-1} = -\sum_1^n (f_{\nu-1} - f_\nu) \sum_\nu^\infty a_\kappa - f_n \sum_n^\infty a_\kappa$$
$$= -(f_0 - f_n) M_n - f_n \sum_n^\infty a_\kappa,$$

where M_n lies between the least and greatest of the values of

$$\sum_\nu^\infty a_\kappa \quad (\nu = 1, 2, \ldots, n).$$

Making n tend to infinity, and observing that $\sum_1^\infty a_{\nu-1} f_{\nu-1}$ is convergent, we obtain

$$-f_0 \sum_0^\infty a_\nu + \sum_1^\infty a_{\nu-1} f_{\nu-1} = -(f_0 - \lim_{n=\infty} f_n) M,$$

where M lies between the least and greatest of all the values of $\sum_\nu^\infty a_\kappa$.

He then makes x tend to A, and (unless his meaning has been entirely obscured by misprints), argues that, because

$$\lim_{x=A} f_0 = \lim_{x=A} f_n$$

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* I have altered Kronecker's notation so as to agree with my own (Kronecker, *l.c.*, pp. 88, 89).

for all values of n, therefore

$$\lim_{x=A} (f_0 - \lim_{n=\infty} f_n) = 0\,;$$

and therefore $$\lim_{x=A} \sum_1^\infty a_{\nu-1} f_{\nu-1} = \lim_{x=A} f_0 \times \sum_0^\infty a_\nu.$$

But it is obvious that all that he is justified in asserting is that

$$\lim_{x=A} f_0 = \lim_{n=\infty} (\lim_{x=A} f_n),$$

and *not* $$\lim_{x=A} f_0 = \lim_{x=A} (\lim_{n=\infty} f_n),$$

the two repeated limits only being equal in exceptional circumstances. And, in fact, in the very simplest case, when $f_n(x) = x^n$ and $A = 1$,

$$\lim_{n=\infty} \lim_{x=1} x^n = 1, \qquad \lim_{x=1} \lim_{n=\infty} x^n = 0\,;$$

so that his argument does not even suffice to prove Abel's theorem itself. And a careful examination of the passage will, I think, lead any reader to the conclusion that the flaw in it is fundamental and not to be repaired by any alterations merely of detail.

6. I shall now consider the case in which the series Σa_n is divergent but summable by Cesàro's method of mean values. I use the following notation and terminology. We shall say that Σa_n is *summable* if

$$\frac{s_0 + s_1 + \ldots + s_n}{n+1},$$

where $$s_n = a_0 + a_1 + \ldots + a_n,$$

tends to a finite limit for $n = \infty$; and, if the terms a_n are functions of a variable x, and the convergence of this mean value to its limit is uniform throughout a certain interval or region, we shall say that Σa_n is *uniformly summable*. It is evident that the sum of a uniformly summable series of continuous terms is a continuous function of x.

THEOREM 2.—*If the functions f_n are finite, real, and positive, and $f_n - f_{n+1}$ and $f_n - 2f_{n+1} + f_{n+2}$, their first and second differences, are positive for $0 \leqslant x \leqslant 1$ and for all values of n, and if the series Σa_n is summable, then the series $\Sigma a_n f_n$ is uniformly summable throughout* (0, 1).

COROLLARY.—*If the functions f_n are continuous, the sum of the series $\Sigma a_n f_n$ is a continuous function of x.*

The proof of this theorem presents somewhat greater difficulties than those of the simpler theorems of § 2. We shall find it a necessary preliminary to establish a series of lemmas.

LEMMA 1.—*If s_n tends uniformly to a limit s, the series Σa_n is uniformly summable and has the sum s.*

If we omit "uniformly," this is a well known theorem* asserting the consistency of the new definition with the old. The insertion of "uniformly" in no way affects the proof.

LEMMA 2.—*If*
$$\lim \frac{s_0+s_1+\ldots+s_n}{n+1} = 0,$$
we can determine a series of positive quantities $\epsilon_1, \epsilon_2, \ldots$, whose limit is zero, such that
$$\left|\frac{s_p+s_{p+1}+\ldots+s_{p+r}}{p+r+1}\right| < \epsilon_p$$
for all values of r.

For we may write $s_0+s_1+\ldots+s_n = (n+1)\,\eta_n$,

where $\lim \eta_n = 0$. And then
$$s_p+s_{p+1}+\ldots+s_{p+r} = (p+r+1)\,\eta_{p+r}-p\eta_{p-1},$$
from which the lemma follows; for we can choose p so that, for $\nu \geqslant p-1$, $|\eta_\nu| < \epsilon$, however small be ϵ, and then
$$\left|\frac{s_p+s_{p+1}+\ldots+s_{p+r}}{p+r+1}\right| < 2\epsilon$$
for all values of r. In particular, as is well known,
$$\lim s_p/(p+1) = 0.$$

LEMMA 3.—*If f_n is finite, real, and positive and $f_n \geqslant f_{n+1}$ for all values of n and x, and*
$$\lim \frac{s_0+s_1+\ldots+s_n}{n+1} = 0,$$
then
$$\lim \frac{s_0 f_0+s_1 f_1+\ldots+s_n f_n}{n+1} = 0$$
uniformly for all values of x.

For
$$\begin{aligned} s_0 f_0+\ldots+s_n f_n &= \sum_{\nu=0}^{n-1} (s_0+\ldots+s_\nu)(f_\nu-f_{\nu+1})+(s_0+\ldots+s_n)f_n \\ &= \left(\sum_{\nu=0}^{r-1}+\sum_{r}^{n-1}\right)(s_0+\ldots+s_\nu)(f_\nu-f_{\nu+1})+(s_0+\ldots+s_n)f_n \\ &= (f_0-f_r)\,M_{0,\,r-1}+f_r M_{r,\,n}, \end{aligned}$$

* See, *e.g.*, Bromwich and Hardy, *Proceedings*, Vol. II., p. 172.

where $M_{0,\,r-1}$ lies between the least and greatest of

$$s_0,\ s_0+s_1,\ \ldots,\ s_0+s_1+\ldots+s_{r-1},$$

and $M_{r,\,n}$ between the least and greatest of

$$s_0+s_1+\ldots+s_r,\ \ldots,\ s_0+s_1+\ldots+s_n.$$

Let ϵ be an assigned positive small quantity. We can choose r so that for $\nu \geqslant r$

$$\left|\frac{s_0+s_1+\ldots+s_\nu}{\nu+1}\right| < \epsilon,$$

and, *a fortiori*,

$$\left|\frac{s_0+s_1+\ldots+s_\nu}{n+1}\right| < \epsilon$$

for $n \geqslant \nu \geqslant r$; and therefore we can choose r so that

$$\left|\frac{M_{r,\,n}}{n+1}\right| < \epsilon$$

for all values of $n \geqslant r$. But when r is fixed we can obviously choose n so that

$$\left|\frac{M_{0,\,r-1}}{n+1}\right| < \epsilon.$$

When r and n are thus chosen

$$\left|\frac{s_0 f_0+s_1 f_1+\ldots+s_n f_n}{n+1}\right| < 2M\epsilon,$$

where M is the maximum of $f_0(x)$. The lemma is therefore proved.

LEMMA 4.—*If the conditions of* 3 *are satisfied except that*

$$\lim_{n=\infty} \frac{s_0+s_1+\ldots+s_n}{n+1} = s\ (\neq 0),$$

then

$$\lim_{n=\infty} \frac{s_0 f_0+s_1 f_1+\ldots+s_n f_n}{n+1} = s \lim_{n=\infty} f_n\,;$$

but the convergence to this limit will in general not be uniform.

For let $s_0 = s+t_0$, $s_1 = s+t_1$, ... Then

$$\lim_{n=\infty} \frac{t_0+t_1+\ldots+t_n}{n+1} = 0\,;$$

and therefore

$$\frac{t_0 f_0+t_1 f_1+\ldots+t_n f_n}{n+1}$$

converges *uniformly* to zero. Also

$$\lim_{n=\infty} \frac{s(f_0+f_1+\ldots+f_n)}{n+1} = s \lim_{n=\infty} f_n;$$

but the convergence to this limit will not in general be uniform unless f_n converges to its limit uniformly, which will not generally be the case.

LEMMA 5.—*If*
$$\lim_{n=\infty} \frac{s_0+s_1+\ldots+s_n}{n+1} = 0,$$

and the f_n's satisfy the further condition

$$f_n - f_{n+1} \geqslant f_{n+1} - f_{n+2}$$

for all values of n and x in question, then the series

$$\sum_0^\infty s_n(f_n - f_{n+1})$$

is uniformly convergent.

In the first place

$$f_0 - f_n = (f_0 - f_1) + \ldots + (f_{n-1} - f_n) \geqslant n(f_{n-1} - f_n).$$

Hence a constant K can be assigned so that for all values of x and n

$$f_{n-1} - f_n < K/n.$$

Now $s_p(f_p - f_{p+1}) + s_{p+1}(f_{p+1} - f_{p+2}) + \ldots + s_{q-1}(f_{q-1} - f_q)$

$$\begin{aligned} &= s_p(f_p - 2f_{p+1} + f_{p+2}) + (s_p + s_{p+1})(f_{p+1} - 2f_{p+2} + f_{p+3}) \\ &\qquad + \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ &\qquad + (s_p + s_{p+1} + \ldots + s_{q-2})(f_{q-2} - 2f_{q-1} + f_q) \\ &\qquad + (s_p + s_{p+1} + \ldots + s_{q-1})(f_{q-1} - f_q), \end{aligned}$$

the modulus of which is less than

$$\begin{aligned} &\epsilon_p \{(p+1)(f_p - 2f_{p+1} + f_{p+2}) + (p+2)(f_{p+1} - 2f_{p+2} + f_{p+3}) + \ldots \\ &\qquad \ldots + (q-1)(f_{q-2} - 2f_{q-1} + f_q) + q(f_{q-1} - f_q)\} \\ &= \epsilon_p\{p(f_p - f_{p+1}) + f_p - f_q\} < \epsilon_p\{K + 2M\}, \end{aligned}$$

where M is the maximum of $f_0(x)$. The lemma is therefore proved.

LEMMA 6.—*If the f_n's satisfy the conditions of* 5, *but*

$$\lim_{n=\infty} \frac{s_0+s_1+\ldots+s_n}{n+1} = s\,(\neq 0),$$

the series

$$\sum_0^\infty s_n(f_n-f_{n+1})$$

is convergent (but, in general, not uniformly convergent).

Let

$$s_n = s+t_n;$$

then, by 3, the series $\Sigma t_n(f_n-f_{n+1})$ is uniformly convergent. On the other hand, the series $\Sigma s(f_n-f_{n+1})$ is convergent, but not uniformly convergent, unless f_n tends to its limit uniformly.

7. *Proof of Theorem* 2.—Let s be the sum of the divergent series Σa_n, and let

$$a_0' = a_0-s,\ a_1' = a_1,\ a_2' = a_2,\ \ldots,\ s_n' = a_0'+a_1'+\ldots+a_n' = s_n-s;$$

then $\Sigma a_n'$ is summable, and its sum is zero; *i.e.*,

$$\lim \frac{s_0'+s_1'+\ldots+s_n'}{n+1} = 0.$$

By Lemma 3,

$$\frac{s_0'f_0+s_1'f_1+\ldots+s_n'f_n}{n+1}$$

tends uniformly to 0 for $n=\infty$; and, by Lemma 5, the series

$$\Sigma s_n'(f_n-f_{n+1})$$

is uniformly convergent. Hence, if

$$S_n' = \sum_0^n s_\nu(f_\nu-f_{\nu+1}),$$

S_n' tends uniformly to a limit for $n=\infty$, and so, by Lemma 1,

$$\frac{S_0'+S_1'+\ldots+S_n'}{n+1}$$

does the same.

Now $a_\nu'f_\nu = (s_\nu-s_{\nu-1}')f_\nu = s_\nu'f_\nu-s_{\nu-1}'f_{\nu-1}+s_{\nu-1}'(f_{\nu-1}-f_\nu).$

Hence, if $\sigma_n = a_0f_0+a_1f_1+\ldots+a_nf_n$, $\sigma_n' = a_0'f_0+a_1'f_1+\ldots+a_n'f_n$,

$$\sigma_n' = s_n'f_n+\sum_1^n s_{\nu-1}'(f_{\nu-1}-f_\nu) = s_n'f_n+S_{n-1}',$$

and $$\frac{\sigma_0'+\sigma_1'+\ldots+\sigma_n'}{n+1} = \frac{s_0'f_0+\ldots+s_nf_n}{n+1}+\left(\frac{n}{n+1}\right)\frac{S_0'+S_1'+\ldots+S_{n-1}}{n},$$

and therefore tends uniformly to a limit for $n=\infty$. But

$$\frac{\sigma_0+\sigma_1+\ldots+\sigma_n}{n+1}=sf_0+\frac{\sigma_0'+\sigma_1'+\ldots+\sigma_n'}{n+1},$$

and therefore also tends uniformly to a limit for $n=\infty$. Hence *the series $\Sigma a_n f_n$ is uniformly summable, and, if the functions f_n are continuous, its sum is a continuous function of n.* The theorem is therefore proved.

8. In order to show more precisely the relations of the preceding lemmas and theorem I take a very simple example.

Let $$a_0=1,\quad a_1=-2,\quad a_2=2,\quad a_3=-2,\quad \ldots,$$

so that $$s_{2n}=1,\quad s_{2n+1}=-1,$$

and $$\lim\frac{s_0+s_1+\ldots+s_n}{n+1}=0;$$

and suppose $f_n(x)=x^n$. Then

(i.) $$s_n f_n=(-)^n x^n,$$

$$\frac{s_0 f_0+s_1 f_1+\ldots+s_n f_n}{n+1}=\frac{1+(-)^n x^{n+1}}{(n+1)(1+x)},$$

which converges *uniformly* to 0 for $n=\infty$ (*Lemma* 3).

(ii.) Again $\sum_1^n s_{\nu-1}(f_{\nu-1}-f_\nu)=\sum_1^n(-)^{\nu-1}x^{\nu-1}(1-x)=(1-x)\{1+(-)^{n-1}x^n\}/(1+x)$,

which tends *uniformly* to $(1-x)/(1+x)$ for $n=\infty$ (*Lemma* 5). For, although x^n does *not* tend uniformly to its limit,

$$x^n-x^{n+1}-(x^{n+1}-x^{n+2})=x^n(1-x)^2\geqslant 0,$$

and $$1-x^{n+1}=(1-x)+(x-x^2)+\ldots+(x^n-x^{n+1})\geqslant(n+1)(x^n-x^{n+1}),$$

so that $$x^n(1-x)<\frac{1}{n+1},$$

and therefore does tend uniformly to zero.

(iii.) Finally, $$\sigma_n=1-2x+2x^2-\ldots+(-)^n 2x^n=\frac{1-x}{1+x}+2(-)^n\frac{x^{n+1}}{1+x},$$

and $$\frac{\sigma_0+\sigma_1+\ldots+\sigma_n}{n+1}=\frac{1-x}{1+x}+\frac{2\{x+(-)^n x^{n+2}\}}{(n+1)(1+x)^2},$$

which tends *uniformly* to $(1-x)/(1+x)$ for $n=\infty$ (*Theorem* 2).

If the conditions were altered by changing a_0 into $1+a$ $(a\neq 0)$, we should have

$$s_n f_n=\{a+(-1)^n\}x^n,$$

and $$\frac{s_0 f_0+s_1 f_1+\ldots+s_n f_n}{n+1}=\phi+\frac{1+(-)^n x^{n+1}}{(n+1)(1+x)},$$

where $$\phi=\frac{a}{n+1}\,\frac{1-x^{n+1}}{1-x}\quad(x<1),$$

$$\phi=a\qquad(x=1),$$

and the convergence of ϕ to its limit is not uniform (*Lemma* 4). Similarly $\Sigma s_{\nu-1}(f_{\nu-1}-f_\nu)$ is increased by the addition of the non-uniformly convergent series $\Sigma a(x^{n-1}-x^n)$ (*Lemma* 6); but it is easily verified that the uniformity of convergence which is prescribed by Theorem 2 is not affected, the two non-uniformities (so to say) cancelling one another.

9. *Applications of Theorem* 2.—(i.) If $f_n(x) = x^n$,

$$f_n - 2f_{n+1} + f_{n+2} = x^n(1-x)^2 \geqslant 0$$

for $0 \leqslant x \leqslant 1$ and all values of n. Hence, if Σa_n is summable, $\Sigma a_n x^n$ is uniformly summable for $0 \leqslant x \leqslant 1$; and its sum is a continuous function of x for $x = 1$, which is Frobenius's theorem cited in § 1.

(ii.) If $f_n(x) = n^{-x}$ ($n \geqslant 1$, $x \geqslant 0$), it is easy to see that the first and second differences of f_n are positive (or zero). Hence we obtain the theorem that, if $\sum_1^\infty a_n$ is summable, $\sum_1^\infty a_n n^{-x}$ is uniformly summable for all positive values of x, including zero, and its sum is a continuous function of x for $x = 0$. That is to say

$$\lim_{x=0}\left(\frac{a_1}{1^x}+\frac{a_2}{2^x}+\frac{a_3}{3^x}+\dots\right) = \lim_{n=\infty}\frac{s_0+s_1+\dots+s_n}{n+1},$$

if the latter limit exists. For example,

$$\lim_{x=0}\left(\frac{1}{1^x}-\frac{1}{2^x}+\frac{1}{3^x}-\dots\right) = \tfrac{1}{2}.$$

(iii.) If
$$f_n(q) = \frac{q^n}{1+q^n} \quad (0 \leqslant q \leqslant 1),$$

$$f_n - 2f_{n+1} + f_{n+2} = \frac{q^n(1-q)^2(1-q^{n+1})}{(1+q^n)(1+q^{n+1})(1+q^{n+2})} \geqslant 0.$$

Hence, if Σa_n is summable, $\Sigma a_n q^n/(1+q^n)$ is uniformly summable for $0 \leqslant q \leqslant 1$, and represents a continuous function of q, in particular for $q = 1$.

For instance, from the formula

$$\frac{2k'K}{\pi} = 1 - \frac{4q}{1+q} + \frac{4q^3}{1+q^3} - \frac{4q^5}{1+q^5} + \dots *$$

we deduce that
$$\lim_{q=1}\frac{2k'K}{\pi} = 1 - 4\left(\tfrac{1}{2}-\tfrac{1}{2}+\dots\right) = 1 - 4\,.\,\tfrac{1}{4} = 0.\dagger$$

(iv.) Consider the series
$$\frac{q}{1-q^2} - \frac{2q^2}{1-q^4} + \frac{3q^3}{1-q^6} - \dots,$$

whose sum is easily found‡ to be
$$\frac{K}{2\pi^2}(E - k'^2K).$$

We may write this in the form
$$\frac{q}{1-q^2}\Sigma a_n f_n(q),$$

where
$$a_n = (-1)^n \quad \text{and} \quad f_n(q) = \frac{(n+1)q^n}{1+q^2+\dots+q^{2n}},$$

and it is easy to verify that the first and second differences of f_n are positive. Hence $\Sigma a_n f_n$ is uniformly summable. For $q = 1$ it takes the form

$$1 - 1 + 1 - \dots = \tfrac{1}{2}.$$

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* *Fundamenta Nova*, § 40, (6).

† Strictly speaking, the divergent series should be written

$$\tfrac{1}{2} + 0 - \tfrac{1}{2} - 0 + \tfrac{1}{2} + 0 - \tfrac{1}{2} - \dots.$$

‡ *E.g.*, by making $x = \frac{1}{2}\pi$ in formula (1) of § 41 of the *Fundamenta Nova*.

We deduce that
$$K(E-k'^2K) \sim \frac{\pi^2}{2(1-q)}$$
for $q = 1$.

10. It would be easy to multiply instances of interesting applications of Theorem 2. Those which I have given are fair examples of some of the simplest types which naturally occur, and the length of this paper forbids that I should attempt to treat them in a more systematic manner. I shall conclude by indicating briefly certain actual or possible further generalisations.

In the first place we may at once enunciate

THEOREM 2 *a* 1.—*The conclusions of Theorem* 2 (*and the lemmas preliminary to it*) *are still valid if the functions* $f_n(x)$ *are not restricted to be real and positive, and the condition that the first and second differences of the functions are not negative is replaced by the conditions*

$$\sum_m^n |f_\nu - f_{\nu+1}| < K, \qquad \sum_m^n (\nu+1)\,|f_\nu - 2f_{\nu+1} + f_{\nu+2}| < K,$$

for all values of m, n, *and* x.

The course of the proof is unaffected save for slight modifications in the case of Lemmas 3 and 5.

Consider, for example, the series

$$\vartheta_4(v, q) = 1 + 2\sum_1^\infty (-)^n q^{n^2} \cos 2n\pi v.$$

Taking $a_n = 2(-)^n \cos 2n\pi v$ $(n > 0)$ and $f_n = q^{n^2}$, we may verify without difficulty that the conditions of the theorem are satisfied. Since the series

$$1 - 2\cos 2\pi v + 2\cos 4\pi v - \ldots$$

has the sum zero when summed by Cesàro's method, we deduce that

$$\lim_{q=1} \vartheta_4(v, q) = 0.^*$$

THEOREM 2 *a* 2.—*The preceding conclusions are not affected if the terms of the series* Σa_n *are functions of* x, *provided the series be uniformly summable.*

A much more interesting and more difficult question is that of the extension of Theorem II. to cases in which the summation of Σa_n requires

* See Borel, *Leçons sur les Séries divergentes*, p. 7; L. Fejér, *Math. Annalen*, Bd. LVIII., p. 66; Hardy, "Note on Divergent Fourier Series," *Messenger*, Vol. XXXIII., p. 144. I refer later to Herr Fejér's investigations.

one of the extended forms of the mean value process, *e.g.*, when, if

$$s_n^{(1)} = \frac{s_0+s_1+\ldots+s_n}{n+1},$$

$s_n^{(1)}$ oscillates for $n=\infty$, but

$$s_n^{(2)} = \frac{s_0^{(1)}+s_1^{(1)}+\ldots+s_n^{(1)}}{n+1}$$

has a limit.

The following more general theorem is naturally suggested, and I have no doubt that it is true. We define "summable" to mean "summable by k repetitions of the mean value process." Then,

If the first, second, ..., $(k+1)$-th differences of the functions $f_n(x)$ are positive (or zero) for all values of x and n in question, and the series Σa_n is summable, then the series $\Sigma a_n f_n(x)$ is uniformly summable, and therefore its sum is a continuous function of x

—with corollaries and generalisations in every way analogous to those of Theorems I. a and II. Such a theorem would be related to Hölder's extensions of Frobenius's theorem as is II. to Frobenius's and I. a to Abel's theorem. But I have not up to the present succeeded in overcoming the algebraical difficulties attendant upon a complete and rigorous proof.

In the most interesting cases Theorem II. is generally sufficient. But the latter theorem does not cover such cases as those in which Σa_n is a series like $1-2+3-4+\ldots$ or $1^2-2^2+3^2-4^2+\ldots$.

An example in which a result more general than that of II. is needed may be found in the theory of two electrified spheres. In the paper already referred to, Prof. Bromwich, seeking a rigorous proof of Lord Kelvin's theorem that the force acting between two spheres in contact and at potential V is $\frac{1}{6}V^2(\log 2-\frac{1}{4})$, requires to show that, for small values of θ,

$$f(\theta) = \Sigma\frac{(-)^{n-1}}{\sinh n\theta} = \frac{\log 2}{\theta} - \tfrac{1}{24}\theta + \ldots.$$

The first approximation was established in § 3 (iv.). To obtain the second we must prove that

$$\lim_{\theta=0}\frac{1}{\theta}\Sigma(-)^{n-1}\left(\frac{1}{n\theta}-\frac{1}{\sinh n\theta}\right) = \tfrac{1}{24}.$$

The limiting form of the series is $\quad \frac{1}{6}(1-2+3-4+\ldots)$,

which is summable by *two* repetitions of the mean value process, and has the sum $\frac{1}{24}$. Here we could take $a_n = (-)^{n-1}n$ and $f_n(\theta) = \frac{1}{n\theta}\left(\frac{1}{n\theta}-\frac{1}{\sinh n\theta}\right)$, and so obtain the result desired.

Although I have not succeeded in proving the suggested general theorem, I have, starting from a theorem of Herr Fejér's, succeeded in proving a number of theorems of a more special character which do enable us to deal effectively with cases such as these: *e.g.*, to assign the limit of

$$\frac{q}{1+q} - \frac{2q^2}{1+q^2} + \frac{3q^3}{1+q^3} - \ldots$$

for $q = 1$. I confine myself at present to stating one of these theorems. Herr Fejér's theorem (modified so as to correspond to Theorem 2*) runs as follows:—*If*

(i.) Σa_n *is summable* (*to the sum* s),

(ii.) *the functions* $f_n(x)$ *and their first and second differences are positive* (*or zero*),

(iii.) $\Sigma n f_n(x)$ *is convergent for* $x > 0$,

(iv.) $\lim_{x=0} f_n(x) = 1$ *for all values of* n,

then $\Sigma a_n f_n(x)$ *is absolutely convergent for* $x > 0$, *and its limit for* $x = 0$ *is* s.

The more general theorem is that the same conclusion holds when k repetitions of the mean value process are necessary in order to sum the series Σa_n, and

(ii.)′ the first, second, ..., $(k+1)$-th differences of the functions $f_n(x)$ are positive (or zero),

(iii.)′ $\Sigma n^k f_n(x)$ is absolutely convergent.

The proof is not difficult. The other theorems relate to cases in which condition (ii.) or (ii.)′ is not satisfied. I have included proofs of those theorems in a paper which will be published in the *Mathematische Annalen*.

* The conditions actually stated by Herr Fejér differ from the above in the restriction of $f_n(x)$ to be of the form $\phi(nx)$, and the substitution for (ii.) and (iii.) of the conditions

$$|\phi(t)| < \frac{K}{t^{2+\rho}}, \qquad |\phi''(t)| < \frac{K}{t^{2+\rho}},$$

where $\rho > 0$. The proof of the theorem as I state it may be made a good deal simpler than Herr Fejér's proof.

ON THE QUESTION OF THE EXISTENCE OF TRANSFINITE NUMBERS

By PHILIP E. B. JOURDAIN.

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IN a recent paper in these *Proceedings** Dr. Hobson has initiated a discussion of the existence of certain transfinite numbers. His arguments may conveniently be divided into two classes. Firstly, while agreeing with me that the series (W) of all ordinal numbers, arranged in order of magnitude, has no type and no associated cardinal number,† he puts forward the suggestion that some *segment* of W may be "inconsistent," in the sense in which I used this word.‡

Secondly, there is his requirement of a "norm" for the definition of an aggregate,§ and the objections connected therewith to an infinite series of acts of arbitrary selection.‖

With regard to the first class of arguments, I give (§ 1) an exact statement of what I meant by the term "inconsistent," which seems to have been misunderstood by many people. In fact, an "inconsistent" aggregate is an aggregate (which is itself defined in a manner free from self-contradiction) of which the cardinal number (and type, if it is ordered) is contradictory; thus, I see no reason for denying the existence of W,¶ but I do see reason for denying that W has a type or associated cardinal number. Further, Hobson's remark** on the possible introduction of contradiction by Cantor's second principle of generation ignores the character of this principle; for the essence of it is, as shown in §§ 3–5, the constant avoidance of contradiction, while the difficulty as to the type of W is simply that the second principle is not applicable. And this requires the discussion of § 4 to show that there are no ordinal numbers other than those to which Cantor's third principle applies. Lastly,

* "On the General Theory of Transfinite Numbers and Order Types," *Proc. Lond. Math. Soc.*, Ser. 2, Vol. 3 (1905), pp. 170–188.

† *Loc. cit.*, pp. 170–171, 180, 185.

‡ *Loc. cit.*, pp. 171–172.

§ *Loc. cit.*, pp. 172–175.

‖ *Loc. cit.*, pp. 182–188.

¶ Indeed, I admit a well-ordered series such that W is ordinally similar to merely a *segment* of it (§ 1).

** *Loc. cit.*, pp. 171–172, 178, 180.

Hobson's own criteria of existence,* and his objections to Russell,† are easily shown to fail (§ 6).

With regard to the second class of arguments, Hobson's requirement of a "norm" (§§ 8–9) can be proved rigorously to be too narrow (§ 10), but his objections to an infinite series of arbitrary selections, although intermixed with a psychology which, I think, is irrelevant to mathematics, lead to a discussion of what is known as the "multiplicative axiom" (§§ 11–13), which is necessary and sufficient to justify the process of arbitrary selection. Finally (§ 14), I have tried to state the grounds there are for admitting that the axiom in question is both evident and true.

1.

Hobson's statement‡ of the contradiction arising from supposing the series W to have a type does not seem to me to be the best possible. If, he says, we suppose that every well-ordered series has an ordinal number, W has one, which must be the maximum ordinal β. But, if we place any element after W, we form a series of the greater type $\dot{\beta}+1$.

My own form was: If W has a type (β), we have, by this very supposition, indicated a series (W, β) of type (if it has one) $\beta+1$. This form is advantageous, because one emphasises, by using it, the fact that the supposition that W has a type implies that W is a segment of a well-ordered series; and this fact is necessary to refute a certain explanation of Burali-Forti's contradiction.§

But there is no reason for making the contradiction depend on the additional assumption that a series, like (W, β), which transcends W, has a type. For,‖ if W has an ordinal number β, β must occur in W, the series of *all* ordinals, and consequently

$$\beta > \beta,$$

a manifest contradiction.

Now, I see no reason for denying, with Hobson,¶ the existence of the series W (and the "field of the relation"** W, the class w of all ordinal numbers); for not only does there appear to be nothing objectionable in the definition of a series (W, γ) transcending W, so long as γ is not said to be the type of its preceding series W, but also, if there were no w, it is not evident what meaning could be given to the phrase "γ is an ordinal

* *Loc. cit.*, pp. 177–179, 180–181.

† *Loc. cit.*, pp. 179–180.

‡ *Loc. cit.*, p. 170.

§ Due to Bernstein ("Uber die Reihe der transfiniten Ordnungszahlen," *Math. Ann.*, Bd. LX., 1905, pp. 187–193; cf. *ibid.*, pp. 469–470).

‖ See *Math. Ann.*, Bd. LX., 1905, p. 466.

¶ *Loc. cit.*, p. 180.

** See Russell, *The Principles of Mathematics*, Vol. I., Cambridge, 1903, p. 97.

number," or $\gamma\epsilon w$, in the symbolic logic of Peano and Russell, which is a necessary hypothesis to any theorem on ordinal numbers.*

There seemed, indeed, to be a ground for the assertion of the non-existence of w and W when it was assumed that every propositional function defines a class; † for, if u is a class, the propositional function—"the z's such that u is similar ‡ to z"—would define a class, the cardinal number of u.§ But when the hypothesis is abandoned, as it is now ‖ by Russell, from the non-existence of a cardinal number (or type, as the case may be) of u does not follow the non-existence of u.

As regards the above definition of series transcending W, W must, it appears, be distinguished from any of the series such that every well-ordered series is ordinally similar either to it or to a segment of it.¶ There *is* a contradiction involved in supposing any one of these series** ($\mathfrak{W}$) to be followed by a term (for, if it could, the new series would not be ordinally similar to $\mathfrak{W}$ or to any segment of it),†† and hence W is ordinally similar to a *segment* merely of $\mathfrak{W}$.

Thus, what I denoted as "inconsistent" aggregates or classes are not non-entities (which would make it absurd to speak of them as a kind of class), but classes such that cardinal numbers of them (and types, if the classes are ordered) are non-entities. The terms "non-entity" and "non-existent" must be kept distinct; Russell, following Peano, applied the term "non-existent" to classes which are null,‡‡ and considered that all the "existence-theorems" of mathematics were proofs that the various classes defined are not null. It was only later that "non-entities" were discovered; these were the results of giving propositional functions certain values,§§ looked like classes, and, if *every* propositional function were to

* [However, this difficulty is, I hear, avoided by Russell in his "No-Classes Theory." Here "γ is an ordinal number" is only a short way of stating something which does not imply a conception of *class*.—April 30th, 1906.]

† Russell, *op. cit.*, p. 20.

‡ *Op. cit.*, p. 113. This word replaces Cantor's *äquivalent*.

§ *Op. cit.*, p. 115.

‖ The first intimation of this I had in a letter of May 15th, 1904.

¶ Cf. *Phil. Mag.*, Jan. 1904, p. 67, and Jan. 1905, pp. 51, 53.

** Mr. Russell drew my attention to the fact that such a series is not unique (in fact, given one such series, we can get another such by interchanging two of its terms), and hence that my use of the words "*the* series," in the passages quoted in the preceding note, is incorrect.

†† Thus, we cannot remove the first term and put it after $\mathfrak{W}$; for there is no "after $\mathfrak{W}$," though there *is* an "after W."

‡‡ *Op. cit.*, pp. 32, 73-76.

§§ For example, such non-entities are "all the x's such that x is not a member of x" (Russell's contradictory "class"), and "all the x's such that x is similar to the class of ordinal numbers," which is, according to Russell, "the cardinal number of w" (a part of Burali-Forti's contradiction).

define a class, would be classes, and, indeed, *existent* (not null) classes. But "non-entities" are not classes, not even the null class, but are nothing at all, because self-contradictory, and have the characteristic property that they can be proved formally to be members of the null class.*

2.

Hobson finds the following difficulty:—The series of ordinal numbers and Alephs arise, says he, from the fundamental principle that every well-ordered aggregate has both a type and a cardinal number, but this principle fails with the series W of all ordinal numbers, owing to Burali-Forti's contradiction, and yet it is by means of this very principle that the existence of the successive ordinal numbers which make up W is regarded as having been established. And, further, it is not clear that *every segment* of W must have a type and cardinal number, and a criterion is needed which will enable us to distinguish (at least, theoretically) aggregates which have numbers from those which have not, at some less advanced stage than that at which we define W.

I have departed somewhat from Hobson's phraseology,† but I think I have reproduced accurately the substance of his difficulties. As regards the last, it is evident that we cannot, in a systematic exposition of the theory, use W itself as such a criterion. Historically, W was the first series without a type to be discovered, but we cannot say that we are to judge whether a well-ordered series has a type or not by seeing whether it is similar to a segment of W or not, without a palpable vicious circle. I pointed this out in my first paper‡ on the series in question, and suggested another criterion which will be discussed below (§ 7).

But the other difficulties seem to arise from a misconception of the relation of Burali-Forti's contradiction to Cantor's principles of generation of the ordinal numbers, which is fully discussed in the next section (§ 3), while this is illustrated, in § 4, by the incompleteness of Burali-Forti's argument, and the completion which I gave it.§

* See an article by myself entitled "De Infinito in Mathematica" in Peano's *Rivista di Matematica* (t. VIII., 1906).

† In particular, I avoid the use which Hobson makes, and I formerly made, of the term "inconsistent aggregate." In fact the term is misleading; for I see no reason to think that the aggregate is inconsistent (does not exist), but only that it has no type and no cardinal number (see § 1).

‡ *Phil. Mag.*, Jan., 1904, p. 67.

§ *Ibid.*, Jan., 1905, pp. 51–53.

3.

Cantor's first and second principles of generation of ordinal numbers may be combined and stated in the form: Whenever, starting with the ordinal number 1, we have a finite or infinite series, we posit (or create) a *new* number which is the next greater number to all the numbers of the series, provided always* that the new entity which we postulate forms, together with the old ones, a logically consistent scheme.† In fact, Cantor ‡ has stated his belief that "mathematics is completely free in its development, and has only to pay attention to the self-evident condition that its conceptions are both free from self-contradiction and, in determinate relations, fixed by definitions, to the conception already present and verified. In particular, with the introduction of new numbers, it has only to give definitions of them by which such a definiteness, and, under circumstances, such a relation to the older numbers, is afforded that they can be distinguished from one another in given cases." § Thus we form, successively, the numbers

$$1,\ 2,\ \ldots,\ \nu,\ \ldots,\ \omega,\ \omega+1,\ \ldots,\ \omega+\nu,$$
$$\ldots,\ \omega.2,\ \ldots,\ \omega.\nu,\ \ldots,\ \omega^2,\ \ldots,\ \omega^{\omega},\ \ldots,\ \omega^{\omega^{\omega}},\ \ldots,\ \epsilon,\ \ldots,\ a,\ \ldots,$$

of the second number-class, and then

$$\omega_1,\ \omega_2,\ \ldots,\ \omega_{\omega},\ \ldots,\ \omega_{\omega_\omega},\ \ldots,\ \omega_{\gamma},\ \ldots,$$

where ω_γ is the first number of the γ-th number-class [or the $(\gamma+2)$-th class, if γ is less than ω].

In particular, ω is not the greatest finite number (which is a self-contradictory conception, since we can easily prove that there is no greatest finite number), but is the first number which follows (is greater than) all the finite numbers and, consequently, is *transfinite*. Similarly, ω_1 is a number, transfinite indeed like the numbers of the second class, but of the *third class*. It is highly important to dwell on these distinctions; for, as I will show, Burali-Forti's contradiction (in its completed form) is analogous to the contradiction arising from the statement that there is a greatest finite ordinal number, in that both contradictions arise from supposing a next greater to all the numbers of a certain class to exist which is itself a member of that class.

* This addition shows that Cantor did not use Hobson's "principle" that *every* well-ordered series has a type.

† Cantor, *Grundlagen einer allgemeinen Mannichfaltigkeitslehre*, Leipzig, 1883, p. 33.

‡ *Ibid.*, pp. 19, 45–46.

§ It cannot, I think, be maintained as an historical fact that any advance in mathematics has been brought about by an arbitrary creation of the mind, of which the fruitfulness has only been discovered afterwards, but one cannot object, on logical grounds, to such a creation if non-contradictory.

(*a*) Suppose that there is a greatest finite ordinal number n. Let) be the class of finite integer ordinal numbers; then the proposition m is a member of w" implies that $n > m$. But n is a member of w; onsequently $n > n$, a palpable contradiction.

(*b*) Suppose that there is a greatest ordinal number β. Then, as in 1, we get, if w is the class of ordinal numbers and β is a member of), $\beta > \beta$.

Now (*b*) is a form of Burali-Forti's contradiction, and it is evident hat both contradictions arise from the hypothesis that n [in (*a*)] or β in (*b*)] is a member of w. If this is denied, the contradiction is avoided; hus there is no contradiction in the inequality $n > m$, for every finite ι, if, for example, $n = \omega$; and the generalisation of this argument to he case of w being some or all of the numbers of, or preceding, a ertain number-class, while n is the first number of the next number-lass, is immediate.

Now, if we attempt a like alteration in (*b*), we fail, unless all ordinal umbers γ, which were defined or indicated by Cantor, show themselves s merely particular cases of a more general class of ordinal numbers. 'hat this possibility was not to be thrown aside at once, the following onsiderations show.*

Suppose that the series W of all the numbers defined or indicated y Cantor has a type, which is an ordinal number, β; then the cardinal umber corresponding to W is easily proved to be $\aleph_\beta$. On the other and, we can show that β, if it exists, is the first number of a number-lass, and thus of the form $\beta = \omega_\gamma$; while, if γ is a Cantor's ordinal umber, the cardinal number of the ordinal numbers less than ω_γ is $\aleph_\gamma$.

Thus, then, unless

$$\aleph_{\omega_\gamma} = \aleph_\gamma, \tag{1}$$

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* Cf. *Phil. Mag.*, Jan., 1905, p. 52. We see without difficulty that, if u be a class of ordinal umbers (and, if γ is a u, all less than γ are to be members of u), then, in order that there may e a type of the series of numbers of u, arranged in order of magnitude, it is necessary and ıfficient that we should be able to define without contradiction a class of numbers containing as a *proper* part. Now Cantor's advance lies in the perception that it is useful and indeed ecessary to introduce what he called "number-classes" other than that of the finite integers; nd I think it is clearly shown in the place referred to that Cantor's creation of "number-classes" not possible *beyond* the series W. Hence it is not the case that Cantor's method of postulation ads to contradiction; such an idea arises from a neglect of the essential in Cantor's method, amely, that, if γ is an ordinal number, it is of the ζ-th class (ζ being some ordinal number), and ence there are numbers greater than γ [of the $(\zeta+1)$-th class, for example].

The first number of the $(\zeta+2)$-th class (or ζ-th, if $\zeta \geqslant \omega$) is ω_ζ, and ω_γ never $< \gamma$, it may be γ, but I see no reason why there should not be γ's such that $\omega_\gamma = \gamma$ (in other words, why the mit of $\omega, \omega_\omega, \omega_{\omega_\omega}, \ldots$ should not exist). Hence my construction in the *Messenger of Maths.*,)05, pp. 56–58, appears false.

we have no grounds for asserting that the series of Cantor's ordinal numbers has not a definite type (β); β would be, not the greatest ordinal number [a conception which leads to the contradiction (b)], but the least ordinal number which is greater than all of Cantor's ordinal numbers.

4.

The contradiction of Burali-Forti was, then, incompletely stated. Either it is a statement about the series of Cantor's numbers, in which case it is necessary for its validity to prove the equation (1); or else it is a statement about the *whole* series of ordinal numbers, in which case it is valid, but appears to use implicitly the conception of certain numbers whose existence has never been contemplated by Cantor or (if I except myself, for a short time) any one else.

However, we *can* prove (1), and hence that every ordinal number is a Cantor's ordinal number. In fact, if there is such a type as β or ω_γ, γ cannot be a Cantor's number, and hence, since γ can never be greater than ω_γ,

$$\gamma = \beta \quad \text{or} \quad \omega_\beta = \beta,$$

whence (1) follows.

5.

It will now, I think, be clear that Cantor's second principle of generation creates new numbers* by the constant avoidance of such contradictions as (a). Hence, instead of saying that the unrestricted application of the second principle leads to contradiction, it seems to me to be more correct to say that in the case (b) the second principle cannot be applied.

Further, I think that the above considerations show that doubts as to the existence (that is to say, the non-contradictory nature) of the type of any segment of W can hardly be maintained seriously.†

6.

Hobson proposes to adopt the "less ambitious procedure of postulating the existence of definite ordinal numbers of a limited number of classes

* I may be permitted to protest again here (cf. *Phil. Mag.*, Jan., 1905, p. 51) against the arbitrary restriction of Cantor's second principle to the *second* class, and the pretended necessity of a *third* principle to create ω_1, and so on. The second principle (together with the first), as stated by Cantor, suffices to create all the ordinal numbers.

† By this I mean that "the type of a segment of W" is not contradictory in the same way as "the type of W," and the latter is the only type of a series of ordinals which has been shown to lead to contradictions. It cannot, I think, be serious mathematics to doubt the existence of a thing without any reason for doing so, but, if anything, with reasons against doing so.

in accordance with Cantor's earlier method.* So long as the postulation of the existence of ordinal numbers does not go beyond some definite point, no contradiction will arise, and the utility of the scheme, for purposes of representation, will suffice to justify the postulations which have been made." † In this way, Burali-Forti's contradiction is certainly avoided, but Hobson's view does not, on this account, appear to have "an advantage over that of Russell," ‡ unless, indeed, discretion is the better part of valour.

Further, we must guard against confusing the utility which a mathematical conception may have—and but for which it would hardly have been conceived—with the purely logical question of whether the conception is possible (exists, or is non-contradictory).§

But, apart from these considerations, Hobson's criterion for the existence of a number ‖ does not appear to be above criticism. This criterion (for ordinal numbers) seems to be capable of statement in the form: "The existence of an ordinal number cannot be inferred from the existence of that *single* series of the preceding ordinal numbers, but it can if, and only if, other series other than the above number-series (and similarly ordered to it), such as series of points on a straight line, can be exhibited." The motive for this requirement of *other* series was that, if only one series is considered, we must leave out of account that conception of a number as the common characteristic of *many* similarly ordered series.

But, in the first place, if we have one series A, we can obtain other series similarly ordered to A by interchanging two terms of A or by replacing a term by something else.¶ In the second place, since a point-series is merely a picturesque way of describing a series of real numbers, I am at a loss to understand why a series of integers should not be allowed a type until a series of real numbers can be found to support its claim. Moreover, it seems inconsequent to accept the number continuum (as a sort of substratum), if the existence of $\aleph_1$ is doubted because

* That is to say, the method of 1882 (see Cantor's *Grundlagen*, pp. 32–35, and § 3 above). In Cantor's memoir of 1897 (*Math. Ann.*, Bd. XLIX., pp. 207–246) the first order of things (which now seems preferable, because of Burali-Forti's contradiction) is inverted, and the principles of generation given a secondary place (p. 226).

† *Loc. cit.*, p. 176.

‡ *Loc. cit.*, p. 180.

§ See § 3 of this paper.

‖ *Loc. cit.*, pp. 176–181.

¶ This is sufficient to show that those classes which are called "numbers" by Russell all have (except 0) more than one element; whereas the contrary was stated by Hobson (*loc. cit.*, p. 179).

Hobson doubts Hardy's construction of an aggregate of points of this cardinal number.

The criticism* of Russell's† objection to the assumption that a class of similar classes has "a common characteristic" contained in the words: "The mind does, however, in point of fact, in the case of finite aggregates at least, recognize the existence of such single entity, the number of the aggregates," must be mentioned. This is an appeal to common sense, and is quite irrelevant here. Russell, by defining numbers as classes, avoided the introduction of new indefinables, while such numbers satisfy all the formal laws.‡ Hobson would, apparently, introduce a new indefinable, "the mind," into mathematics, and make psychology a foundation of mathematics. It is, I think, true that in the history (and teaching) of mathematics one should endeavour to present the development of the science as a succession of human documents (of mathematicians), but such psychological information is irrelevant and intolerable in mathematics regarded as a body of logical doctrine.

7.

It is important to define a series ordinally similar to W, but in which numbers are not mentioned. My two attempts both failed: the first§ because $\mathfrak{W}$ is not a substitute for W, the second‖ (as already mentioned in a note to § 3) because it seems that we can find genuine ordinal numbers (γ) for which

$$\omega_\gamma = \gamma.$$

I see no alternative but to define such a series as any well-ordered series such that its "type" is a non-entity, while that of any segment of it is not. If we suppose the elements of this series to form a class, we can formally prove that class to be a member of the null-class, as indicated at the end of § 1.

8.

For the definition of an aggregate,¶ a "norm" is, according to Hobson,

* *Loc. cit.*, pp. 179–180.

† *Op. cit.*, pp. 114–115.

‡ I only mention this one advantage of Russell's definition; the other "common sense" objections were already sufficiently dealt with in *The Principles*, pp. 114–115, 304–307. [See also the Note at the end of this paper.]

§ *Phil. Mag.*, Jan. 1904, p. 67.

‖ "The Definition of a Series similarly ordered to the Series of all Ordinal Numbers," *Mess. of Math.*, 1905, pp. 56–58.

¶ Hobson also deals with *series*, and says (p. 174): "In order that a transfinite aggregate may be capable of being ordered, a principle of order must be explicitly or implicitly contained in the norm by which the aggregate is defined." I am unable to see in this more than the truism: "In

necessary. A "norm" is the word he uses for "a law or set of laws by which the aggregate is defined," and he proceeds: "It is, however, convenient to admit the case of two or more alternative sets of conditions: thus an aggregate may contain all objects each of which satisfies either the conditions A or else one of the sets of conditions B, C, ..., K. The conditions forming the norm by which the aggregate is defined must be of a sufficiently precise character to make it logically determinate as regards any particular object whatever, whether such object does, or does not, belong to the aggregate."* And,† "In the case of a finite aggregate, the norm may take the form of individual specification of the objects which form the aggregate."

This, it appears to me, may be expressed more simply by the words: "A class (or aggregate) is all the entities x such that a certain propositional function $\phi(x)$ is true of each."‡ It should be noted that a definable class need not be an existent class. Thus, if $\phi(x)$ is not true for any x (say: "x is not identical with x"), the class is the *null-class*.§

Hobson's definition seems to be (i.) redundant, since with *every* class it is logically determined whether a particular thing is or is not a member of it, and (ii.) incomplete, since it neglects the fact, which is vital in the explanation of Burali-Forti's (and Russell's ‖) contradiction, that there are propositional functions which do not define classes.

As regards (i.), it amounts to an assertion that *every* propositional function is as "sufficiently precise" as Hobson requires it to be. If I may venture to interpret Hobson, it seems to me that he was thinking of propositional functions in which an indefinable occurs which is not one of logic and mathematics (is not a logical constant).¶ Thus "x is a poet" is a function which can hardly be said to divide humanity into two classes —poets and non-poets. For the term "poet" is probably incapable of

order that a transfinite aggregate may be capable of being ordered, it must be capable of being ordered." For I see no reason why an aggregate may not be ordered in accordance with "some law extrinsically imposed upon the aggregate," and, indeed, Zermelo's proof shows this to be possible, provided that his axiom be granted. In fact, Hobson seems to have been led to his condition by such observations as: "It is difficult, if not impossible, to see how order could be imposed upon" the aggregate of all functions of a real variable. But it is well known that it is possible to arrange all *continuous* functions of a real variable (which appears, *a priori*, just as little capable of order) in a series of type θ (for example).

* *Loc. cit.*, pp. 172, 173. Cf. Cantor, *Math. Ann.*, Bd. xx. (1882), p. 114.

† *Loc. cit.*, p. 173.

‡ Russell, *op. cit.*, p. 20.

§ *Ibid.*, pp. 22, 23.

‖ *Ibid.*, pp. 79, 80, 101-107, 366-368.

¶ *Ibid.*, p. 3.

definition: if "poetry" is defined as "metrical composition," the present Poet Laureate would be a poet and Walt Whitman would not. And both statements might provoke discussion. Thus, also, the function by which du Bois-Reymond's decimal* is defined is "x is either 0 or 1, as determined by the chance throw of dice," and is not expressed in logical constants (and "chance" is, perhaps, as obscure in meaning as "poetry"). The "series of arbitrary selections,"† to which Hobson compares du Bois-Reymond's decimal, is, on the contrary, to be justified by an axiom (whose truth can hardly be denied seriously) which is expressible in terms of the logical constants (§§ 11–14).

9.

But the real meaning of Hobson's requirement of a "norm" appears, I think, later. Thus, in order to show that the totality of the numbers of the second number-class, taken in order, has a type or a cardinal number, "it would be necessary to show that a finite set of rules can be set up which will suffice to define a definite object corresponding to each ordinal number of the second class." And Hobson maintains† that an aggregate the elements of which are regarded as being successively defined by an endless series of separate acts of choice cannot be contemplated as existing, but that some "norm" must be assigned by which the successive elements are defined. On these grounds, he objects to Cantor's‡ proof that every transfinite aggregate has an enumerable component; to Hardy's§ proof that every cardinal number is either an Aleph or is greater than all Alephs, and, in particular, that

$$2^{\aleph_0} \geqslant \aleph_1;$$

to my‖ proof, that every cardinal number must be an Aleph; and to Zermelo's¶ proof, that every aggregate can be well-ordered.

10.

This restriction of the "norm" to a *finite* set of rules seems to be demonstrably too narrow. In fact, I will show that the class of all entities which are definable by a finite set of rules is of cardinal number

* Hobson, *loc. cit.*, p. 182.

† *Loc cit.*, pp. 182–185.

‡ *Math. Ann.*, Bd. XLVI. (1895), p. 493.

§ *Quart. Journ. of Math.*, Vol. XXXV. (1903), pp. 87–94. Cf. Hobson, *loc. cit.*, pp. 183, 178, 185–188 (where also a method of Young's is noticed; cf. *Proc. Lond. Math. Soc.*, Ser. 2, Vol. 1, p. 243).

‖ *Phil. Mag.*, Jan. 1904, pp. 63, 64, 70 (in particular).

¶ *Math. Ann.*, Bd. LIX. (1904), pp. 514–516.

$\aleph_0$,* and we can always define perfectly definitely an enitity not in this class.

The predicate "definable by a finite set of rules" is expressed, more clearly, I think, by the words: "definable by what is, symbolically, a finite series of variables and logical constants,† in which any constant may be repeated a finite number of times," and the theorem in question can be proved in a manner quite analogous to my earlier theorem referred to in the note, as follows.

If $\mathfrak{m}$ and $\mathfrak{n}$ are finite cardinal numbers, and μ is the type of a finite series to which the cardinal number $\mathfrak{m}$ belongs, then the cardinal number of entities definable by a series of type μ of indefinables and variables with $\mathfrak{n}$ different indefinables and variables is at most‡

$$\mathfrak{n}^{\mathfrak{m}}.$$

We get certainly all such definable entities by letting μ take all values less than ω, in succession, and add the results; thus, if $\mathfrak{a}$ is the cardinal number of all such definable entities,

$$\mathfrak{a} \leqslant \sum_{\mu<\omega} \mathfrak{n}^{\mathfrak{m}} \leqslant \sum_{\mu<\omega} \aleph_0^{\mathfrak{m}} = \aleph_0, \tag{2}$$

and we see that, for our purpose, we may take $\mathfrak{n}$ as variable, but finite. Again, since an aggregate of cardinal number $\aleph_0$ (such as the aggregate of finite cardinal numbers) is so definable, we have also

$$\mathfrak{a} \geqslant \aleph_0. \tag{3}$$

From (2) and (3), now, $\mathfrak{a} = \aleph_0.$

Thus, only $\aleph_0$ real numbers are definable by "norms," and yet, by an obvious modification of Cantor's§ process for proving that

$$2^{\aleph_0} > \aleph_0,$$

we can construct a definite real number not defined by a "norm." Thus Hobson's "norms" cannot define all the real numbers which we have seen to be capable of definition.

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* Cf. my proof, in *Phil. Mag.*, Jan., 1905, that the cardinal number of all "actually" (better, *practically*) representable real numbers is $\aleph_0$. Soon after this (April 28th, 1905) Russell communicated to me what is, in effect the above generalisation, and which appears to me to be identical with König's somewhat obscurely expressed theorem (*Math. Ann.*, Bd. LXI., 1905, pp. 156–160) that the cardinal number of all "finitely defined" numbers is $\aleph_0$. [Cf. also A. C. Dixon, *Proc. Lond. Math. Soc.*, Ser. 2, Vol. 4, 1906, pp. 18–20.]

† See Russell, *op. cit.*, pp. 3, 5, 6.

‡ This is an (unattained) upper limit; for some of the "definitions" thus formed are meaningless, just as some "words" of μ letters are.

§ *Jahresber. der Deutsch. Math.-Ver.*, Bd. I., 1892, p. 77.

11.

With regard to Cantor's argument* that it is possible to pick an enumerable aggregate out of *any* given transfinite aggregate (M), from which we may conclude, by the use of the Schröder-Bernstein theorem,† that

$$\mathfrak{m} \geqslant \aleph_0,$$

it rests on an endless series of acts of arbitrary selection of elements of M. But this can be avoided if only we admit the possibility of fixing on a single (arbitrary) element m_1 of M, and on a definite one-one correspondence between M and that part of M which arises by taking m_1 from M.‡ Let, then, $\phi(m_1) = m_2$ be the correlate of m_1, $\phi(m_2) = m_3$ that of m_2, and so on; it is easily seen that the sequence

$$m_1,\ m_2,\ m_3,\ \ldots,\ m_\nu,\ \ldots \tag{4}$$

consists of different elements of M, and is enumerable. Cantor's theorem is then proved.§

It is easily seen that, in order to continue the sequence (4) to obtain other elements

$$m_\omega,\ m_{\omega+1},\ m_{\omega+2},\ \ldots,$$

it is necessary to establish a *new* correspondence ϕ_1, if this is possible, between M and the part obtained by taking all the elements of (4) from M. Then

$$m_\omega = \phi_1(m_1), \quad m_{\omega+1} = \phi_1(m_\omega), \quad \ldots.$$

And when we try to take from M a sequence of elements of type ω_1 it evidently becomes necessary to perform an endless series of acts of arbitrary selection, in forming the series

$$\phi_1,\ \phi_2,\ \phi_3,\ \ldots;$$

for each of these ϕ's is one out of an infinity of equivalent ones—for there are an infinity of ways of establishing a one-one correspondence between two similar infinite aggregates.

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* *Math. Ann.*, Bd. XLVI., 1895, p. 493.

† It has often been thought that this theorem is: If $\mathfrak{a} \geqslant \mathfrak{b}$ and $\mathfrak{b} \geqslant \mathfrak{a}$, then $\mathfrak{a} = \mathfrak{b}$. But this is a simple logical conclusion; while the Schröder-Bernstein theorem may be stated: If a part of the aggregate A is similar (equivalent) to B, then $\mathfrak{a} \geqslant \mathfrak{b}$, the signs $>$ and $=$ having been defined by Cantor (*Math. Ann.*, Bd. XLVI., 1895, pp. 482–484).

‡ This property may, of course, be taken for the *definition* of the "infinite" character of M.

§ The above method, as I subsequently recognised, is that in which Dedekind ("Was sind und was sollen die Zahlen?", 1887 and 1893, § 6) derives a "simply infinite system" (an aggregate arranged in type ω) from any infinite system, and is, in essentials, the method of Schröder and Bernstein, in the proof of the theorem known by their names (cf. Schoenflies, *Die Entwickelung der Lehre von den Punktmannigfaltigkeiten*, Leipzig, 1900, pp. 16–18; Zermelo, *Gött. Nachr.*, 1901, pp. 34–38; A. E. Harward, *Phil. Mag.*, Oct., 1905, p. 457).

Thus Hardy's extension of Cantor's argument, in which elements corresponding to *higher* number classes are successively removed, and from which follows, by the Schröder-Bernstein theorem, that any cardinal number is either an Aleph or is greater than all Alephs (follows them all when arranged in order of magnitude), cannot be replaced, with any advantage, by an extension of the former method.

It is one of the greatest triumphs of the symbolic logic developed, for the most part, by Peano and Russell to have showed clearly that there is an assumption involved in what is popularly expressed as the performance of any infinite series of acts of arbitrary selection. Before Russell had noticed this, the need of such an assumption to make such arguments rigorous had, indeed, been pointed out by Zermelo, Beppo Levi, and Bernstein,* and, less clearly, Borel; obscurely felt by Hobson and Harward; and disregarded by all others who have written on the subject.

12.

Let us examine more closely the first stage of Hardy's argument. If M be a given transfinite aggregate, we can always, by what precedes, remove an enumerable aggregate from M. Hence, by the Schröder-Bernstein theorem,

$$\mathfrak{m} \geqslant \aleph_0.$$

If we dismiss the case of equality, we are to prove that

$$\mathfrak{m} \geqslant \aleph_1. \tag{5}$$

For this purpose, we remark that, firstly, if γ is any fixed number of the second number class, there is always a part of M similar to the (enumerable) aggregate of ordinal numbers less than γ; secondly, if γ' is any number of the second number class greater than γ, there is always a part of M similar to the aggregate of ordinal numbers less than γ', *and which contains the former part as a proper part.* This last condition is easily seen to be essential, since we cannot prove (5) from the knowledge that M has an enumerable part, by arguing that this *same* part can always be reordered in the type of any number of the second class.

Now, if ϕ is a definite one-one correspondence which images a definite part u of M on the numbers less than γ, there is not only one, but *an infinity*, of correspondences ψ which fulfil the two conditions which we require: (i.) $\psi(u) = \phi(u)$, (ii.) ψ images a part (v) of M on the numbers

* Cf. Bernstein, "Bemerkung zur Mengenlehre," *Gött. Nachr.*, 1904, pp. 1–4.

less than γ';* from which we deduce that u is a proper part of v, and, if z is that part of v which is left when u is taken away, $\psi(z)$ is the aggregate of ordinal numbers ζ, such that

$$\gamma' > \zeta \geqslant \gamma.$$

We have to pick out *one* definite ψ for *each* γ' in order to define a part of M which is of cardinal number $\aleph_1$. That this is possible, since for each γ' there is an infinite choice of such ψ's, though no pre-eminent one (for example, there is not, in general, a "first" or a "last" ψ for each γ'), may be true, and, in fact, has frequently been considered as obvious; that it is a supposition, which needs either a proof or a new axiom asserting it, becomes evident on a closer logical consideration.

13.

If we have a class u defined as "the entities x such that some propositional function (ϕx) is true of them," we cannot derive the legitimacy of some proposition "x is a u" without the premiss "u is not the null-class."† The latter premiss, which may also be expressed: "the class u exists" (or, more popularly, "the class u has at least one member"), plays a very important part in Russell's work, since it is one of his chief objects to prove the "existence theorems of mathematics," that is to say, to prove that the various classes (such as numbers and types) defined in mathematics are not null.

In order to define the product of any class (not necessarily finite) of cardinal numbers, we form, with Whitehead,‡ the conception of the "multiplicative class" of a class of classes no two of which have any term in common. Let k be such a class of exclusive classes; the multiplicative class of k is the class each of whose terms is a class formed by choosing one and only one term from each of the classes (p) which are terms of k. Then the cardinal number of terms in the multiplicative class of k is defined to be the product of all the numbers of

* That there *are* such imagings as ψ is evident from the consideration that, if (ii.) could not be fulfilled at the same time as (i.), there would be a *lowest* ordinal number amongst those equal to or greater than γ and less than *or equal to* γ', to which *no* imaging ψ such that $\psi(u) = \phi(u)$ can correlate an element of M. But this is impossible.

We cannot say, instead of (ii.): "ψ images a part of M on the numbers less than ω_1"; for, although we can prove that there is no number less than ω_1 which is the lowest of those to which no imaging χ such that $\chi(u) = \phi(u)$ corresponds, we cannot prove that ω_1 itself is not.

† Cf. Whitehead, "On Cardinal Numbers," *Amer. Jour. of Math.*, Vol. xxiv., 1902, p. 373.

‡ *Loc. cit.*, pp. 369, 383, 385.

the various classes composing k, and is denoted:

$$\prod_{p \epsilon k} p.$$

The theorem or axiom of the existence of the multiplicative class was necessary to the validity of many of Whitehead's proofs, but was probably considered by him as not needing a proof. A multiplicative class occurs, as we have seen, in the proof that any aggregate has either a cardinal number which is an Aleph or a part similar to the aggregate of ordinal numbers (or Alephs). In this case our supposition may be expressed as follows (an axiom which is equivalent to Zermelo's):—

Consider as argument of a function an aggregate of existent classes (not necessarily exclusive*); we imagine a *many*-valued function defined for the whole of this argument-aggregate in such a way that, if of the argument-classes, $f(x)$ is a class which is any x. Then our axiom is that there exists† a *one*-valued function F of the same argument, such that $F(x)$ is a member of the class $f(x)$.‡

14.

It remains to consider the evidence for the truth of the axiom. It is that this type of argument has been independently used and considered valid by almost all writers on the theory of aggregates. We have seen it used by Cantor, Zermelo, Hardy, and Whitehead; it is involved in some theorems and conceptions of Schoenflies§ and König,‖ and Bernstein¶ has, in consequence of a remark made by Beppo Levi,** explicitly formulated and adopted the axiom, which was involved in his proof that the class of closed aggregates is similar to the continuum; and we may remark that Levi's proof that there is a *definite* way of putting all closed aggregates in a one-one correspondence with the continuum if metrical properties (which Bernstein avoids) are used, supports the truth of the axiom.

* They must be exclusive for Whitehead's purpose.

† That is to say, can be defined "sub specie aeternitatis," if I may so express myself.

‡ We do not assume that *all* such functions F form a class, but merely that the statement that F is a function of the kind required is not false for every F.

§ *Op. cit.*, pp. 9, 13–14, 26, 41.

‖ "Zum Kontinuum-Problem," *Verhandl. des dritten internat. Math.-Kongr.*, 1905, pp. 144–147.

¶ "Bemerkung zur Mengenlehre," *Gött. Nachr.*, 1904, pp. 1–4.

** "Intorno alla teoria degli aggregati," *Lomb. Ist. Rend.*, (2), t. xxxv., 1902, pp. 863–869: cf. Bernstein in *Math. Ann.*, Bd. LXI., 1905, pp. 132, 146.

Also my proof* of the equality†

$$\aleph_\gamma \cdot \aleph_\gamma = \aleph_\gamma;$$

a proof which, I have learnt,‡ is supported by the authority of Cantor, utilises this axiom; as also does my proof§ that a series with no first term contains a part of type

$$^*\omega,$$

a very evident theorem.

[*Note added April* 30*th*, 1906.

Since writing the above, some papers have appeared which must be briefly referred to here.

Russell|| has discussed Hobson's paper, and many of his criticisms are, in substance, the same as mine.¶ In particular, Russell and I were led independently to the observation that certain propositional functions do not define classes: he, from working on his (cardinal) contradiction; I, from working on the ordinal contradiction of Burali-Forti. My own result is implicitly contained in my first paper of 1903,** in which I maintained that *W* has no type. This point of view is obviously different from the one attributed to me by Russell,†† that the alternative that the type of *W* does not exist is neglected, while the alternative that *W* does not exist is accepted. In fact, it has always seemed to me absurd to suppose that there is no such thing as *W*; for we can, without contradiction, define classes such that there are parts of them similar to the class of ordinal numbers, and the latter class would not exist if *W* did not.

What justifies the naming of my own theory (with some others, like Cantor's) the "theory of the limitation of size" is that I differ from upholders of the "zig-zag theory," in not admitting that a contradictory "class," like "the x's such that x is not an x," can become an irreproachable class, like "the class of all classes," by adding new members to it. For this reason, then, a supporter of any "limitation of size" theory must

* *Phil. Mag.*, March, 1904, pp. 295–301.

† This equality must be proved before it can be proved that the Alephs form a series (*ibid.*, Jan., 1904, p. 74).

‡ Bernstein (*Math. Ann.*, Bd. LXI., 1905, pp. 150, 151) states that Cantor had communicated this result to him, having proved it in what is, apparently, the method I followed.

§ *Phil. Mag.*, Jan., 1904, p. 65.

|| "On some Difficulties in the Theory of Transfinite Numbers and Order Types," *Proc. Lond. Math. Soc.*, Ser. 2, Vol. 4, pp. 29–53.

¶ *Ibid.*, pp. 30, 31, 40–43.

** *Phil. Mag.*, Jan., 1904, p. 66.

†† *Loc. cit.*, p. 44.

deny both the notion of a class of all classes and the definition of numbers as classes.*

With regard to the "multiplicative axiom" and the more general Zermelo's axiom,† the following remark may be made. Zermelo's axiom is equivalent to the axiom that, if u is a class and v is the class of all non-null classes which are contained in u, then the complex of propositions: w is a class; w is contained in u;‡ x is a v implies, for every such x, that x and w have one, and only one, term in common;—is not false for every value of w. This axiom (cf. § 13 above), which I am accustomed to call the "multiplicative axiom," has the advantage of being quite analogous to the "multiplicative axiom" of Russell, which is used in the conception of the product of an infinity of cardinal numbers, and which results from the above, if we substitute "v is any class of *mutually exclusive* classes" and "w is contained in the logical sum of v."

Hobson§ has devoted a paper to showing that König's distinction between "finitely defined" entities and those which are not so defined is not valid. This is not the place to examine Hobson's arguments, but I may remark that his method‖ is the same as the one I have given in § 10 of my above paper for showing that Hobson's requirement of a "norm"¶ is too narrow.

Finally, Schönflies** suggests that Russell's contradiction is really the statement: "the class of all classes is a member of itself." I fail to see that this is contradiction, and Schönflies†† only assumes that it is so. Also Schönflies'‡‡ remark that such "classes" as Russell's are contradictory can hardly be regarded, as he states, as a solution of the contradiction.]

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* See Russell, *loc. cit.*, p. 39.

† *Ibid.*, pp. 47–53.

‡ Here u is the "logical sum of v."

§ "On the Arithmetic Continuum," *Proc. Lond. Math. Soc.*, Ser. 2, Vol. 4, 1906, pp. 21–28.

‖ *Loc. cit.*, pp. 24, 25.

¶ *Ibid.*, Vol. 3, 1905, p. 180.

** "Über die logischen Paradoxien der Mengenlehre," *Jahresber. d. Deutsch. Math.-Ver.*, Bd. xv., 1906, pp. 19–25.

†† *Loc. cit.*, p. 22.

‡‡ *Loc. cit.*, p. 20.

ON CERTAIN FUNCTIONS DEFINED BY TAYLOR'S SERIES OF FINITE RADIUS OF CONVERGENCE

By E. W. BARNES.

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1. The function $g_\beta(x;\theta)$ is defined when $|x|<1$ by the Taylor's series

$$\sum_{n=0}^{\infty} \frac{x^n}{(n+\theta)^\beta}.$$

When β is a positive integer the function can be derived from the case when $\beta=1$ by differentiation with regard to θ. The function

$$g(x;\theta) = \sum_{n=0}^{\infty} \frac{x^n}{n+\theta}$$

has been separately studied by the author.*

We shall therefore assume in the present investigation that β is not equal to zero or a positive integer. The subsequent theory is a development of the investigation given in the author's memoir "On the Asymptotic Expansion of Integral Functions defined by Taylor's Series."† Some of the following results were originally communicated in that paper. On account of its length they were merely stated ‡ in brief without proofs; the complete investigation, with some extensions, is now given. I refer to the introduction to that paper for an account of the general history and literature of the subject.

We shall assume that θ is not zero or a negative integer; in such cases the function $g_\beta(x;\theta)$ evidently does not exist.

We shall also assume that, in the definition of $g_\beta(x;\theta)$,

$$(n+\theta)^\beta = \exp\{\beta \log(n+\theta)\},$$

wherein $$0 < |I\{\log(n+\theta)\}| < \pi.$$

This definition completely specifies the function when $|x|<1$ and θ is not real and negative. In the latter case we may conveniently take

$$I\{\log(n+\theta)\} = \pi$$

when $(n+\theta)$ is negative. We thus arbitrarily specify at most only a finite number of terms of the series.

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* *Quarterly Journal of Mathematics*, Vol. XXXVII., pp. 289–313.

† *Philosophical Transactions of the Royal Society* (A), Vol. 206, pp. 249–297.

‡ *Loc. cit.*, Parts IV. and VI.

We use throughout $I\{f(x)\}$ to denote the imaginary part of $f(x)$, $R\{f(x)\}$ denoting its real part. Thus the condition

$$0 < |I\{\log(n+\theta)\}| < \pi$$

is equivalent to
$$-\pi < \frac{1}{\iota} I\{\log(n+\theta)\} < \pi.$$

2. I propose to establish the following propositions:—

(1) The function $g_\beta(x;\theta)$ has a single singularity in the finite part of the plane. The singularity occurs at $x=1$, and is not an essential singularity.

(2) The function $g_\beta(x;\theta)-g_\beta(x;1)/x^{\theta-1}$ has no singularities in the finite part of the plane, and, if $|\log x| < 2\pi$, it admits the expansion

$$\frac{1}{x^\theta}\sum_{n=0}^{\infty}\frac{(\log x)^n}{n!}\{\zeta(\beta-n,\theta)-\zeta(\beta-n,1)\}.$$

(3) Near $x=1$, $g_\beta(x;\theta)$ is many-valued.

(4) The function $g_\beta(x;\theta)-\Gamma(1-\beta)(-\log x)^{\beta-1}x^{-\theta}$ is one-valued near $x=1$, and in the vicinity of this point admits the convergent expansion

$$\sum_{n=0}^{\infty}\frac{(x-1)^n}{x^{n+1}}\bar{\zeta}_{n+1}(\beta,\theta),$$

where $\bar{\zeta}_{n+1}(\beta,\theta)$ denotes the $(n+1)$-ple Riemann ζ function of equal parameters unity.

(5) If θ be not real, the function

$$g_\beta(x;\theta)+g_\beta(x^{-1};-\theta)\,e^{\mp\pi\iota\beta},$$

the negative or positive sign being taken as $I(\theta) >$ or < 0, is one-valued near $x=1$, and has no singularity at this point.

(6) If θ be not real and a positive or negative integer (zero included), $g_\beta(x;\theta)$ admits, when $|x|$ is very large, the asymptotic expansion

$$\left\{\frac{1}{(-\theta)^\beta}-g_\beta\left(\frac{1}{x};-\theta\right)\right\}e^{\mp\pi\iota\beta}+\frac{[\log(-x)]^{\beta-1}}{(-x)^\theta}\sum_{n=0}^{\infty}\frac{(-)^n\left(\dfrac{\pi}{\sin\pi\theta}\right)^{(n)}}{n!\,\Gamma(\beta-n)[\log(-x)]^n}.$$

The modification of the previous theorem, when θ is zero or a negative integer, will be indicated.

Spence's formulæ connecting the functions $\sum\limits_{m=1}^{\infty}\dfrac{x^m}{m^n}$ and $\sum\limits_{m=1}^{\infty}\dfrac{1}{x^m m^n}$ when n is an integer will be deduced.

It will be shown that the proposition (4) leads to the result previously obtained when $\beta=1$.

In Part II. of the paper similar results are established for the more general function $f_\beta(x;\theta)$, defined, when $|x|<1$, by the series

$$\sum_{n=0}^{\infty} \frac{x^n \chi(n+\theta)}{(n+\theta)^\beta},$$

when, outside a circle outside which the points $n+\theta$ $(n = 0, 1, \ldots, \infty)$ all lie, $\chi(x)$ admits the convergent expansion $\sum\limits_{r=0}^{\infty} b_r/x^r$.

PART I.—*The Function* $g_\beta(x;\theta)$.

3. *To shew that* $g_\beta(x;\theta)$ *has no singularities except possibly on the real axis between* $x=1$ *and* $x=+\infty$*, the limits included.*

We have

$$\sum_{n=0}^{N-1} \frac{x^n}{(n+\theta)^\beta} = \int_0^\infty e^{-z} \sum_{n=0}^{N-1} \frac{(xz)^n}{(n+\theta)^\beta\, n!}\, dz$$

$$= \int_0^\infty e^{-z} G_\beta(xz;\theta)\, dz - \int_0^\infty e^{-z} \sum_{n=N}^{\infty} \frac{(xz)^n}{(n+\theta)^\beta\, n!}\, dz,$$

where
$$G_\beta(xz;\theta) = \sum_{n=0}^{\infty} \frac{(xz)^n}{(n+\theta)^\beta\, n!},$$

and the integration is along the real axis.

Now, when N is large

$$\left| \sum_{n=N}^{\infty} \frac{(xz)^n}{(n+\theta)^\beta\, n!} \right| < \eta_N \frac{|xz|^N}{(N-k)!} e^{|xz|},$$

where η_N tends to zero as N tends to infinity, if $k > R(-\beta)$. Hence

$$\left| \int_0^\infty e^{-z} \sum_{n=N}^{\infty} \frac{(xz)^n}{(n+\theta)^\beta\, n!}\, dz \right| < \eta_N |x|^N \int_0^\infty e^{-(1-|x|)z} \frac{z^N}{(N-k)!}\, dz,$$

and, if $\dfrac{|x|}{1-|x|} < 1$, this expression tends to zero as N tends to infinity.

Therefore, if $|x|$ be sufficiently small,

$$g_\beta(x;\theta) = \int_0^\infty e^{-z} G_\beta(xz;\theta)\, dz.$$

Now, when $|xz|$ is large, both Mr. Hardy and I have shown that

$$G_\beta(xz;\theta) = \left\{ \frac{e^{xz}}{(xz)^\beta} P(xz) + (-xz)^{-\theta} [\log(-xz)]^{\beta-1} Q(xz) \right\},$$

where $|P(xz)|$ and $|Q(xz)|$ tend to definite finite limits as $|xz|$ tends to infinity.

Therefore the integral $\int_0^\infty e^{-z} G_\beta(xz;\theta)\,dz$

is finite for all values of x such that $R(x) < 1$. It is evidently an analytic function of x for all such values, and therefore it represents the continuation of $g_\beta(x;\theta)$ for all such values of x.

Again, if $\int_0^\infty (B)$ denote an integral along an axis in the positive half of the z-plane, $\int_0^\infty (A)$ denoting the original integral along the positive half of the real axis,

$$\int_0^\infty (A) = \int_0^\infty (B)$$

when $R(x) < 1$; for they differ by an integral along a contour at infinity which vanishes. Therefore $\int_0^\infty (B)$ represents the continuation of $g_\beta(x;\theta)$ for all values of x for which it is finite and continuous.

By taking suitable directions for the B-integral, we see that $g_\beta(x;\theta)$ can be continued for all values of x such that $|\arg(1-x)| < \pi$, and that it has no singularities in this region. We therefore have the given theorem. The line $(1, \infty)$ serves as a cross-cut to render the function $g_\beta(x;\theta)$ one-valued.

4. We will now shew that *the function*

$$g_\beta(x;\theta) - \frac{g_\beta(x;1)}{x^{\theta-1}}$$

has no singularities in the finite part of the plane except $x=0$, and that, near $x=1$, it admits the expansion

$$\frac{1}{x^\theta}\sum_{n=0}^{\infty}\frac{(\log x)^n}{n!}\{\zeta(\beta-n,\theta)-\zeta(\beta-n,1)\}.$$

Let $1/L$ be an axis from the origin within 90° of the axis to the point a, and let L be the image of $1/L$ in the real axis. Then, if the integral be taken round a Gamma function contour embracing the axis L,

$$\frac{\iota\Gamma(1-\beta)}{2\pi}\int_L (-y)^{\beta-1} e^{-ay}\,dy = \frac{1}{a^\beta},$$

where $(-y)^{\beta-1}$ has a cross-cut along the axis L, and $\log(-y)$ is real when y is real and negative, and where a^β has a cross-cut along $-1/L$ (*i.e.*, the negative direction of the axis $1/L$), and is real when a is real and positive. We assume that a is not real and negative.

Consider the integral

$$I = \frac{\iota\Gamma(1-\beta)}{2\pi}\int_L (-y)^{\beta-1}\frac{e^{-y\theta}}{1-xe^{-y}}\,dy,$$

where the contour excludes the poles $\log x \pm 2n\pi\iota$ of the subject of integration. If θ be not real and negative, we can determine L so that it is within $90°$ of the axes to the points $\theta+n$, $n = 0, 1, 2, \ldots, \infty$. We have

$$I = \frac{\iota\Gamma(1-\beta)}{2\pi}\int_L (-y)^{\beta-1} e^{-y\theta} \sum_{n=0}^{N-1} x^n e^{-ny}\, dy$$
$$+\frac{\iota\Gamma(1-\beta)}{2\pi}\int_L (-y)^{\beta-1} \frac{x^N e^{-y(\theta+N)}}{1-xe^{-y}}\, dy;$$

and therefore $$I = \sum_{n=0}^{N-1} \frac{x^n}{(n+\theta)^\beta} + I_N, \text{ let us say.}$$

If $|x| < 1$, the series tends to a definite finite limit as N tends to infinity.

Also, if $|x| < 1$, we shall have $R(\log x) < 0$.

If $\log x$ lies outside the contour and $|1-x|$ be small, we may near $y = 0$ deform the contour so that the minimum value of $|1-xe^{-y}|$ is finite and occurs when $y = \log x + \eta$, where $\eta > 0$, and so that for other values of y on the contour we have $R(y-\log x) > \eta$.

Then we shall have $$I_N = Ke^{-\eta N} + Lx^N,$$

where $|K|$ tends to a finite limit as N tends to infinity and $|L|$ tends to zero.

Therefore, if $\log x$ be outside the contour and $|1-x|$ be small, $|I_N|$ tends to zero as N tends to infinity, provided $|x| < 1$.

Hence, when these conditions hold,
$$I = g_\beta(x;\theta).$$

Hence

$$g_\beta(x;\theta) - \frac{g_\beta(x;1)}{x^{\theta-1}} = \frac{\iota\Gamma(1-\beta)}{2\pi}\int_L (-y)^{\beta-1} \frac{e^{-\theta y}-e^{-y}x^{1-\theta}}{1-xe^{-y}}\, dy$$
$$= -\frac{1}{x^{\theta-1}}\frac{\iota\Gamma(1-\beta)}{2\pi}\int_L (-y)^{\beta-1} e^{-y} \frac{1-(xe^{-y})^{\theta-1}}{1-xe^{-y}}\, dy. \quad \text{(A)}$$

For it is evident that any axis L as previously chosen is in the positive half of the y plane, and is therefore a possible axis when $\theta = 1$.

But in the latter integral the points $y = \log x \pm 2n\pi\iota$ are no longer singularities of the subject of integration: therefore we may drop the condition that such points shall lie outside the contour of integration. We shall assume that $x^{\theta-1}$ is completely specified, as will be the case if we assign a cross-cut along the negative half of the real axis.

The integral (A) is evidently finite and continuous when x takes any range of values limited by this cross-cut. It represents, therefore, the

continuation of $g_\beta(x;\theta)-g_\beta(x;1)/x^{\theta-1}$ for all values of x so limited. Therefore this function has no singularities in the finite part of the plane except the singularity at the origin due to $x^{\theta-1}$.

5. Put $y=\log x+t$, and suppose that $|\log x|$ is small. The integral (A) may be written

$$-\frac{1}{x^\theta}\,\frac{\iota\Gamma(1-\beta)}{2\pi}\int(-\log x-t)^{\beta-1}\,\frac{e^{-t}-e^{-\theta t}}{1-e^{-t}}\,dt.$$

Expand the original contour so that it includes P, a parallel to the axis L from the point $\log x$, and so that it also includes a circle of radius $|\log x|$ whose centre is $y=\log x$. Change the specification of $(-y)^{\beta-1}$ so that it is unaltered on the contour, but has a cross-cut along the parallel inside the contour, so that

$$(-\log x-t)^{\beta-1}=(-t)^{\beta-1}\sum_{n=0}^{\infty}\frac{\beta-1)\ldots(\beta-n)}{n!}\left(\frac{\log x}{t}\right)^n$$

when $|t|>|\log x|$, and is the continuation of the function represented by the series when $|t|<|\log x|$. Now close up the contour till it embraces P, as the original contour embraced L. The integral in (A) will be unaltered in value by these operations.

Hence

$$g_\beta(x;\theta)-g_\beta(x;1)/x^{\theta-1}=-\frac{1}{x^\theta}\,\frac{\iota\Gamma(1-\beta)}{2\pi}\int_L(-\log x-y)^{\beta-1}\,\frac{e^{-y}-e^{-\theta y}}{1-e^{-y}}\,dy. \tag{B}$$

If now the bulb of the contour be a circle of radius $>|\log x|$ and centre $y=0$, and if the remainder of the contour be the double description of that part of the axis L outside this circle, we have on the contour

$$(-\log x-y)^{\beta-1}=(-y)^{\beta-1}\sum_{n=0}^{N}\frac{(\beta-1)\ldots(\beta-n)}{n!}\left(\frac{\log x}{y}\right)^n+R_N,$$

where $|R_N|$ tends to zero as N tends to infinity. Thus the integral (B) is equal to

$$-\frac{1}{x^\theta}\sum_{n=0}^{N}\frac{(\beta-1)\ldots(\beta-n)}{n!}(-\log x)^n\,\frac{\iota\Gamma(1-\beta)}{2\pi}\int_L(-t)^{\beta-n-1}\,\frac{e^{-t}-e^{-\theta t}}{1-e^{-t}}\,dt$$
$$-\frac{1}{x^\theta}\,\frac{\iota\Gamma(1-\beta)}{2\pi}\int_L R_N\,\frac{e^{-y}-e^{-\theta y}}{1-e^{-y}}\,dy$$

$$=\frac{1}{x^\theta}\sum_{n=0}^{N}\frac{(\beta-1)\ldots(\beta-n)}{n!}(-\log x)^n\,\frac{\Gamma(1-\theta)}{\Gamma(1-\beta+n)}\{\zeta(\beta-n,\theta)-\zeta(\beta-n,1)\}$$
$$-J_N\quad\text{(say)}$$

$$=\frac{1}{x^\theta}\sum_{n=0}^{N}\frac{(\log x)^n}{n!}[\zeta(\beta-n,\theta)-\zeta(\beta-n,1)]-J_N,$$

where $\zeta(s, \theta)$ denotes the simple Riemann ζ function of parameter unity.

6. Now, if r be so chosen that $R(\theta+r)$ is positive,

$$\frac{\zeta(\beta-n, \theta)}{n!} = \frac{\sum_{m=0}^{r-1} (\theta+m)^{n-\beta}}{n!} + \frac{\iota\Gamma(1+n-\beta)}{2\pi\Gamma(1+n)} \int_L (-x)^{\beta-n-1} \frac{e^{-(\theta+r)x}}{1-e^{-x}}\, dx.$$

The contour of the integral embraces the axis L and excludes the points $2n\pi\iota$ $(n \neq 0)$. Hence on the contour we may take the minimum value of $|x|$ to be k, where $k < 2\pi$.

Hence, when n is large and θ not real and a negative integer,

$$\left| \frac{\zeta(\beta-n, \theta)}{n!} \right| = K \frac{n^{-R(\beta}}{k^n}$$

where K is finite when n is very large.

The series

$$\sum_{n=0}^{N} \frac{(\log x)^n}{n!} \zeta(\beta-n, \theta)$$

therefore tends to a finite limit as n tends to infinity, provided

$$|\log x| < k < 2\pi.$$

Finally, therefore, if $\log x$ is defined by a cross-cut along the negative half of the real axis, if θ be not real and negative, and if $|\log x| < 2\pi$,

$$g_\beta(x; \theta) - g_\beta(x; 1)/x^{\theta-1} = \frac{1}{x^\theta} \sum_{n=0}^{\infty} \frac{(\log x)^n}{n!} \{\zeta(\beta-n, \theta) - \zeta(\beta-n, 1)\}.$$

By means of the relation

$$g_\beta(x; \theta-1) = \frac{1}{(\theta-1)^\beta} + x g_\beta(x; \theta)$$

we may enunciate the previous theorem with the narrower restriction that θ shall not be zero or a negative integer. Compare the investigation in § 9.

7. We proceed now to shew that $g_\beta(x; \theta)$ *has a single singularity in the finite part of the plane, that this singularity occurs at* $x = 1$, *and that at this point the function branches infinitely often.*

We have seen in § 4 that, provided β be not a positive integer, $|x| < 1$, and the contour excludes the points $\log x \pm 2n\pi\iota$ $(n = 0, 1, \ldots, \infty)$,

$$g_\beta(x; \theta) = \frac{\iota\Gamma(1-\beta)}{2\pi} \int_L (-y)^{\beta-1} \frac{e^{-y\theta}}{1-x e^{-y}}\, dy.$$

Suppose now that the contour includes $\log x$, but excludes the points

$$\log x \pm 2n\pi\iota \quad (n \neq 0),$$

so that, if $x = re^{\iota\phi}$,

$$|(\phi \pm 2\pi)/\log r - \arg L| > |\phi/\log r - \arg L|;$$

then the integral is equal to

$$g_\beta(x;\theta) - \Gamma(1-\beta)(-\log x)^{\beta-1}x^{-\theta}.$$

Now suppose that $|x| > 1$, and that $(-\log x)^{\beta-1}$ is made one-valued by a cross-cut along the axis of integration, $\log(-\log x)$ being real when $\log x$ is real and negative. The integral remains finite and continuous. Hence the equality

$$g_\beta(x;\theta) - \Gamma(1-\beta)(-\log x)^{\beta-1}x^{-\theta} = \frac{\iota\Gamma(1-\beta)}{2\pi}\int_L (-y)^{\beta-1}\frac{e^{-y\theta}}{1-xe^{-y}}\,dy \quad \text{(A)}$$

continues to hold good, even for values of x which are real and greater than unity, provided we regard $g_\beta(x;\theta)$ as representing the continuation of the function defined by the original Taylor's series where $|x| < 1$. Hence the function $g_\beta(x;\theta)$ has no singularities on the positive part of the real axis between $x = 1$ and $x = \infty$. It has, therefore (§ 3), a single singularity in the finite part of the plane, viz., at $x = 1$. Near this point the function is many-valued.

8. We proceed now to show that, *near* $x = 1$, *the function*

$$g_\beta(x;\theta) - \Gamma(1-\beta)(-\log x)^{\beta-1}x^{-\theta}$$

is one-valued and admits the expansion

$$\sum_{n=0}^{\infty}\frac{(x-1)^n}{x^{n+1}}\bar{\zeta}_{n+1}(\beta,\theta),$$

valid when $\left|\dfrac{x-1}{x}\right| < 1$. *We thus see that* $x = 1$ *is a singularity of specifiable branching of* $g_\beta(x;\theta)$.

From the equality (A) of the previous paragraph we obtain

$$g_\beta(x;\theta) - \Gamma(1-\beta)(-\log x)^{\beta-1}x^{-\theta}$$

$$= \sum_{n=0}^{N-1}\frac{\iota\Gamma(1-\beta)}{2\pi}\int_L (-y)^{\beta-1}\frac{e^{-y\theta}}{(1-e^{-y})^{n+1}}\frac{(x-1)^n}{x^{n+1}}\,dy$$

$$+ \frac{\iota\Gamma(1-\beta)}{2\pi}\int_L (-y)^{\beta-1}\frac{e^{-y\theta}}{1-xe^{-y}}\left\{\frac{x-1}{x(1-e^{-y})}\right\}^N dy.$$

The first series may be written

$$\sum_{n=0}^{N-1}\frac{(x-1)^n}{x^{n+1}}\bar{\zeta}_{n+1}(\beta,\theta),$$

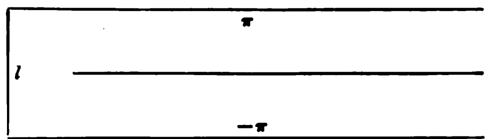
where $\bar{\zeta}_{n+1}(\beta, \theta)$ is the $(n+1)$-ple Riemann ζ function of equal parameters unity defined by the integral

$$\frac{\iota\Gamma(1-\beta)}{2\pi}\int_L (-y)^{\beta-1}\,\frac{e^{-y\theta}}{(1-e^{-y})^{n+1}}\,dy.$$

We have assumed that θ is not real and negative.

Suppose now that $R(\theta) > 0$; then we may replace the contour L by a contour C, embracing the positive half of the real axis, which now serves as a cross-cut to make the function $(-\log x)^{\beta-1}$ one-valued.

Deform this contour till it consists of two lines above and below the real axis and distant π from it, and a line l parallel to the imaginary axis, cutting the real axis in a point whose distance from the origin is $> \log 2$ on the negative side of the origin.



Since $I\{\log x\}$ lies between $\pm\pi\iota$, the point $\log x$ can always be taken to lie within the contour.

The minimum value of $|1-e^{-y}|$ on the contour will be unity. For, if $y = (\cos\phi+\iota\sin\phi)r$, we have on the two infinite lines $r\sin\phi = \pm\pi$, and therefore $\cos(r\sin\phi) = -1$. Hence

$$|1-e^{-y}| = \sqrt{[1-2e^{-r\cos\phi}\cos(r\sin\phi)+e^{-2r\cos\phi}]} = 1+e^{-r\cos\phi} > 1;$$

and on the line l $\qquad e^{-r\cos\phi} > 2,$

and therefore $\qquad |1-e^{-y}| > \sqrt{[(2-1)^2]} > 1.$

Hence, when n is large

$$|\bar{\zeta}_{n+1}(\beta, \theta)| \leqslant \frac{\Gamma(1-\beta)}{2\pi\mu^{n+1}}\int |(-y)^{\beta-1}|\,|e^{-y\theta}|\,|dy| < \mu^{-(n+1)}K,$$

where $\mu > 1$ and K is finite and independent of n if β be not a positive integer and $R(\theta) > 0$.

Hence the series $\qquad \sum_{n=0}^{N-1}\frac{(x-1)^n}{x^{n+1}}\,\bar{\zeta}_{n+1}(\beta, \theta)$

tends to a definite finite limit as N tends to infinity if $|(x-1)/x| < 1$.

This can be otherwise seen since the integral

$$\frac{\iota\Gamma(1-\beta)}{2\pi}\int_C (-y)^{\beta-1}\,\frac{e^{-y\theta}}{1-xe^{-y}}\left\{\frac{x-1}{x(1-e^{-y})}\right\}^N dy$$

will tend to zero as N tends to infinity if $|(x-1)/x| < 1$.

Therefore, if $R(\theta) > 0$ and $|(x-1)/x| > 1$, and if the principal value of $(-\log x)^{\beta-1}$ [which is such that $\log(-\log x)$ is real when $\log x$ is real and negative and has a cross-cut along the positive half of the real axis,

$|I(\log x)|$ being less than π, since $|(x-1)/x|<1$] be taken,

$$g_\beta(x;\theta)-\Gamma(1-\beta)(-\log x)^{\beta-1}x^{-\theta}=\sum_{n=0}^{\infty}\frac{(x-1)^n}{x^{n+1}}\bar{\zeta}_{n+1}(\beta,\theta).$$

Thus the nature of the singularity of $g_\beta(x;\theta)$ at $x=1$ is given by

$$\Gamma(1-\beta)(-\log x)^{\beta-1}x^{-\theta}.$$

This singularity is not essential*, and is not even an infinity unless $R(\beta)<1$, or β is a positive integer, or we wind infinitely often round the point. When $\beta=0$, $g_\beta(x;\theta)=(1-x)^{-1}$, and the nature of its singularity near $x=1$ is given by $-x^{-\theta}/\log x$. This result, though somewhat paradoxical at first sight, is evidently true.

9. We will now *remove the limitation* $R(\theta)>0$ *introduced into the proof of the preceding proposition, and show that the theorem is true if* θ *be not zero or a negative integer.*

We evidently have

$$g_\beta(x;\theta-1)=\sum_{n=0}^{\infty}\frac{x^n}{(n-1+\theta)^\beta}=(\theta-1)^{-\beta}+xg_\beta(x;\theta).$$

Hence, by the preceding theorem, if $R(\theta)>0$,

$$g_\beta(x;\theta-1)-\Gamma(1-\beta)(-\log x)^{\beta-1}x^{-(\theta-1)}$$
$$=\frac{1}{(\theta-1)^\beta}+\sum_{n=0}^{\infty}\frac{(x-1)^n}{x^n}\bar{\zeta}_{n+1}(\beta,\theta).$$

For brevity put

$$u_{n+1}=\bar{\zeta}_{n+1}(\beta,\theta-1),\qquad v_{n+1}=\bar{\zeta}_{n+1}(\beta,\theta),$$

and
$$z=\frac{x-1}{x}.$$

I have elsewhere shewn† that

$$v_{n+1}=u_{n+1}-u_n.$$

[Let
$$S_N=\sum_{n=0}^{N}v_{n+1}z^n,\qquad S'_N=\sum_{n=0}^{N}u_{n+1}z^n.$$

* It must, of course, be counted as an essential singularity if we say that $\log x$ has an essential singularity at $x=0$. Essential singularity is defined in such a negative manner that it will probably be ultimately convenient to class such points as the one in question under another title.

† "The Theory of the Multiple Gamma Function," *Transactions of the Cambridge Philosophical Society*, Vol. XIX., pp. 374–425, § 26.

Then $S_N = \sum_{n=0}^{N} (u_{n+1}-u_n) z^n = -u_0 + \sum_{n=0}^{N} u_{n+1}(z^n - z^{n+1}) + u_{N+1} z^{N+1}$

$$= -u_0 + (1-z) S'_N + u_{N+1} z^{N+1}.$$

But $$u_{N+1} = v_{N+1} + u_N = \ldots = v_{N+1} + v_N + \ldots + v_1 + u_0.$$

Therefore $$|u_{N+1}| \leqslant \sum_{r=1}^{N+1} |v_r| + |u_0|.$$

We have seen that, when r is large and $R(\theta) > 0$,

$$|v_r| < K/\mu^{r+1},$$

where $\mu > 1$.* Hence, when N is large,

$$|u_{N+1}| < K'N,$$

where K' is finite, if β be not an integer and $R(\theta) > 0$.

Hence, if $|z| < 1$, $|u_{N+1} z^{N+1}|$ tends to zero as N tends to infinity. But S_N tends to a definite finite limit as N tends to infinity. Therefore the same is true of S'_N. Hence

$$\frac{1}{(\theta-1)^\beta} + \sum_{n=0}^{\infty} \left(\frac{x-1}{x}\right)^n \bar{\zeta}_{n+1}(\theta, \beta)$$
$$= \frac{1}{(\theta-1)^\beta} - u_0 + \frac{1}{x} \sum_{n=0}^{\infty} \left(\frac{x-1}{x}\right)^n \bar{\zeta}_{n+1}(\beta, \theta-1).]$$

And the latter series is convergent.

Now $u_0 = (\theta-1)^{-\beta}$. Therefore, if $R(\theta) > -1$, and θ be not zero, the theorem of the preceding paragraph is valid. Proceeding by successive stages, we shew that it is valid for all values of θ, provided θ be not zero or a negative integer.

10. If we compare the results of the preceding paragraphs with the expansion obtained in § 4, we see that when x is in the immediate vicinity of the point 1 we have the equality of the two expansions

$$\frac{1}{x^\theta} \sum_{n=0}^{\infty} \frac{(\log x)^n}{n!} \{\zeta(\beta-n, \theta) - \zeta(\beta-n, 1)\},$$

and $$\sum_{n=0}^{\infty} \frac{(x-1)^n}{x^{n+1}} \{\bar{\zeta}_{n+1}(\beta, \theta) - x^{1-\theta} \bar{\zeta}_{n+1}(\beta, 1)\},$$

for each is convergent when $|x-1|$ is small and equal to

$$g_\beta(x;\ \theta) - x^{1-\theta} g_\beta(x;\ 1).$$

* The previous argument can be used to show that, when $R(\theta) > 0$ and β is not an integer, $|v_r|$ tends to zero as r tends to infinity.

The Case of $\beta = 1$.

11. I have previously* shewn that, when $\beta = 1$,

$$x^\theta g(x;\theta)+\log(1-x)$$
$$= \psi(1)-\psi(\theta)+x^\theta \sum_{n=1}^{\infty} \frac{(1-x)^n}{x^{n+1}} \frac{(\theta-1)\dots(\theta-n)}{n!}\left(\frac{1}{1}+\dots+\frac{1}{n}\right),$$

provided $|(1-x)/x| < 1$.

This result, at first sight, seems very different from that previously obtained for general values of β. It is now proposed to shew that as β tends to unity the result of § 8 leads to that just quoted.

It is necessary to introduce certain properties of $(n+1)$-ple Riemann ζ functions of equal parameters: these are taken from an unpublished chapter of a forthcoming book by the author on *Gamma Functions and Allied Transcendents.* The reader will, however, find little difficulty in deducing them from the author's memoirs dealing with the general multiple Riemann ζ function.†

If we put $\beta = 1-\epsilon$ in the result of the preceding paragraph, we obtain

$$g_{1-\epsilon}(x;\theta)-\Gamma(\epsilon)(-\log x)^{-\epsilon}x^{-\theta} = \sum_{n=0}^{\infty} \frac{(x-1)^n}{x^{n+1}} \bar{\zeta}_{n+1}(1-\epsilon,\theta). \tag{A}$$

Now $$\operatorname*{Lt}_{\epsilon=0}\left\{\bar{\zeta}_{n+1}(1-\epsilon,\theta)+(-)^n\frac{{}_{n+1}\bar{S}_1^{(2)}(\theta)}{\epsilon}\right\} = -\bar{\psi}'_{n+1}(\theta),$$

when ${}_{n+1}\bar{S}_1(\theta)$ denotes the first $(n+1)$-ple Bernoullian function of θ of equal parameters unity, and $\bar{\psi}'_{n+1}(\theta)$ denotes

$$\frac{d}{d\theta}\log\{\bar{\Gamma}_{n+1}(\theta)\},$$

$\bar{\Gamma}_{n+1}(\theta)$ denoting the $(n+1)$-ple gamma function of equal parameters unity.

I have elsewhere shewn that‡

$${}_{n+1}\bar{S}_1^{(2)}(\theta) = \frac{(\theta-1)\dots(\theta-n)}{n!}.$$

Hence, if we expand the result (A) in ascending powers of ϵ, as is

* *Quarterly Journal of Mathematics*, Vol. XXXVII., p. 308.

† *Loc. cit.*, § 9.

‡ *Transactions of the Cambridge Philosophical Society*, Vol. XIX., p. 431.

evidently legitimate if ϵ be very small, we get, on equating coefficients of $1/\epsilon$,

$$-x^{-\theta} = -\sum_{n=0}^{\infty} \frac{(x-1)^n}{x^{n+1}} {}_{n+1}\bar{S}_1^{(2)}(\theta),$$

and, on equating the terms independent of ϵ,

$$g(x, \theta) + x^{-\theta} \log(-\log x) - \psi(1) x^{-\theta} = -\sum_{n=0}^{\infty} \frac{(x-1)^n}{x^{n+1}} \bar{\psi}'_{n+1}(\theta).$$

The result of equating higher powers of ϵ is to give us the nature of the behaviour of functions

$$\sum_{n=0} \frac{x^n [\log(n+\theta)]}{n+\theta}$$

near $x = 1$.

12. The first result is equivalent to

$$x^{-\theta} = \sum_{n=0}^{\infty} \frac{(x-1)^n}{x^{n+1}} \frac{(\theta-1)\dots(\theta-n)}{n!}$$

$$= \frac{1}{x}\left(1+\frac{1-x}{x}\right)^{\theta-1}, \tag{1}$$

and is evidently true.

The second result may be written

$$g(x;\theta) + x^{-\theta}\log(1-x) - \psi(1) x^{-\theta}$$
$$= x^{-\theta} \log\left(\frac{1-x}{-\log x}\right) - \sum_{n=0}^{\infty} \frac{(x-1)^n}{x^{n+1}} \bar{\psi}'_{n+1}(\theta).$$

We have then to prove that the right-hand side of this equality is equal to

$$-x^{-\theta}\psi(\theta) + \sum_{n=1}^{\infty} \frac{(1-x)^n}{x^{n+1}} \frac{(\theta-1)\dots(\theta-n)}{n!} \left(\frac{1}{1}+\frac{1}{2}+\dots+\frac{1}{n}\right).$$

Put now $(x-1)/x = z$; then we have to shew, when $|z| < 1$, that

$$(1-z)^{\theta-1}\left[\psi(\theta) - \log\left\{\frac{(1-z)\log(1-z)}{-z}\right\}\right]$$
$$= \psi(\theta) + \sum_{n=1}^{\infty} z^n \left\{\bar{\psi}'_{n+1}(\theta) + \frac{(1-\theta)\dots(n-\theta)}{n!}\left(\frac{1}{1}+\dots+\frac{1}{n}\right)\right\}.$$

Now $\quad (-)^n \bar{\psi}'_{n+1}(\theta) = \sum_{k=1}^{n} \frac{(-)^k}{k!} {}_{n+1}\bar{S}_0^{(k+1)}(\theta) \frac{S'_k(\theta)}{k} + {}_{n+1}\bar{S}_0(\theta)\psi'(\theta),$

where $\quad S_k(\theta) = {}_1S_k(\theta) \quad$ and $\quad \psi(\theta) = \psi_1(\theta).$

Therefore

$$\bar{\psi}'_{n+1}(\theta) = \psi_1(\theta) \frac{(1-\theta)\dots(n-\theta)}{n!} + (-)^n \sum_{k=1}^{n} \frac{(-)^k}{k!} {}_{n+1}\bar{S}_0^{(k+1)}(\theta) \frac{S'_k(\theta)}{k}.$$

Using the identity (1), we see that the terms involving $\psi_1(\theta)$ vanish from the equality, and we have to establish that, if $|z| < 1$,

$$-(1-z)^{\theta-1}\log\left\{\frac{(1-z)\log(1-z)}{-z}\right\}$$
$$=\sum_{n=1}^{\infty}(-z)^n\left\{\sum_{k=1}^{n}\frac{(-)^k}{k!}\,{}_{n+1}\bar{S}_0^{(k+1)}(\theta)\,\frac{S_k'(\theta)}{k}+{}_{n+1}\bar{S}_0'(\theta)\left(\frac{1}{1}+\ldots+\frac{1}{n}\right)\right\}.$$

13. We will first shew that

$${}_{n+1}\bar{S}_0'(\theta)\left(\frac{1}{1}+\ldots+\frac{1}{n}\right)+\sum_{k=1}^{n}\frac{(-)^k}{k!}\,{}_{n+1}\bar{S}_0^{(k+1)}(\theta)\,\frac{S_k'(\theta)-S_k'(0)}{k}$$
$$+{}_{n+1}\bar{S}_0^{(2)}(\theta)=0.$$

Denote the function on the left-hand side of this equality by $F_{n+1}(\theta)$.

Then, since $\qquad {}_{n+1}\bar{S}_0'(\theta+1)={}_{n+1}\bar{S}_0'(\theta)+{}_n\bar{S}_0'(\theta)$

and $\qquad S_k'(\theta)-S_k'(0)=kS_{k-1}(\theta),$

we have

$$F_{n+1}(\theta)+F_n(\theta)$$
$$=F_{n+1}(\theta+1)+\frac{1}{n}\,{}_n\bar{S}_0'(\theta)-\sum_{k=1}^{n}\frac{(-)^{k-1}}{k!}\,{}_{n+1}\bar{S}_0^{(k+1)}(\theta+1)\,\theta^{k-1}$$
$$=F_{n+1}(\theta+1)+\frac{1}{n}\,{}_n\bar{S}_0'(\theta)+\frac{1}{\theta}\left\{1-\exp\left(-\theta\frac{d}{d\theta}\right)\right\}{}_{n+1}\bar{S}_0'(\theta+1)$$

[for ${}_{n+1}\bar{S}_0(\theta+1)$ is a polynomial in θ]

$$=F_{n+1}(\theta+1)+\frac{1}{n}\,{}_n\bar{S}_0'(\theta)-\frac{1}{\theta}\,{}_{n+1}\bar{S}_0'(\theta+1)$$
$$=F_{n+1}(\theta+1),$$

for $\qquad {}_{n+1}\bar{S}_0'(\theta)=\dfrac{(\theta-1)\ldots(\theta-n)}{n!}.$

Now, $F_n(\theta)$ is a polynomial in θ. Therefore, if

$$F_n(\theta)=0,$$

we have $\qquad F_{n+1}(\theta)=\text{constant}.$

But, when $\theta=0$,

$$F_n(\theta)={}_n\bar{S}_0'(0)\left\{\frac{1}{1}+\ldots+\frac{1}{n-1}\right\}+{}_n\bar{S}_0^{(2)}(0)=0.$$

Therefore, if $$F_n(\theta) = 0,$$
we have $$F_{n+1}(\theta) = 0.$$

Now
$$\begin{aligned} F_1(\theta) &= {}_2\bar{S}_0'(\theta) + {}_2\bar{S}_0^{(2)}(\theta) - {}_2\bar{S}_0^{(2)}(\theta)\, S_0(\theta) \\ &= \theta - 1 + 1 - \theta \\ &= 0. \end{aligned}$$

Therefore, by induction, $$F_{n+1}(\theta) = 0.$$

14. We now have to shew, if $|z| < 1$, that
$$-(1-z)^{\theta-1} \log \left\{ \frac{(1-z)\log(1-z)}{-z} \right\}$$
$$= \sum_{n=1}^{\infty} (-z)^n \left\{ \sum_{k=2}^{n} (-)^k \, {}_{n+1}\bar{S}_0^{(k+1)}(\theta) \frac{S_k(0)}{k \, . \, k!} + S_1'(0)\, {}_{n+1}\bar{S}_0^{(2)}(\theta) \right\}.$$

The function on the left-hand side can evidently be expanded in a series of ascending powers of z, if $|z|$ be sufficiently small, and the coefficient of $(-z)^n$ is given by
$$\frac{1}{2\pi\iota} \int \frac{(1-z)^{\theta-1}}{(-z)^{n+1}} \log \left\{ \frac{(1-z)\log(1-z)}{-z} \right\} dz,$$
taken round a small circle including $z = 0$, on which the subject of integration is one-valued.

Put $1-z = e^y$; then, when z makes a small circuit round the origin, y will do the same, and the integral becomes
$$-\frac{1}{2\pi\iota} \int \frac{e^{\theta y}}{(e^y-1)^{n+1}} \log \left\{ \frac{e^y y}{e^y-1} \right\} dy.$$

Now, when y is small, $y + \log \frac{y}{e^y-1}$ admits the expansion $\sum_{n=1}^{\infty} c_n y^n$.

Differentiating, we have
$$\sum_{n=1}^{\infty} n c_n y^{n-1} = 1 + \frac{1}{y} - \frac{e^y}{e^y-1} = 1 + \frac{1}{y} + \sum_{n=0}^{\infty} (-y)^{n-1} \frac{S_n'(0)}{n!}.$$

Hence $$1 + S_1'(0) = c_1;$$
and therefore $$c_1 = -S_1'(0),$$
and $$c_n = (-)^{n-1} \frac{S_n'(0)}{n \, . \, n!}, \text{ when } n > 1.$$

The integral is therefore equal to
$$-\frac{1}{2\pi\iota} \int \frac{e^{\theta y}}{(e^y-1)^{n+1}} \left\{ -S_1'(0)\, y + \sum_{k=2}^{\infty} (-)^{k-1} \frac{S_k'(0)}{k \, . \, k!} y^k \right\} dy.$$

Now* $\dfrac{e^{\theta y}}{(e^y-1)^{n+1}} = \sum_{s=1}^{n+1} \dfrac{{}_{n+1}\bar{S}_0^{(k)}(\theta)}{y^k}$

+terms which are finite when y vanishes.

Therefore the integral is equal to

$$S_1'(0)\ {}_{n+1}\bar{S}_0^{(2)}(\theta) + \sum_{k=2}^{n} (-)^k\ {}_{n+1}\bar{S}_0^{(k+1)}(\theta)\, \frac{S_k'(0)}{k\,.\,k!}.$$

We thus have the required equality.

15. We proceed now to shew that, *if θ be not real, the function*

$$g_\beta(x\,;\,\theta) + g_\beta\left(\frac{1}{x};\,-\theta\right) e^{\mp\pi\iota\beta},$$

the positive or negative sign being taken according as $I(\theta)$ is $<$ or $>$ 0, is one-valued† *near $x=1$, and has no singularity at this point.*

In the investigation of § 7, we have seen that

$$g_\beta(x\,;\,\theta) - \Gamma(1-\beta)(-\log x)_L^{\beta-1} x^{-\theta} = \frac{\iota\Gamma(1-\beta)}{2\pi}\int_L (-y)^{\beta-1}\frac{e^{-y\theta}}{1-xe^{-y}}\,dy,$$

where $1/L$ is an axis within 90° of the points θ, $\theta+1$, ..., $\theta+n$, ..., and where $\arg(-\log x)_L$ lies between $-(\pi-\psi)\iota$ and $(\pi+\psi)\iota$, ψ being the angle between L and the positive half of the real axis, and ranging in value from $-\pi$ to π.

In exactly the same way we may prove that

$$g_\beta\left(\frac{1}{x};\,-\theta\right) - \Gamma(1-\beta)\left(-\log\frac{1}{x}\right)_\lambda^{\beta-1} x^{-\theta}$$
$$= \frac{\iota\Gamma(1-\beta)}{2\pi}\int_\lambda (-y)^{\beta-1}\frac{e^{y\theta}}{1-x^{-1}e^{-y}}\,dy,$$

where $1/\lambda$ is an axis within 90° of the points $-\theta$, $-\theta+1$, ..., $-\theta+n$, ..., and where $\arg(-\log 1/x)_\lambda$ lies between $-(\pi-\phi)\iota$ and $(\pi+\phi)\iota$, ϕ being the angle between λ and the positive half of the real axis, and ranging in value from $-\pi$ to π, $-\log 1/x$ being real when x is real and positive.

In the former case L is a cross-cut for $\log x$ to make $g_\beta(x;\,\theta)$ one-valued; in the latter case $-\lambda$ is a cross-cut for $\log x$ to make $g_\beta(x^{-1};\,-\theta)$ one-valued.

These cross-cuts are in general not the same, but within the region common to the two expansions it is readily seen by constructing a

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* *Cambridge Philosophical Transactions*, Vol. XIX., p. 378.

† Its actual value depends, of course, on which branches of the original functions we choose.

figure that, if $I(\theta) > 0$,

$$\arg(-\log 1/x)_\Lambda = \arg(-\log x)_L + \pi\iota,$$

and, if $I(\theta) < 0$, $\arg(-\log 1/x)_\Lambda = \arg(-\log x)_L - \pi\iota$.

When the final integrals are expressed by convergent series in powers of $(1-x)/x$, we may rotate the cross-cuts till they coincide along an imaginary axis; and then, since

$$(-\log x)_L^{\beta-1} + (-\log 1/x)_\Lambda^{\beta-1} e^{\mp\beta\pi\iota} = 0$$

$$[-, I(\theta) > 0; \ +, I(\theta) < 0],$$

we see that $$g_\beta(x;\theta) + g_\beta(x^{-1}; -\theta)\, e^{\mp\pi\iota\beta}$$

may be represented by a series convergent when $|1-x|$ is small. Therefore this function is uniform near $x = 1$, and has no singularity at this point.

16. The preceding proposition indicates a close connection between the functions $g_\beta(x;\theta)$ and $g_\beta(x^{-1}; -\theta)$, the former of which can be expressed by a Taylor's series when $|x| < 1$, and the latter by a Taylor's series when $|x| > 1$.

We may readily shew that, when $|x| > 1$,

$$g_\beta\left(\frac{1}{x}; -\theta\right) e^{\mp\pi\iota\beta} - \frac{1}{\theta^\beta} = \sum_{n=1}^{\infty} \frac{1}{x^n(\theta-n)^\beta},$$

the $-$ or $+$ sign being taken according as $I(\theta) >$ or < 0, provided $(\theta-n)$ has values which correspond to a cross-cut along the negative half of the real axis.

For, by definition, when $|x| > 1$,

$$g_\beta\left(\frac{1}{x}; -\theta\right) = \sum_{n=0}^{\infty} \frac{1}{x^n(n-\theta)^\beta},$$

where $$|\arg(n-\theta)| < \pi$$

and $$(\theta-n)^\beta = (n-\theta)^\beta e^{\pm\pi\iota\beta},$$

according as $I(\theta) >$ or < 0.

17. We may now shew that, $g_\beta(x;\theta)$ *admits, when* $|x|$ *is very large, the asymptotic expansion*

$$-\sum_{n=1}^{\infty} \frac{1}{x^n(\theta-n)^\beta} + \frac{[\log(-x)]^{\beta-1}}{(-x)^\theta} \sum_{n=0}^{\infty} \frac{(-)^n \left(\dfrac{\pi}{\sin\pi\theta}\right)^{(n)}}{n!\,\Gamma(\beta-n)\,[\log(-n)]^n},$$

provided θ be not real, and $\log(-x)$ *be defined with respect to a cross-cut along the positive half of the real axis, being real when x is real and negative. If the argument of* $\log(-x)$ *is ϕ ($|\phi| < \pi$), so that*

$$\log(-x) = |\log(-x)|\, e^{\iota\phi},$$

the argument of $[\log(-x)]^{\beta-1}$ *is* $\phi[R(\beta)-1] + \log\{|\log(-x)|\}\, I(\beta)$.

If $|x| < 1$,

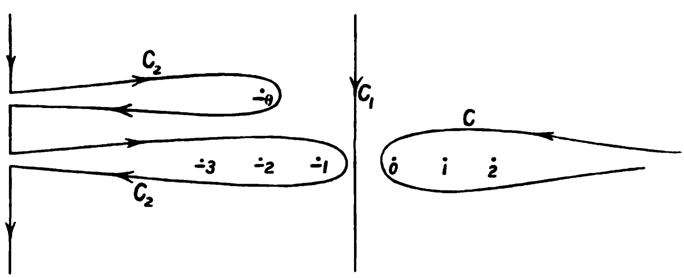
$$g_\beta(x;\theta) = \sum_{n=0}^{\infty} \frac{x^n}{(n+\theta)^\beta} = \frac{1}{2\pi\iota}\int_C \frac{\pi(-x)^s\, ds}{\sin \pi s\,(s+\theta)^\beta} \tag{A}$$

where the contour C encloses the origin but not the points -1 or $-\theta$, and embraces the positive half of the real axis.

Let C_1 be a straight contour parallel to the imaginary axis, cutting the real axis between 0 and -1, with a loop, if necessary, to ensure that $-\theta$ lies to the left of this contour. Then, if $|\arg(-x)| < \pi$,

$$\int_C = \int_{C_1};$$

for the integral vanishes along the infinite contour which is the difference of the contours C and C_1. Hence the integral (A) taken along the contour C_1 represents the continuation of the function $g_\beta(x;\theta)$ for all values of x such that $|\arg(-x)| < \pi$, $|x|$ having any value greater than, equal to, or less than unity.



Suppose now that $|x| > 1$, and that C_2 is the contour of the figure. Then, if $|\arg(-x)| < \pi$, the integral along the contour C_1 will equal that along the contour C_2, and the integral along the straight parts of this contour at infinity will vanish.

We shall therefore have, if $|x| > 1$ and $|\arg(-x)| < \pi$,

$$g_\beta(x;\theta) = -\sum_{n=1}^{\infty} \frac{1}{x^n(\theta-n)^\beta} + \frac{\iota}{2\pi}\int_C (-x)^{-y-\theta}(-y)^{-\beta}\frac{\pi}{\sin\pi(y+\theta)}\, dy.$$

In obtaining the final integral we have employed the transformation $s = -y-\theta$. The values of $(\theta-n)^\beta$ correspond to a cross-cut ($-\theta$ to

$-\infty$) parallel to the real axis. In the final integral $\pi/\sin\pi(y+\theta)$ has no poles within or on the contour of integration. It is therefore represented by the summable divergent series

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\pi}{\sin \pi\theta}\right)^{(n)} y^n$$

on the contour, and, in fact, over the whole of the y-plane dissected by lines passing from the zeros of $\sin\pi(y+\theta)$ away from the origin to infinity.

Therefore, by a proposition proved in my original memoir,* this integral may be represented by the asymptotic series

$$\frac{\iota(-x)^{-\theta}}{2\pi} \sum_{n=0}^{N} \frac{(-)^n}{n!} \left(\frac{\pi}{\sin\pi\theta}\right)^{(n)} \int_C e^{-y\log(-x)} (-y)^{n-\beta}\, dy + J_N,$$

where $|J_N\{\log(-x)\}^{n-\beta+1}|$ tends to zero as $|\log(-x)|$ tends to infinity; and this series is equal to

$$(-x)^{-\theta} \sum_{n=0}^{N} \frac{(-)^n}{n!} \left(\frac{\pi}{\sin\pi\theta}\right)^{(n)} \frac{1}{\Gamma(\beta-n)[\log(-x)]^{n-\beta+1}} \tag{1}$$

where $$[\log(-x)]^{n-\beta} = \exp\{(n-\beta)\log[(-x)]\},$$

where, when $\log(-x)$ is specified, $\arg\{\log(-x)\}$ lies between $\pm\pi$, and is zero when $\log(-x)$ is real and positive.

We thus have the given expansion.

We notice that, when $R(\theta) < 1$, the series (1) gives the asymptotic expansion near $|x| = \infty$ of $g_\beta(x;\theta)$. When $R(\theta) > 1$, let ν be the integer next greater than $R(\theta)$. Then the asymptotic expansion of $g_\beta(x;\theta)$ is

$$\sum_{n=1}^{\nu-1} \frac{1}{x^n(\theta-n)^\beta} + \text{the series (1)}.$$

18. From the previous theorem we see that, when $|x|$ is very large and θ is not real, we have the asymptotic equality

$$g_\beta(x;\theta) + \left\{g_\beta\left(\frac{1}{x}; -\theta\right) - \frac{1}{(-\theta)^\beta}\right\} e^{\mp\pi\iota\beta}$$
$$= \frac{[\log(-x)]^{\beta-1}}{(-x)^\theta} \sum_{n=0}^{\infty} \frac{(-)^n \left(\dfrac{\pi}{\sin\pi\theta}\right)^{(n)}}{n!\,\Gamma(\beta-n)[\log(-x)]^n}, \tag{1}$$

the $-$ or $+$ sign being taken as $I(\theta) >$ or < 0.

* *Philosophical Transactions of the Royal Society* (A), Vol. 206, pp. 249–297, § 5.

Changing x into $1/x$, we get the asymptotic expansion of $g(1/x;\theta)$ at the origin. The relation (1) holds when $|\log(-x)|$ is very large, that is, whether $|x|$ or $1/|x|$ be very large. It is easy to verify that the preceding formula remains unchanged when we change x into $1/x$ and θ into $-\theta$.

The series (1) becomes convergent when we multiply the general term by $1/n!$: it is therefore what I have elsewhere* proposed to call an asymptotic series in $\log(-x)$ of the second order, similar to the well known series for $\log\Gamma(z+a)$ when $|z|$ is large.

19. *When β is a positive integer and θ is not real, the previous investigation will hold.*

But now the asymptotic series (1) is replaced by the finite series

$$\frac{[\log(-x)]^{\beta-1}}{(-x)^{\theta}}\sum_{n=0}^{\beta-1}\frac{(-)^n\left(\dfrac{\pi}{\sin\pi\theta}\right)^{(n)}}{n!\,\Gamma(\beta-n)\,[\log(-x)]^n};$$

for this series is the residue of $\dfrac{\pi(-x)^s}{\sin\pi s(s+\theta)^{\beta}}$ at $s=-\theta$. Thus, when θ is not real and β is a positive integer, we have the absolute equality

$$g_{\beta}(x;\theta)+\left\{g_{\beta}\left(\frac{1}{x};\ -\theta\right)-\frac{1}{(-\theta)^{\beta}}\right\}e^{\mp\pi i\beta}$$
$$=\frac{[\log(-x)]^{\beta-1}}{(-x)^{\theta}}\sum_{n=0}^{\beta-1}\frac{(-)^n\left(\dfrac{\pi}{\sin\pi\theta}\right)^{(n)}}{n!\,\Gamma(\beta-n)\,[\log(-x)]^n}.$$

20. We must now *investigate the asymptotic expansion of $g_{\beta}(x;\theta)$ when θ is real.*

In this case difficulties arise from the fact that the specification of $(s+\theta)^{-\beta}$ when s is real and less than $-\theta$ is arbitrary. We have (§ 1) adopted the convention that in this case we will take $\arg(s+\theta)=+\pi$. We must therefore take a cross-cut from $-\theta$ to $-\infty$ inclined at a small angle to the real axis and work with the contours of the modified figure (p. 304).

We see that, if $|\arg(-x)|<\pi$ and $|x|>1$, and if θ be not zero or a positive or negative integer,

$$g_{\beta}(x;\theta)=-\sum_{n=1}^{\infty}\frac{1}{x^n(\theta-n)^{\beta}}-\frac{1}{2\pi\iota}\int_{C'}(-x)^{-y-\theta}(-y)^{-\beta}\frac{\pi}{\sin\pi(y+\theta)}\,dy,$$

the contour C' being derived from the part of the contour C_2 which

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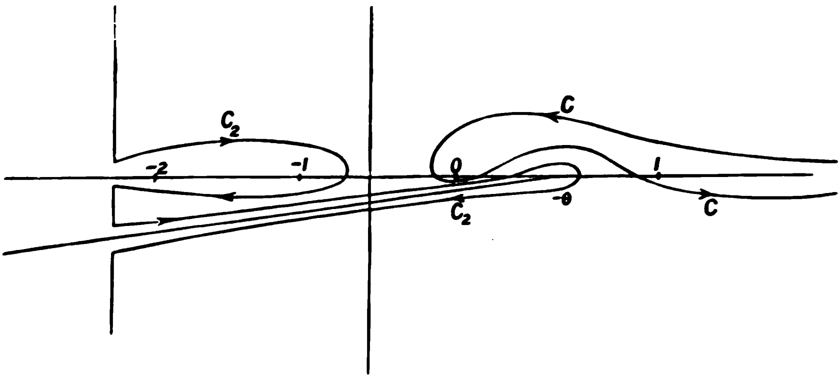
* "A Memoir on Integral Functions," *Phil. Trans. Roy. Soc.* (A), Vol. 199, pp. 411–500, § 32.

encloses $-\theta$ by the transformation $s = -(y+\theta)$, so that C' embraces an axis from the origin which lies above the real axis and is inclined at a small angle to it.



We obtain the same result as before for the asymptotic expansion of this integral. And this is to be expected, since the terms of

$$\sum_{n=-\infty}^{\infty} \frac{x^n}{(n+\theta)^\beta},$$

for which $n < -\theta$, are, when $|x|$ is large, of order less than that of any term of the asymptotic expansion.

21. Consider next *the case where θ is a positive integer.*

If we put $\theta = p$, we now obtain

$$g_\beta(x;\theta) = -\sum_{n=1}^{\infty}{}' \frac{1}{x^n(p-n)^\beta} - \frac{1}{2\pi\iota}\int_C (-x)^{-y-p}(-y)^{-\beta}\frac{(-)^p\pi}{\sin \pi y}\,dy,$$

the accent denoting that the term corresponding to $n = p$ is to be omitted from the summation.

On the contour C' we have the summable divergent expansion

$$\frac{\pi y}{\sin \pi y} = 1 + \sum_{n=1}^{\infty}\left(\frac{\pi x}{\sin \pi x}\right)^{(n)}_{x=0}\frac{y^n}{n!}.$$

Therefore the integral gives rise to the asymptotic series

$$\frac{1}{x^p}\sum_{n=0}^{\infty}\left(\frac{\pi x}{\sin \pi x}\right)^{(n)}_{x=0}\frac{(-)^{n-1}}{n!}\frac{1}{2\pi\iota}\int e^{-y\log(-x)}(-y)^{n-\beta-1}\,dy.$$

We therefore have the asymptotic expansion

$$g_\beta(x;p) = -\sum_{n=1}^{\infty}{}' \frac{1}{x^n(p-n)^\beta}$$
$$+\frac{[\log(-x)]^\beta}{x^p}\sum_{n=0}^{\infty}\left(\frac{\pi x}{\sin \pi x}\right)^{(n)}_{x=0}\frac{(-)^{n-1}}{n!\,\Gamma(1+\beta-n)[\log(-x)]^n},$$

22. Suppose, finally, *that $\theta = p$, a positive integer, and that β is a positive integer.*

The result just obtained becomes the actual equality

$$g_\beta(x;p) = -\sum_{n=1}^{\infty}{}' \frac{1}{x^n(p-n)^\beta}$$
$$-\frac{[\log(-x)]^\beta}{x^p}\sum_{n=0}^{\beta-1}\left(\frac{\pi x}{\sin \pi x}\right)_{x=0}^{(n)} \frac{(-)^n}{n!\,\Gamma(1+\beta-n)[\log(-x)]^n}.$$

Suppose now that $\theta = 1$.

Then, if we put

$$P_\beta(x) = \sum_{n=1}^{\infty} n^{-\beta}x^n,$$

so that $P_\beta(x)$ is one of the series which Leau makes fundamental in his researches, we have

$$g_\beta(x;\theta) = x^{-1}P_\beta(x).$$

Also
$$\sum_{n=1}^{\infty}{}' \frac{1}{x^n(1-n)^\beta} = e^{-\pi\iota\beta}\sum_{n=2}^{\infty}\frac{1}{x^n(n-1)^\beta} = \frac{e^{-\pi\iota\beta}}{x}\,P_\beta\left(\frac{1}{x}\right).$$

Now we know that, when $|x|$ is small, we have the expansion

$$\frac{\pi x}{\sin \pi x} = 1+\sum_{n=1}^{\infty}\frac{2^{2n-1}-1}{2^{2n-2}}\,S_{2n}x^{2n}$$

where
$$S_{2n} = \frac{1}{1^{2n}}+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\dots.$$

Hence
$$\left(\frac{\pi x}{\sin \pi x}\right)_{x=0}^{(n)} = 1 \qquad \text{(when } n=0)$$
$$= 0 \qquad \text{(when } n \text{ is odd)}$$
$$= (2m)!\,\frac{2^{2m-1}-1}{2^{2m-2}}\,S_{2m} \qquad \text{(when } n=2m).$$

Hence, if β be even,

$$P_\beta(x)+P_\beta\left(\frac{1}{x}\right) = -\frac{[\log(-x)]^\beta}{\beta!} - \sum_{m=1}^{\frac{1}{2}\beta}\frac{[\log(-x)]^{\beta-2m}}{(\beta-2m)!}\,\frac{2^{2m-1}-1}{2^{2m-2}}\,S_{2m};$$

and, if β be odd,

$$P_\beta(x)-P_\beta\left(\frac{1}{x}\right) = -\frac{[\log(-x)]^\beta}{\beta!} - \sum_{m=1}^{\frac{1}{2}(\beta-1)}\frac{[\log(-x)]^{\beta-2m}}{(\beta-2m)!}\,\frac{2^{2m-1}-1}{2^{2m-2}}\,S_{2m}.$$

These are two little known formulæ due to Spence, and quoted by De Morgan.*

23. As has been suggested in § 11, it is evident that we may apply the preceding methods to series of the type

$$h_\beta(x) = \sum_{n=0}^{\infty} \frac{x^n \{\log(n+\theta)\}^k}{(n+\theta)^\beta},$$

where k is a positive integer. We have merely to write $\beta+\epsilon$ for β and expand the various functions of β in ascending powers of ϵ. The analysis will evidently be laborious.

When k is not restricted to be an integer, but is any complex quantity of finite modulus, the function may be expressed by the contour integral

$$\frac{1}{2\pi\iota}\int_C \frac{[\log(s+\theta)]^k \pi(-x)^s ds}{(s+\theta)^\beta \sin \pi s}.$$

And this in turn, as in § 17, can, when $|x| > 1$ and $|\arg(-x)| < \pi$, be expressed in the form

$$-\sum_{n=1}^{\infty} \frac{[\log(\theta-n)]^k}{x^n(\theta-n)^\beta} - \frac{1}{2\pi\iota}\int_C \frac{[\log(-y)]^k \pi(-x)^{-y-\theta}}{\sin \pi(y+\theta)(-y)^\beta} dy$$

provided θ be not real.

If $R\log(-x) > 0$, the integral is convergent. Thus the function $h_\beta(x)$ has no singularities outside a circle of radius unity except possibly on the line $(1, +\infty)$. An asymptotic expansion in the neigbourhood of this singularity at infinity can be obtained.

We can also shew that $x = 1$ and $x = \infty$ are the only singularities of the function.

PART II.—*The Function* $f_\beta(x; \theta)$.

24. We proceed now to obtain analogous theorems for the very general function $f_\beta(x; \theta)$. This function is defined when $|x| < 1$ by the expansion

$$\sum_{n=0}^{\infty} \frac{x^n \chi(n+\theta)}{(n+\theta)^\beta}$$

* De Morgan, *Differential and Integral Calculus* (1842), p. 659, Formulæ I. and II. In the notation of De Morgan

$$2s_n = \frac{2^{n-1}-1}{2^{n-2}} S_n.$$

where outside a circle of radius $l < \mu$, where μ is the least of the quantities $|n+\theta|$, $n = 0, 1, 2, \ldots, \infty$, $\chi(x)$ admits the absolutely convergent expansion

$$\sum_{r=0}^{\infty} \frac{b_r}{x^r}.$$

As before, we assume that β is not a positive integer, and we make the further restriction that it shall not be zero or a negative integer. Such limiting cases may be dealt with by more elementary methods, or by applying the calculus of limits to the formulæ which arise.

25. We will first shew *that, when* $|x| < 1$, $f_\beta(x;\theta)$ *can be written in the form*

$$\sum_{r=0}^{\infty} b_r g_{\beta+r}(x;\theta).$$

We have $$f_\beta(x;\theta) = \sum_{n=0}^{\infty} x^n \left\{ \sum_{r=0}^{R} \frac{b_r}{(n+\theta)^{\beta+r}} + \sum_{r=1}^{\infty} \frac{b_{R+r}}{(n+\theta)^{\beta+R+r}} \right\}.$$

Now, for values of r greater than an assignable quantity R,

$$|b_r| < l'^r,$$

where $l' = l+\epsilon$, and ϵ may be as small as we please.

Hence $$\left| \sum_{r=1}^{\infty} \frac{b_{R+r}}{(n+\theta)^{\beta+R+r}} \right| \leqslant \sum_{r=1}^{\infty} \frac{l'^{R+r}}{|(n+\theta)^{\beta+R}|\,|n+\theta|^r}$$

$$< \frac{l'^R}{|(n+\theta)^{\beta+R}|} \sum_{r=1}^{\infty} \frac{l'^r}{\mu^r}$$

$$< \frac{l'^{R+1}}{|(n+\theta)^{\beta+R}|\,(\mu-l')}, \qquad \text{if } \mu > l'.$$

Therefore $$f_\beta(x;\theta) = \sum_{r=0}^{R} b_r g_{\beta+r}(x;\theta) + J_R,$$

where $$|J_R| < \sum_{n=0}^{\infty} \frac{|x^n|\, l'^{R+1}}{|(n+\theta)^{\beta+R}|\,(\mu-l')}.$$

Thus $|J_R|$ tends to zero as R tends to infinity if $|x| < 1$.
We thus have the theorem stated.

26. We will now shew that $f_\beta(x;\theta)$ *has no singularities except possibly when x lies on the positive part of the real axis between* $1-\epsilon$ $(\epsilon > 0)$ *and* $+\infty$.

Suppose that $F_\beta(x;\theta)$ denotes the integral function

$$\sum_{n=0}^{\infty} \frac{x^n \chi(n+\theta)}{n!\,(n+\theta)^\beta}.$$

Then I have shewn, in my fundamental memoir,* that, when $R(x) > 0$,

$$F_\beta(x;\theta) = \frac{e^x J(x)}{x^\beta},$$

where $|J(x)|$ tends to a definite finite limit as $|x|$ tends to infinity.

And, when $R(x) < 0$, $F_\beta(x;\theta) = (-x)^{l-\theta} J_1(x)$, where l is the radius of convergence of $\chi(y)$, and where $|J_1(x)|$ is at most finite when $|x|$ tends to infinity.

As in § 3, we may therefore show that

$$f_\beta(x;\theta) = \int_0^\infty e^{-z} F_\beta(xz;\theta)\,dz.$$

The integral may be taken along any axis for which $R(z) > 0$ and $R[(1-x)z] > 0$. We thus obtain continuations of $f_\beta(x;\theta)$ which are finite and continuous for all values of x such that $|\arg(1-x)| < \pi$.

We thus have the given theorem, which is true for all values of β of finite modulus.

27. We will now show that, *provided β be not an integer, and if $R(\theta) > 0$, and if θ lies outside the circle of convergence of $\chi(x)$, and provided* $\left|\frac{x-1}{x}\right| < 1$,

$$f_\beta(x;\theta) - (-\log x)_{1/\theta}^{\beta-1} x^{-\theta}\Gamma(1-\beta) \sum_{r=0}^\infty b_r \frac{(\log x)^r \Gamma(\beta)}{\Gamma(\beta+r)} = \sum_{n=0}^\infty \frac{(x-1)^n}{x^{n+1}} \psi_{n+1}(\beta,\theta)$$

where† $$\psi_{n+1}(\beta,\theta) = \frac{\iota\Gamma(1-\beta)}{2\pi}\int (-y)^{\beta-1}\Phi(y)\frac{e^{-y\theta}}{(1-e^{-y})^{n+1}}\,dy$$

and $$\Phi(y) = \sum_{r=0}^\infty \frac{\Gamma(\beta)\, b_r y^r}{\Gamma(\beta+r)}.$$

The contour of the integral embraces the axis to $1/\theta$, which is the cross-cut which makes $(-\log x)_{1/\theta}^{\beta-1}$ one-valued.

If $|x| < 1$, we have, by § 4,

$$\begin{aligned} f_\beta(x;\theta) &= \sum_{r=0}^\infty b_r g_{\beta+r}(x;\theta) \\ &= \sum_{r=0}^\infty \frac{\iota\Gamma(1-\beta-r)\, b_r}{2\pi}\int (-y)^{\beta+r-1}\frac{e^{-y\theta}}{1-xe^{-y}}\,dy, \end{aligned}$$

provided $R(\theta) > 0$, and provided the contour of the integral excludes the

* *Loc. cit.*, § 1, Part V., §§ 31 and 36.

† The reader will, of course, not confuse this function with the logarithmic derivates of the multiple gamma function considered in §§ 11 *et seq.*

poles $y = \log x \pm 2n\pi\iota$ $(n = 0, 1, 2, \ldots, \infty)$. Hence, if $|I\{\log x\}| < \pi$,

$$f_\beta(x;\theta) - \sum_{r=0}^{\infty} b_r \Gamma(1-\beta-r)(-\log x)^{\beta+r-1} x^{-\theta}$$

$$= \sum_{r=0}^{\infty} \frac{\iota\Gamma(1-\beta-r)\, b_r}{2\pi} \int (-y)^{\beta+r-1} \frac{e^{-y\theta}}{1-xe^{-y}}\, dy, \quad (1)$$

where now the contour of the integral includes $\log x$, but not $\log x \pm 2n\pi\iota$ $(n = 1, 2, \ldots, \infty)$.

In this equality $(-\log x)^{\beta+r-1}$ has its principal value with respect to a cross-cut along the axis of integration. And the first series converges if $|-\log x|$ is finite. The second series is therefore convergent if $|-\log x|$ be finite and $|x| < 1$.

28. We will next shew that, if the contour of integration embraces the axis to $1/\theta$, and if $R(\theta) > 0$ and $|\theta| > l$, the integral

$$\frac{\iota\Gamma(1-\beta)}{2\pi} \int \sum_{r=0}^{R} \frac{b_r \Gamma(1-\beta-r)(-y)^{\beta+r-1}}{\Gamma(1-\beta)} \frac{e^{-y\theta}}{1-xe^{-y}}\, dy \quad (2)$$

tends to a definite finite limit as R tends to infinity.

Evidently $$\sum_{r=0}^{\infty} \frac{b_r(-y)^r \Gamma(1-\beta-r)}{\Gamma(1-\beta)} = \Phi(y),$$

an integral function of y.

Also, if k be an integer such that $R(\beta+k) > 0$, we have

$$|\beta+k+r| > r.$$

Hence $$|\Phi(y)| < \sum_{r=0}^{k-1} \frac{\Gamma(\beta)\, b_r y^r}{\Gamma(\beta+r)} + \frac{1}{|\beta(\beta+1)\ldots(\beta+k)|} \sum_{r=0}^{\infty} \frac{b_{r+k} y^{r+k}}{r!}.$$

If, now, k be sufficiently large, $|b_{r+k}| < l'^{r+k}$, where $l' = l + \epsilon$, and ϵ is a positive quantity as small as we please.

Therefore

$$|\Phi(y)| < \text{a polynomial in } |y| + \frac{l'^k |y|^k}{|\beta(\beta+1)\ldots(\beta+k)|} e^{l'|y|}.$$

Therefore the integral (2) tends to a definite finite limit under the conditions assigned. Therefore under these conditions we obtain from (1)

$$f_\beta(x;\theta) - \sum_{r=0}^{\infty} b_r \Gamma(1-\beta-r)(-\log x)_{1/\theta}^{\beta+r-1} x^{-\theta}$$

$$= \frac{\iota\Gamma(1-\beta)}{2\pi} \int (-y)^{\beta-1} \Phi(y) \frac{e^{-y\theta}}{1-xe^{-y}}\, dy. \quad (3)$$

Provided $\log x$ is inside and $\log x \pm 2n\pi\iota$ $(n \neq 0)$ is outside the contour, and provided $(-\log x)_{1/\theta}^{\beta-1}$ has its principal value with respect to the axis of integration, the integral remains finite and continuous when $|x| > 1$, provided $|-\log x|$ is finite. Thus, under these limitations coupled with $R(\theta) > 0$ and $|\theta| > l' > l$, we obtain a continuation of $f_\beta(x;\theta)$ when $|x| > 1$.

It is obvious that, by taking $R(\theta)$ sufficiently large, we may find an infinite number of lines in the positive half of the y-plane which may serve as axes of integration and cross-cuts for $(-\log x)^{\beta-1}$. For instance, if $R(\theta) > l'$, the positive half of the real axis will so serve. We are not limited to the particular cross-cut chosen if $R(\theta)$ be very large.

29. Consider now the integral in the formula (3). It will be equal to

$$\sum_{n=0}^{N-1} \frac{(x-1)^n}{x^{n+1}} \psi_{n+1}(\beta;\theta) + J_N$$

where $$\psi_{n+1}(\beta;\theta) = \frac{\iota\Gamma(1-\beta)}{2\pi} \int_{1/\theta} (-y)^{\beta-1} \Phi(y) \frac{e^{-y\theta}}{(1-e^{-y})^{n+1}} dy$$

and $$J_N = \frac{\iota\Gamma(1-\beta)}{2\pi} \int_{1/\theta} \frac{(-y)^{\beta-1}\Phi(y) e^{-y\theta}}{1-xe^{-y}} \left[\frac{x-1}{x(1-e^{-y})}\right]^N dy.$$

By deforming the contour embracing $1/\theta$, so that it consists near the origin of the contour figured in § 8 and further away embraces a parallel to the axis of $1/\theta$, we can ensure that upon it the minimum value of $|1-e^{-y}|$ is μ, where $\mu = 1-\epsilon$, and ϵ is > 0, but as small as we please.

Then evidently $|J_N|$ tends to zero as N tends to infinity, if

$$|(x-1)/x| < 1-\epsilon.$$

The series $$\sum_{n=0}^{\infty} \frac{(x-1)^n}{x^{n+1}} \psi_{n+1}(\beta;\theta)$$

is therefore absolutely convergent under the same limitation.

We therefore have, if $|(x-1)/x| < 1$,

$$f_\beta(x;\theta) - (-\log x)^{\beta-1} x^{-\theta} \frac{\pi}{\sin\pi\beta} \sum_{r=0}^{\infty} \frac{b_r (\log x)^r}{\Gamma(\beta+r)} = \sum_{n=0}^{\infty} \frac{(x-1)^n}{x^{n+1}} \psi_{n+1}(\beta;\theta).$$

If $(-\log x)^{\beta-1}$, *qua* function of $\log x$, have a cross-cut along the axis to $1/\theta$, we have proved the formula under the limitation $R(\theta) > 0$ and $|\theta| > l' > l$.

And, if $R(\theta)$ be sufficiently large, we may take the cross-cut for $(-\log x)^{\beta-1}$ to be in an infinite number of positions in the positive half of the y plane.

30. From the previous theorem we deduce at once that $f_\beta(x;\theta)$ ***has, when*** $R(\theta)$ ***is sufficiently large, a single singularity in the finite part of***

*the plane. This singularity occurs at $x = 1$, and is specifiable.** *It is a multiform point, and near it $f_\beta(x;\theta)$ behaves like*

$$\frac{\pi}{\sin \pi\beta}(-\log x)^{\beta-1}x^{-\theta} \sum_{r=0}^{\infty} \frac{b_r(\log x)^r}{\Gamma(\beta+r)}.$$

To establish these results the reader has merely to recall § 26, and to notice that the condition $|(x-1)/x| < 1$ is equivalent to $R(x) > \frac{1}{2}$.

31. We give some further theorems before we remove from the previous theorem the condition that $R(\theta)$ must be positive and sufficiently large.

We will first shew that, *if $R(\theta) > 0$, and if θ have any value of finite modulus such that the points $\theta+m$ $(m = 0, 1, 2, \ldots, \infty)$ lie outside the circle of convergence of $\chi(x)$, and if n be finite,*

$$\psi_{n+1}(\beta;\theta) = \sum_{r=0}^{\infty} b_r \bar{\zeta}_{n+1}(\beta+r, \theta).$$

If $|\theta| > l' > l$ and $R(\theta) > 0$,

$$\psi_{n+1}(\beta;\theta) = \frac{\iota\Gamma(1-\beta)}{2\pi}\int_{1/\theta} (-y)^{\beta-1}\Phi(y)\frac{e^{-y\theta}}{(1-e^{-y})^{n+1}}\,dy$$

$$= \sum_{r=0}^{R-1} \frac{\iota\Gamma(1-\beta-r)\,b_r}{2\pi}\int (-y)^{\beta+r-1}\frac{e^{-y\theta}}{(1-e^{-y})^{n+1}}\,dy$$

$$+\frac{\iota}{2\sin\pi\beta}\int (-y)^{\beta-1}\sum_{r=R}^{\infty}\frac{b_r y^r}{\Gamma(\beta+r)}\frac{e^{-y\theta}}{(1-e^{-y})^{n+1}}\,dy$$

$$= \sum_{r=0}^{R-1} b_r \bar{\zeta}_{n+1}(\beta+r, \theta) + J_R \quad \text{(say)}.$$

Now
$$(-)^n \bar{\zeta}_{n+1}(s, \theta) = \sum_{k=0}^{n} (-)^k \frac{{}_{n+1}\bar{S}_1^{(k+2)}(\theta)}{k!}\,\zeta(s-k, \theta);$$

and, if β be not an integer and θ be not equal to $0, -1, -2, \ldots,$ each of the multiple $\bar{\zeta}$ functions is finite.

Hence the series $\sum_{r=0}^{R-1} b_r \bar{\zeta}_{n+1}(\beta+r, \theta)$ tends to a definite finite limit as R tends to infinity, provided series of the type

$$\sum_{r=0}^{\infty} b_r \zeta(\beta+r-k, \theta) \quad (k = 0, 1, \ldots, n)$$

are convergent. But when $R(\beta+r-k)$ is very large and positive

$$|\zeta(\beta+r-k, \theta)| < \mu^{-r} L$$

where L is finite and independent of r, and μ is defined in § 24.

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* See the note to § 8.

The series are therefore convergent if $\mu > l' > l$, where μ is the minimum value of $|\theta+m|$ $(m = 0, 1, 2, \ldots, \infty)$.

This will appear again, and we prove the theorem by considering the integral J_R.

Divide up the contour of integration into two parts:—(1) a circle of radius less than unity round the origin; (2) the double description of the axis of the contour outside this circle.

On (1) the subject of integration is finite, and can be made as small as we please by sufficiently increasing R.

On (2), if we choose k so that $R(\beta+k) > 0$, we have, if $r > k$,

$$|\Gamma(\beta+r)| > K(r-k)!$$

where K is finite, non-zero, and independent of r.

Hence
$$\sum_{r=R}^{\infty} \frac{b_r y^r}{\Gamma(\beta+r)} < \frac{(l'|y|)^R}{K(R-k)!} e^{l'|y|}.$$

Hence the modulus of the integral along this part of the contour tends to zero as R tends to infinity, provided $|\theta| > l'$ and $R(\theta) > 0$.

We thus have the given theorem.

32. We will next shew that, *if* $R(\theta) > l' > l$ *and* $\left|\frac{x-1}{x}\right| < 1$,

$$f_\beta(x;\theta) - x^{-\theta} \sum_{r=0}^{\infty} b_r \Gamma(1-\beta-r)(-\log x)^{\beta+r-1}$$
$$= \sum_{r=0}^{\infty} b_r \sum_{n=0}^{\infty} \frac{(x-1)^n}{x^{n+1}} \bar{\zeta}_{n+1}(\beta+r, \theta).$$

By the result of § 28, the function on the left-hand side of this equality, which for brevity we will denote by $P(x)$, is, if $R(\theta) > l' > l$, equal to

$$\frac{\iota\Gamma(1-\beta)}{2\pi}\int (-y)^{\beta-1}\Phi(y)\frac{e^{-y\theta}}{1-xe^{-y}}\,dy$$

where the contour of integration embraces the positive half of the real axis and includes $\log x$.

This integral may be written

$$\sum_{n=0}^{N-1} \frac{(x-1)^n}{x^{n+1}} \psi_{n+1}(\beta;\theta) + \frac{\iota\Gamma(1-\beta)}{2\pi}\int (-y)^{\beta-1}\Phi(y)\frac{e^{-y\theta}}{1-xe^{-y}}\left\{\frac{x-1}{x(1-e^{-y})}\right\}^N dy$$
$$= \sum_{r=0}^{\infty} b_r \sum_{n=0}^{N-1} \frac{(x-1)^n}{x^{n+1}} \bar{\zeta}_{n+1}(\beta+r, \theta) + J_N \text{ (say)}. \quad (1)$$

Evidently $|J_N|$ tends to zero as N tends to infinity if $\left|\frac{x-1}{x}\right| < 1$.

Again, as in § 8, with the contour there employed,

$$|\bar{\zeta}_{n+1}(\beta+r,\theta)| < \frac{|\Gamma(1-\beta-r)|}{2\pi\mu^{n+1}}\int |(-y)^{r+\beta-1}|\,|e^{-y\theta}|\,dy < \frac{k(1+\epsilon)^r}{\mu^{n+1}[R(\theta)]^r} \quad (2)$$

where k is finite and independent of r and n, and $\epsilon > 0$.

Hence $$\left|\sum_{n=0}^{\infty}\frac{(x-1)^n}{x^{n+1}}\bar{\zeta}_{n+1}(\beta+r,\theta)\right| < \frac{K(1+\epsilon)^r}{[R(\theta)]^r}$$

where K is finite, if $|(1-x)/x| < 1$.

Hence, if $R(\theta) > l'(1+\epsilon)$, the first series in (1) tends to a definite finite limit as N tends to infinity.

We thus obtain the given theorem.

33. We may now shew that *the theorem of* § 30 *is valid for all values of* θ *provided* $\theta+m$ $(m = 0, 1, 2, \ldots, \infty)$ *lies outside the circle of convergence of* $\chi(x)$.

If $R(\theta) > l'$, where $l' > l$, and $|(x-1)/x| < 1$, we have

$$\begin{aligned} f_\beta(x;\theta-1) &= \frac{\chi(\theta-1)}{(\theta-1)^\beta} + xf_\beta(x;\theta) \\ &= x^{1-\theta}\sum_{r=0}^{\infty} b_r\Gamma(1-\beta-r)(-\log x)^{\beta+r-1} + \frac{\chi(\theta-1)}{(\theta-1)^\beta} \\ &\qquad + \sum_{r=0}^{\infty} b_r \sum_{n=0}^{\infty}\frac{(x-1)^n}{x^{n+1}}\bar{\zeta}_{n+1}(\beta+r,\theta) \end{aligned}$$

Hence

$$\begin{aligned} &f_\beta(x;\theta-1) - x^{1-\theta}\sum_{r=0}^{\infty} b_r\Gamma(1-\beta-r)(-\log x)^{\beta+r-1} \\ &= \frac{\chi(\theta-1)}{(\theta-1)^\beta} + \sum_{r=0}^{\infty} b_r\left[\sum_{n=0}^{\infty}\left\{-\bar{\zeta}_n(\beta+r,\theta-1)+\bar{\zeta}_{n+1}(\beta+r,\theta-1)\right\}\frac{(x-1)^n}{x^{n+1}}\right] \\ &= \frac{\chi(\theta-1)}{(\theta-1)^\beta} + \sum_{r=0}^{\infty} b_r\Big[-\bar{\zeta}_0(\beta+r,\theta-1) \\ &\qquad + \sum_{n=0}^{\infty}\bar{\zeta}_{n+1}(\beta+r,\theta-1)\left\{\frac{(x-1)^n}{x^n}-\frac{(x-1)^{n+1}}{x^{n+1}}\right\}\Big] \\ &= \frac{\chi(\theta-1)}{(\theta-1)^\beta} + \sum_{r=0}^{\infty} b_r\left[\frac{-1}{(\theta-1)^{\beta+r}} + \sum_{n=0}^{\infty}\frac{(x-1)^n}{x^{n+1}}\zeta_{n+1}(\beta+r,\theta-1)\right].^* \end{aligned}$$

Now $\sum_{r=0}^{\infty}\frac{b_r}{(\theta-1)^{\beta+r}}$ is convergent and equal to $\frac{\chi(\theta-1)}{(\theta-1)^\beta}$ if $\theta-1$ lies outside the circle of convergence of $\chi(x)$.

* The analysis may be made formally rigorous by coupling § 32 (2) with the results of § 9.

Also, if $\sum_{r=0}^{\infty} b_r(u_r+v_r)$ and $\sum_{r=0}^{\infty} b_r u_r$ are absolutely convergent under any assigned limitations, then will $\sum_{r=0}^{\infty} b_r v_r$ be convergent under the same limitations. For

$$\sum_{r=R}^{\infty} |b_r v_r| = \sum_{r=R}^{\infty} |b_r(u_r+v_r-u_r)| \leqslant \sum_{r=R}^{\infty} \{|b_r(u_r+v_r)|+|b_r u_r|\},$$

and the latter series can be made as small as we please by sufficiently increasing R.

Hence, if $(\theta-1)$ lies outside the circle of convergence of $\chi(x)$, and if $R(\theta) > l'$ where $l' > l$, and $|(1-x)/x| < 1$, the series

$$\sum_{r=0}^{\infty} b_r \sum_{n=0}^{\infty} \frac{(x-1)^n}{x^{n+1}} \bar{\zeta}_{n+1}(\beta+r, \theta-1)$$

is absolutely convergent and equal to

$$f_\beta(x; \theta-1) - x^{1-\theta} \sum_{r=0}^{\infty} b_r \Gamma(1-\beta-r)(-\log x)^{\beta+r-1}.$$

Continue this process indefinitely, and we see that the theorem of § 30 is valid for all values of θ provided $\theta+m$ $(m = 0, 1, 2, \ldots, \infty)$ lies outside the circle of convergence of $\chi(x)$.

34. Finally, let us consider the result of applying the process of § 17 to the function $f_\beta(x; \theta)$.

Assume that the points $\theta \pm n$, $n = 0, 1, 2, \ldots, \infty$ all lie outside the circle of convergence of $\chi(x)$.

If the contour C of § 17 do not enclose any part of the circle of radius l and centre $-\theta$, we evidently have

$$f_\beta(x; \theta) = \frac{1}{2\pi\iota} \int_C \frac{\pi(-x)^s \chi(s+\theta)}{\sin \pi s\,(s+\theta)} ds$$

when $|x| < 1$.

If, now, $|\arg(-x)| < \pi$, the integral will vanish when taken round an infinite contour for which $R(s)$ is greater than a finite negative quantity.

Hence, if the contour C_1 of § 17 have, if necessary, a loop to ensure that the circle round $-\theta$ lies to the left of the contour, we have

$$f_\beta(x; \theta) = \frac{1}{2\pi\iota} \int_{C_1} \frac{\pi(-x)^s \chi(s+\theta)}{\sin \pi s\,(s+\theta)^\beta} ds.$$

The latter integral is valid for all values of $|x|$ provided $|\arg(-x)| < \pi$; and therefore represents the continuation of $f_\beta(x; \theta)$ over the whole plane, except this part near the positive half of the real axis. We thus again arrive at the theorem of § 26.

If now the contour C_2 of § 17 include the circle of radius l round $-\theta$, we see that, when $|\arg(-x)| < \pi$ and $|x| > 1$, $f_\beta(x;\theta)$ is equal to the integral along the contour C_2. The integral along the straight parts of the latter contour tends to zero as these move away to infinity.

Under the assigned limitations we therefore have

$$f_\beta(x;\theta) = -\sum_{n=1}^{\infty} \frac{\chi(-n+\theta)}{x^n(\theta-n)^\beta} - \frac{1}{2\pi\iota}\int_{C'} \frac{\pi(-x)^{-\theta-y}\chi(-y)}{\sin\pi(\theta+y)(-y)^\beta}\,dy. \qquad (1)$$

The contour C' embraces the real axis and encloses within its bulb the circle of convergence of $\chi(-y)$.

35. The equality is true *when β is an integer*. In this case C' may be replaced by a contour just outside the circle of convergence of $\chi(-y)$.

In either case we see that the final integral in the equality (1) is equal to $(-x)^{l'-\theta}J(x)$, where, if $l' > l$, $|J(x)|$ tends to zero as $|x|$ tends to infinity. We thus get a superior limit to the asymptotic value of $f_\beta(x;\theta)$ when $|x|$ is very large and β is or is not an integer.

The problem of obtaining a complete asymptotic expansion for $f_\beta(x;\theta)$ when $|x|$ is large evidently depends upon the consideration of the singularities of $\chi(y)$ within its circle of convergence. The reader will compare the similar property of $F_\beta(x;\theta)$ when $R(x) < 0$, which was established in my original memoir.

[*Note added May*, 1906.—

The first author to consider simple cases of the series which we investigate in the present memoir was William Spence, whose *Essay on the Theory of the Various Orders of Logarithmic Transcendents* was published in 1809. Spence considered series of the type $\sum_{n=0}^{\infty} n^{-r}x^n$, where r is an integer, and by a process of induction obtained the continuation of such series when $|x| > 1$.* This essay, so rare, that no copy is to be found in the library of the University of Cambridge, seems to have been almost entirely forgotten. Such series were also considered by Lambert, Legendre, Abel, and Kummer, among others.

Abel considered such series in two papers. The first paper,† "Sommation de la Série $\sum_{n=0} \phi(n)x^n$, ..., $\phi(n)$ étant une fonction algébrique

* *Loc. cit.*, p. 45.

† *Œuvres Complètes*, 1881, T. II., pp. 14–18.

rationnelle de n," has several points of interest. Abel considers explicitly (p. 16) the function $g_\beta(x;\theta)$ where β is a positive integer; and obtains (p. 18) a rudimentary form of the formula*

$$G(x;\theta)=\Gamma(\theta)(-x)^{-\theta}+e^x\sum_{n=0}^{\infty}\frac{(1-\theta)\dots(n-\theta)}{x^{n+1}}.$$

In the second paper† "Note sur la fonction $\psi(x)=\sum_{n=1}^{\infty}x^n/n^2$," he attains anew several of Spence's results. Both papers were first published posthumously by Holmboe in 1839, and are evidently mere sketches.

But the modern theory of such series is largely due to Leau,‡ whose work, closely associated with the investigations of Hadamard,§ Borel,‖ and Fabry,¶ led to the investigation of series of the type $\Sigma g(1/m)x^m$ where $g(t)$ is holomorphic at the origin.

Then came Le Roy,** to whom appear to be due the theorems (1) and (3) of § 2.

My own developments were largely completed before I saw Hardy's†† paper. Subsequently my attention has been called to Lindelöf's‡‡ monograph, and to another paper by Hardy.§§ In the former will be found series such as occur in theorem (2) of § 2, and in the latter an equation similar to (A) of § 7. Nothing in the second part of the present paper appears to have been anticipated. But so rapid is the development of the subject that it is difficult to assign priority to respective authors, and almost impossible to state that any investigation is new in all its details.]

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* See the author's paper, *Quarterly Journal of Mathematics*, Vol. XXXVII., p. 294.

† *Loc. cit.*, pp. 189–193.

‡ *Liouville*, Sér. 5, T. V. (1899).

§ *Ibid.*, Sér. 4, T. VIII. (1892).

‖ *Acta Mathematica*, T. XXI. (1897); *Liouville*, Sér. 5, T. II. (1896).

¶ *Annales Scientifiques de l'Ecole Normale Supérieure*, Sér. 3, T. XIII. (1896); *Liouville*, Sér. 5, T. IV. (1898); *Acta Mathematica*, T. XXII. (1899).

** *Annales de la Faculté des Sciences de Toulouse*, Sér. 2, T. II. (1900).

†† *Proc. London Math. Soc.*, Ser. 2, Vol. 2, pp. 401–431.

‡‡ *Le Calcul des Résidus* (Paris: Gauthier-Villars, 1905).

§§ *Proc. London Math. Soc.*, Ser. 2, Vol. 3, pp. 381–389.

ON A QUESTION IN THE THEORY OF AGGREGATES

By A. C. DIXON.

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IN the current volume* of the *Proceedings* Dr. Hobson has criticized the views of Prof. König† and myself‡ on entities which can be specified in finite terms: I propose to say something in answer to his objections, which to my mind are not conclusive.

The question§ is as to the validity of the distinction between things that can be finitely defined and those that cannot. Naturally no example of the latter kind can be produced. There is a similar question as to the distinction between those aggregates which can and those which cannot be arranged in type ω by a *finite* set of rules, even when it is possible to prove the cardinal number in the second case to be neither less nor greater than $\aleph_0$. An example of this second category is supplied by the aggregate of all real numbers capable of finite specification, according to my view of the inference to be drawn from Hobson's argument on p. 24.

The passage in question gives a construction for a number N. This construction assumes the existence of a set of rules, say R, by which all adequate definitions of numbers are arranged in the type ω. The rules R are a part of the definition of N, and, if they cannot be stated in finite terms, then N is not finitely specified; so that the contradiction is solved. For instance, if we take the letters and other symbols in a word, phrase, sentence, or chapter to be digits in a scale and arrange all the integers represented in that scale in ascending order of magnitude, it seems at first that we have all possible finite specifications of numbers arranged in type ω, by simply rejecting from the list all words, phrases, &c., which do not purport to specify numbers, or which specify numbers already placed, or which give impossible specifications. Hobson's specification of N offers itself for a place in the list when its turn comes;

* Above, pp. 21-28.
† *Math. Annalen*, Vol. LXI.
‡ Above, pp. 18—20.
§ It is pervaded by the difficulty of disentangling the things we apprehend from the means by which we apprehend them.

suppose the first $n-1$ places to be already filled. Then the first $n-1$ decimal places in N give no trouble, but when we come to the n-th decimal place we are to fill it with a digit a_n, such that

$$a_n = a_n + (-1)^{a_n}.$$

This is impossible, and hence the specification is to be rejected. If, however, it is rejected, it ceases to be impossible, and hence I conclude that the proposed principle does not enable us to arrange in type ω all the numbers that can be described in finite terms. Similarly for any other principle that can be stated in *finite* terms: in fact, "this ordering of the definitions could" not "be" actually "carried out" (Hobson, p. 23), and for that reason I did not say that the cardinal number of the numbers in question was equal to $\aleph_0$, but only that it was neither less nor greater.*

The same answer applies to Hobson's repetition of the argument on p. 25, and, in fact, the finiteness of a specification is not impaired by the use of series of parameters whose law of construction has been already given in finite terms; this is, in fact, what Hobson says on p. 26. The finiteness of which I take account is not in the process of construction, but in the statement of the distinguishing properties of the number, which may consist of a law for its construction or may not. For instance, "the greatest integer whose English name contains only three letters" is adequately defined, although not as simply as it might be. There must always be a basis of definition in a stock of ideas, themselves indefinable for the most part, common to the giver and receiver of the definition. "Finitely defined" must be understood to mean "finitely defined on some particular basis" or "on the usual basis," and the statement of such basis is, as we know, one of the most difficult of problems.

Hobson says that the aggregate of adequately defined numbers is

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* It may be said that the proof of Bernstein's theorem (Borel, *Théorie des Fonctions*, 1898, pp. 104–6) states in finite terms a method by which any aggregate having such a cardinal number can be arranged in type ω. Bernstein's method, however, applies to aggregates that are already defined before the problem of ordering them is attacked; so that the solution of the problem does not affect the actual membership of the aggregates in question. In our case the constitution of the aggregate cannot be decided apart from the ordering, and the problem is not fairly stated unless it is made clear that the unknowns to be found include not only the rule of order, but also in part the actual membership of the aggregate. Compare the case of an algebraic equation: the method by which one can solve an equation $x^2+ax+b=0$ when a, b are constants does not apply when they are functions of x.

The same difficulty arises if we apply Bernstein's theorem to the finitely specified numbers of the second class (see above, p. 20; Borel, *op. cit.*, pp. 121–2; G. Cantor, *Math. Ann.*, Vol. XLIX., p. 227, Theorem D).

perfect (p. 27). Now a perfect aggregate is one that coincides with its derivative. The derivative is the aggregate of limits of convergent sequences of members of the original aggregate. If we are only to take account of sequences formed by adequately defined laws, then no doubt the aggregate of adequately defined numbers is perfect; but the restriction begs the question. The proof (p. 18) that the number of adequate definitions does not exceed $\aleph_0$ is not directly met by Hobson, and, if it is not refuted, we have the deduction that the continuum according to him has a cardinal number not greater than $\aleph_0$.

As to the question of lawless decimals, we have, I think, as much right to postulate numbers represented by them as we have to "take" arbitrary points on a line in geometry, and it is not possible to go far without "taking" at least two. If two, why not three? Then, if three points A, B, C are taken, the ratio AB/AC defines a number and fixes a law for a decimal which before was lawless. The difficulty of specifying the point taken is just as great with the first or second as with the third, and it arises even in the definition of a statute yard, which is the distance between two points whose position has not been, and cannot be, adequately defined from the theoretical point of view.

It may be objected that we have no right to argue from the geometric to the arithmetic continuum, but this is not a case of a deduction from postulates, but rather of a geometrical illustration used to defend the reasonableness of a proposed postulate—a postulate that seems necessary if the arithmetic continuum is to be a proper representation of the geometric. In any case we cannot do without a "basis" of indefinables as a foundation for our definitions, and why may we not at any time add to that basis a new indefinable number, denoting it by a suitable symbol, and ascribing to it a definite, though not completely stated, ordinal relation with the rational numbers?

ON THE ACCURACY OF INTERPOLATION BY FINITE DIFFERENCES

By W. F. SHEPPARD.

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1. If by interpolation from a table giving values of u in terms of x we obtain the value of u for a value of x not given in the table, the result differs from the true value by an error due to two causes. The first is the fact that the formulæ used are only approximate, being based on the assumption of a certain relation between successive values of u; the error due to this cause may be called the "residual error," since it usually represents the remainder of a series the first few terms of which are used for calculating the value of u. The second is the fact that the tabulated values of u are themselves only approximate; if, for instance, u is given to seven places of decimals, we only know that the tabulated value does not differ from the true value by more than $\frac{1}{2}$ of ·0000001. The error due to this cause may be called the "tabular error." The present paper deals with the relative accuracy of the ordinary advancing-difference formula and the central-difference formulæ for interpolation, so far as each of these errors is concerned. The tabular error is considered first, as being of the greater practical importance. It will be found that, as regards both classes of error, central-difference formulæ are in general more accurate than the ordinary formula. The results obtained, so far as the central-difference formulæ are concerned, are believed to be new.

It will be assumed (unless otherwise stated) that the values of x for which u is tabulated proceed by a constant difference h, and that all the values of u are tabulated within the same limits of error $\pm\frac{1}{2}\rho$.

The values of x are represented by $\dots x_{-2}, x_{-1}, x_0, x_1, x_2, \dots$, where $x_n \equiv x_0+nh$; and the corresponding values of u by $\dots u_{-2}, u_{-1}, u_0, u_1, u_2, \dots$. The formulæ considered are for the value u_θ corresponding to $x_\theta \equiv x_0+\theta h$, where θ, in the ordinary formula, lies between 0 and 1, and in the central-difference formulæ lies either between 0 and 1 or between $-\frac{1}{2}$ and $+\frac{1}{2}$.

I. *Tabular Error.*

2. Let the tabulated values of u be ... U_{-2}, U_{-1}, U_0, U_1, U_2, ..., their errors being ... a_{-2}, a_{-1}, a_0, a_1, a_2, ..., so that

$$U_n = u_n + a_n ; \tag{1}$$

and let the calculated value of u_θ be U_θ, with error a_θ, so that (the formula used being assumed correct)

$$U_\theta = u_\theta + a_\theta . \tag{2}$$

In all the formulæ which we shall consider, u_θ is expressed as a series of terms involving finite differences (or "tabular differences") of successively higher orders, but in the first degree only. This series, taken up to any term, will give u_θ as a linear function of a number of consecutive values of u; and it is therefore clear from (1) and (2) that the value of a_θ is obtained* in terms of ... a_{-1}, a_0, a_1, ... or their successive differences by merely substituting a for u or U in the interpolation-formula.

A. *Advancing Differences.*†

3. The ordinary formula for interpolation is‡

$$u_\theta = u_0 + \frac{\theta}{1!}\Delta u_0 - \frac{\theta(1-\theta)}{2!}\Delta^2 u_0 + \frac{\theta(1-\theta)(2-\theta)}{3!}\Delta^3 u_0 - \dots, \tag{3}$$

where $\quad \Delta u_0 = u_1 - u_0, \quad \Delta^2 u_0 = \Delta u_1 - \Delta u_0, \quad \dots ;$

θ being between 0 and 1. Expressing Δu_0, $\Delta^2 u_0$, ... in terms of u_0, u_1, u_2, ..., this becomes

$$u_\theta = u_0 + \frac{\theta}{1!}(u_1 - u_0) - \frac{\theta(1-\theta)}{2!}(u_2 - 2u_1 + u_0) + \dots ; \tag{4}$$

and therefore (§ 2)

$$a_\theta = a_0 + \frac{\theta}{1!}(a_1 - a_0) - \frac{\theta(1-\theta)}{2!}(a_2 - 2a_1 + a_0)$$
$$+ \frac{\theta(1-\theta)(2-\theta)}{3!}(a_3 - 3a_2 + 3a_1 - a_0) - \dots, \tag{5}$$

the series being continued for as many terms as are used in (3).

* It is assumed that the tabulated differences of u are the uncorrected differences of the tabulated values of u (see § 8).

† Cf. H. L. Rice, *Theory and Practice of Interpolation* (1899), pp. 46–52.

‡ A variation of this formula is obtained by taking the *receding* differences Δu_{-1}, $\Delta^2 u_{-2}$, ...; but this need not be specially considered.

If we collect coefficients of $a_0, a_1, a_2, \ldots$ in (5), we find that, except for $r = 0$, the terms containing a_r are all of sign $(-)^{r-1}$. The coefficient of a_0 is

$$1-\frac{\theta}{1!}-\frac{\theta(1-\theta)}{2!}-\frac{\theta(1-\theta)(2-\theta)}{3!}-\ldots, \tag{6}$$

the successive values of which, according to the number of terms taken in (3), are

$$\left.\begin{array}{l} 1 \\ 1-\frac{\theta}{1!}=\frac{1-\theta}{1!} \\ \frac{1-\theta}{1!}-\frac{\theta(1-\theta)}{2!}=\frac{(1-\theta)(2-\theta)}{2!} \\ \frac{(1-\theta)(2-\theta)}{2!}-\frac{\theta(1-\theta)(2-\theta)}{3!}=\frac{(1-\theta)(2-\theta)(3-\theta)}{3!} \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \end{array}\right\}; \tag{7}$$

and therefore, however many terms of (3) are taken, the coefficient of a_0 is always positive.

Hence the greatest possible value of a_θ (a_0 being taken to be positive) is when $a_0, a_1, a_2, \ldots$ are each numerically equal to $\frac{1}{2}\rho$, the signs being

$$\left.\begin{array}{cccccc} a_0, & a_1, & a_2, & a_3, & a_4, & \ldots \\ +, & +, & -, & +, & -, & \ldots \end{array}\right\}. \tag{8}$$

This would give

$$\begin{aligned} a_1-a_0 &= +(2-2)\tfrac{1}{2}\rho, \\ a_2-2a_1+a_0 &= -(2^2-2)\tfrac{1}{2}\rho, \\ a_3-3a_2+3a_1-a_0 &= +(2^3-2)\tfrac{1}{2}\rho, \\ \ldots \quad \ldots \quad \ldots & \quad \ldots \quad \ldots; \end{aligned}$$

so that the limit of error of u_θ is

$$\pm\tfrac{1}{2}\rho\left\{1+\frac{\theta}{1!}(2-2)+\frac{\theta(1-\theta)}{2!}(2^2-2)+\frac{\theta(1-\theta)(2-\theta)}{3!}(2^3-2)+\ldots\right\}, \tag{9}$$

he same number of terms being taken as in the interpolation-formula.

The series inside the brackets in (9), taken up to terms due to use of th differences, is

$$\begin{aligned} &1+0+\theta(1-\theta)+\theta(1-\theta)(2-\theta)+\tfrac{7}{12}\theta(1-\theta)(2-\theta)(3-\theta) \\ &\quad+\tfrac{1}{4}\theta(1-\theta)(2-\theta)(3-\theta)(4-\theta)+\tfrac{31}{360}\theta(1-\theta)(2-\theta)(3-\theta)(4-\theta)(5-\theta). \end{aligned} \tag{10}$$

B. *Central Differences.*

4. *Notation.*—The same notation will be used as in an earlier paper on "Central-Difference Formulæ."* The operator which converts $f(x)$ into $f(x+\frac{1}{2}h)-f(x-\frac{1}{2}h)$ is denoted by $\delta f(x)$, and the operator which converts $f(x)$ into $\frac{1}{2}\{f(x+\frac{1}{2}h)+f(x-\frac{1}{2}h)\}$ is denoted by μ; so that

$$\delta u_{\frac{1}{2}} = u_1 - u_0, \qquad \mu u_{\frac{1}{2}} = \tfrac{1}{2}(u_0+u_1).$$

Thus the tabular differences are as shown in the following table:—

x	u	1st Diff.	2nd Diff.	3rd Diff.	..
$\vdots$	$\vdots$		$\vdots$	$\vdots$	
x_{-2}	u_{-2}		$\delta^2 u_{-2}$		...
		$\delta u_{-\frac{3}{2}}$		$\delta^3 u_{-\frac{3}{2}}$	
x_{-1}	u_{-1}		$\delta^2 u_{-1}$		...
		$\delta u_{-\frac{1}{2}}$		$\delta^3 u_{-\frac{1}{2}}$	
x_0	u_0		$\delta^2 u_0$		..
		$\delta u_{\frac{1}{2}}$		$\delta^3 u_{\frac{1}{2}}$	
x_1	u_1		$\delta^2 u_1$		..
		$\delta u_{\frac{3}{2}}$		$\delta^3 u_{\frac{3}{2}}$	
x_2	u_2		$\delta^2 u_2$		..
$\vdots$	$\vdots$	$\vdots$		$\vdots$	

and, if we take the means of consecutive pairs of terms in each column, and insert them in brackets, we obtain a table

x	u	1st Diff.	2nd Diff.	3rd Diff.	..
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
x_{-1}	u_{-1}	$(\mu\delta u_{-1})$	$\delta^2 u_{-1}$	$(\mu\delta^3 u_{-1})$	...
	$(\mu u_{-\frac{1}{2}})$	$\delta u_{-\frac{1}{2}}$	$(\mu\delta^2 u_{-\frac{1}{2}})$	$\delta^3 u_{-\frac{1}{2}}$	...
x_0	u_0	$(\mu\delta u_0)$	$\delta^2 u_0$	$(\mu\delta^3 u_0)$	...
	$(\mu u_{\frac{1}{2}})$	$\delta u_{\frac{1}{2}}$	$(\mu\delta^2 u_{\frac{1}{2}})$	$\delta^3 u_{\frac{1}{2}}$	...
x_1	u_1	$(\mu\delta u_1)$	$\delta^2 u_1$	$(\mu\delta^3 u_1)$	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

in which the "central differences" of u_n are $\mu\delta u_n$, $\delta^2 u_n$, $\mu\delta^3 u_n$, ..., while

* *Proceedings*, Vol. XXXI., p. 449. The paper contained (p. 465) a brief examination of the accuracy of certain formulæ involving central differences, but it did not deal with the accuracy of interpolation-formulæ.

the term and differences central to the interval between x_n and x_{n-1} are $\mu u_{n+\frac{1}{2}}$, $\delta u_{n+\frac{1}{2}}$, $\mu\delta^2 u_{n-\frac{1}{2}}$, $\delta^3 u_{n-\frac{1}{2}}$, ...

It should be observed that

$$\delta^{2n-1} u_{\frac{1}{2}} = u_n - {}_{2n-1}C_1 u_{n-1} + {}_{2n-1}C_2 u_{n-2} - \ldots - u_{-(n-1)}, \tag{11}$$

$$\delta^{2n} u_0 = u_n - {}_{2n}C_1 u_{n-1} + {}_{2n}C_2 u_{n-2} - \ldots + u_{-n}, \tag{12}$$

where ${}_mC_p$ denotes the coefficient of x^p in the expansion of $(1+x)^m$. Also

$$\begin{aligned} 2\mu\delta^{2n-1} u_0 = u_n - {}_{2n-1}C_1 u_{n-1} + \ldots + (-)^{n-r}\, {}_{2n-1}C_{n-r} u_r \quad + \ldots \\ + u_{n-1} \quad - \ldots + (-)^{n-r-1}\, {}_{2n-1}C_{n-r-1} u_r + \ldots - u_{-n} \end{aligned} \tag{13}$$

$$\begin{aligned} = u_n - \frac{n-1}{n}\, {}_{2n}C_1 u_{n-1} + \frac{n-2}{n}\, {}_{2n}C_1 u_{n-2} - \ldots \\ + (-)^{n-r} \frac{r}{n}\, {}_{2n}C_{n-r} u_r + \ldots - u_{-n}, \end{aligned} \tag{13 A}$$

$$\begin{aligned} 2\mu\delta^{2n} u_{\frac{1}{2}} = u_{n+1} - {}_{2n}C_1 u_n + \ldots + (-)^{n+1-r}\, {}_{2n}C_{n+1-r} u_r + \ldots \\ + u_n \quad - \ldots + (-)^{n-r}\, {}_{2n}C_{n-r} u_r \quad + \ldots + u_{-n} \end{aligned} \tag{14}$$

$$\begin{aligned} = u_{n+1} - \frac{2n-1}{2n+1}\, {}_{2n+1}C_1 u_n + \frac{2n-3}{2n+1}\, {}_{2n+1}C_2 u_{n-1} - \ldots \\ + (-)^{n+1-r} \frac{2r-1}{2n+1}\, {}_{2n+1}C_{n+1-r} u_r + \ldots + u_{-n}. \end{aligned} \tag{14 A}$$

We may combine and generalize (13 A) and (14 A) in the statement that the coefficient of u_{q+p} in $2\mu\delta^{m-1} u_q$ is

$$(-)^{\frac{1}{2}m-p} \frac{p}{\frac{1}{2}m} \frac{m!}{(\frac{1}{2}m-p)!\,(\frac{1}{2}m+p)!} \tag{15}$$

where q is of the form $\pm k$ or $\pm k+\frac{1}{2}$ according as m is even or odd, and $q+p$ is an integer, positive or negative, such that p is not greater than $\frac{1}{2}m$ or less than $-\frac{1}{2}m$.

5. *Interpolation-Formulæ.*—Let the tabulated value of x which is nearest to the value for which u is sought be x_0, so that $x_\theta = x_0 + \theta h$, where θ may have any value from $-\frac{1}{2}$ to $+\frac{1}{2}$. Then the principle of central-difference interpolation is that we express u_θ in terms of the series of differences on the two sides of the half-interval in which x lies, *i.e.*, in terms of u_0, $\delta^2 u_0$, $\delta^4 u_0$, ..., and either $\delta u_{-\frac{1}{2}}$, $\delta^3 u_{-\frac{1}{2}}$, ..., or $\delta u_{\frac{1}{2}}$, $\delta^3 u_{\frac{1}{2}}$, ...

The standard formula, if θ is positive, is*

$$u_\theta = u_0 + \frac{\theta}{1!}\,\delta u_{\frac12} - \frac{\theta(1-\theta)}{2!}\,\delta^2 u_0 - \frac{(1+\theta)\,\theta\,(1-\theta)}{3!}\,\delta^3 u_{\frac12}$$
$$+\frac{(1+\theta)\,\theta\,(1-\theta)(2-\theta)}{4!}\,\delta^4 u_0 + \frac{(2+\theta)(1+\theta)\,\theta\,(1-\theta)(2-\theta)}{5!}\,\delta^5 u_{\frac12} - \ldots, \quad (16)$$

while, if θ is negative, it is †

$$u_\theta = u_0 + \frac{\theta}{1!}\,\delta u_{-\frac12} + \frac{(1+\theta)\,\theta}{2!}\,\delta^2 u_0 - \frac{(1+\theta)\,\theta\,(1-\theta)}{3!}\,\delta^3 u_{-\frac12}$$
$$-\frac{(2+\theta)(1+\theta)\,\theta\,(1-\theta)}{4!}\,\delta^4 u_0 + \frac{(2+\theta)(1+\theta)\,\theta\,(1-\theta)(2-\theta)}{5!}\,\delta^5 u_{-\frac12} + \ldots. \quad (17)$$

The coefficients in the above formulæ are not suitable for actual calculations, and it is obviously inconvenient to have different formulæ according as the half-interval in which x lies is on the negative or the positive side of a tabulated value. The formulæ therefore require adaptation.

Since the tabular difference occurring in each term of (16) or (17) is the difference of two tabular differences of the next lower order, one of which occurs in the preceding term, the successive terms can be combined in pairs, and each pair can be expressed either in terms of two consecutive tabular differences or in terms of the sum and the difference of these two. For instance, $\delta u_{\frac12} = u_1 - u_0$, so that $u_0 + \theta\delta u_{\frac12}$ can be expressed either as $\theta u_1 + (1-\theta)\,u_0$ or as $\frac12(u_1+u_0) - (\frac12-\theta)(u_1-u_0) = \mu u_{\frac12} - (\frac12-\theta)\,\delta u_{\frac12}$. Hence the series in (16) can be converted into four other series, according to one or other of the following schemes:—

$$\text{(A)}\quad \begin{cases} u_0 & \delta^2 u_0 & \delta^4 u_0 & \ldots, \\ u_1 & \delta^2 u_1 & \delta^4 u_1 & \ldots; \end{cases}$$

$$\text{(B)}\quad \mu u_{\frac12} \quad \delta u_{\frac12} \quad \mu\delta^2 u_{\frac12} \quad \delta^3 u_{\frac12} \quad \ldots;$$

$$\text{(C)}\quad u_0 \begin{cases} \delta u_{-\frac12} & \delta^3 u_{-\frac12} & \delta^5 u_{-\frac12} & \ldots, \\ \delta u_{\frac12} & \delta^3 u_{\frac12} & \delta^5 u_{\frac12} & \ldots; \end{cases}$$

$$\text{(D)}\quad u_0 \quad \mu\delta u_0 \quad \delta^2 u_0 \quad \mu\delta^3 u_0 \quad \ldots.$$

It may be shown that the resulting formulæ are as follows:—

$$\text{(A)}\ u_\theta = (1-\theta)\,u_0 - \frac{\theta(1-\theta)(2-\theta)}{3!}\,\delta^2 u_0 + \frac{(1+\theta)\,\theta\,(1-\theta)(2-\theta)(3-\theta)}{5!}\,\delta^4 u_0 - \ldots$$
$$+\theta u_1 - \frac{(1+\theta)\,\theta\,(1-\theta)}{3!}\,\delta^2 u_1 + \frac{(2+\theta)(1+\theta)\,\theta\,(1-\theta)(2-\theta)}{5!}\,\delta^4 u_1 - \ldots, \quad (18)$$

* *Loc. cit.*, p. 473, formula (101). † *Ibid.*, formula (102).

$$(B)\ u_\theta = \mu u_{\frac12} - (\tfrac12-\theta)\,\delta u_{\frac12} - \frac{\theta(1-\theta)}{2!}\,\mu\delta^2 u_{\frac12} + (\tfrac12-\theta)\,\frac{\theta(1-\theta)}{3!}\,\delta^3 u_{\frac12}$$

$$+\frac{(1+\theta)\,\theta(1-\theta)(2-\theta)}{4!}\,\mu\delta^4 u_{\frac12} - (\tfrac12-\theta)\,\frac{(1+\theta)\,\theta(1-\theta)(2-\theta)}{5!}\,\delta^5 u_{\frac12} - \ldots \tag{19}$$

$$(C)\ u_\theta = u_0 + \frac{\theta(1-\theta)}{2!}\,\delta u_{-\frac12} - \frac{(1+\theta)\,\theta(1-\theta)(2-\theta)}{4!}\,\delta^3 u_{-\frac12} + \ldots$$

$$+\frac{(1+\theta)\,\theta}{2!}\,\delta u_{\frac12} - \frac{(2+\theta)(1+\theta)\,\theta(1-\theta)}{4!}\,\delta^3 u_{\frac12} + \ldots, \tag{20}$$

$$(D)\ u_\theta = u_0 + \theta\mu\delta u_0 + \frac{\theta^2}{2!}\,\delta^2 u_0 - \frac{(1+\theta)\,\theta(1-\theta)}{3!}\,\mu\delta^3 u_0$$

$$-\frac{(1+\theta)\,\theta^2(1-\theta)}{4!}\,\delta^4 u_0 + \ldots. \tag{21}$$

The formula (17) would also give (20) and (21), while the corresponding formula for interpolating between $u_{\frac12}$ and u_1 instead of between $u_{-\frac12}$ and u_0, if expressed in terms of $x_\theta - x_0$ instead of in terms of $x_\theta - x_1$, would give (18) and (19). Thus either of the formulæ (18) and (19) includes both (16) and (17) for interpolating through the interval from u_0 to u_1, while either (20) or (21) includes both (16) and (17) for interpolating through the interval from $u_{-\frac12}$ to $u_{\frac12}$. The formulæ (18) and (19) are the result of taking (16) and (17) up to tabular differences of an odd order, while (20) and (21) are the result of taking them up to tabular differences of an even order.

If in (A) we write $\phi \equiv (1-\theta)$, it becomes

$$u_\theta = \phi u_0 - \frac{\phi(1^2-\phi^2)}{3!}\,\delta^2 u_0 + \frac{\phi(1^2-\phi^2)(2^2-\phi^2)}{5!}\,\delta^4 u_0 - \ldots$$

$$+\theta u_1 - \frac{\theta(1^2-\theta^2)}{3!}\,\delta^2 u_1 + \frac{\theta(1^2-\theta^2)(2^2-\theta^2)}{5!}\,\delta^4 u_1 - \ldots, \tag{22}$$

which is Everett's formula.* This is by far the most convenient formula for construction of tables by subdivision of intervals, since, when θ has a series of values corresponding to the subdivision of the interval h into a number of equal portions, ϕ has the same series of values, but in the reverse order. Thus each term in (22) appears in the calculation of two values of u; so that the number of calculations of separate terms in u is halved.

(B) is Bessel's formula, slightly modified. If we write

$$\psi \equiv \theta - \tfrac12,$$

* J. D. Everett in *Journ. Inst. Actuaries*, Vol. xxxv., p. 452, and *British Association Report*, 1900, p. 648.

so that ψ may have any value from $-\frac{1}{2}$ to $+\frac{1}{2}$, it becomes

$$u_{\frac{1}{2}+\psi} = \mu u_{\frac{1}{2}} + \psi \delta u_{\frac{1}{2}} - \frac{(\frac{1}{2})^2-\psi^2}{2!}\mu\delta^2 u_{\frac{1}{2}} - \frac{\psi\{(\frac{1}{2})^2-\psi^2\}}{3!}\delta^3 u_{\frac{1}{2}}$$

$$+\frac{\{(\frac{1}{2})^2-\psi^2\}\{(\frac{3}{2})^2-\psi^2\}}{4!}\mu\delta^4 u_{\frac{1}{2}} + \frac{\psi\{(\frac{1}{2})^2-\psi^2\}\{(\frac{3}{2})^2-\psi^2\}}{5!}\delta^5 u_{\frac{1}{2}} - \dots \quad (23)$$

for interpolation between u_0 and u_1.

(C) is a somewhat similar formula to (A) for subdivision of the interval from $u_{-\frac{1}{2}}$ to $u_{\frac{1}{2}}$. If we write

$$\chi \equiv \tfrac{1}{2}+\theta, \quad \omega \equiv \tfrac{1}{2}-\theta,$$

so that the interval from $u_{-\frac{1}{2}}$ to $u_{\frac{1}{2}}$ is divided into portions χ and ω, it becomes

$$u_\theta = u_0 + \frac{(\frac{1}{2})^2-\omega^2}{2!}\delta u_{-\frac{1}{2}} - \frac{\{(\frac{1}{2})^2-\omega^2\}\{(\frac{3}{2})^2-\omega^2\}}{4!}\delta^3 u_{-\frac{1}{2}} + \dots$$

$$-\frac{(\frac{1}{2})^2-\chi^2}{2!}\delta u_{\frac{1}{2}} + \frac{\{(\frac{1}{2})^2-\chi^2\}\{(\frac{3}{2})^2-\chi^2\}}{4!}\delta^3 u_{\frac{1}{2}} - \dots . \quad (24)$$

This is not so convenient as (22), since, as χ increases from 0 to 1 or ω decreases from 1 to 0, some of the coefficients of each tabular difference are positive and others negative.

(D) is usually known as Stirling's formula. It is the most useful formula for isolated interpolations if written in the form of Taylor's theorem

$$u_\theta = u_0 + \theta h u_0' + \frac{\theta^2}{2!}h^2 u_0'' + \frac{\theta^3}{3!}h^3 u_0''' + \dots, \quad (25)$$

$$= u_0 + \theta[hu_0' + \tfrac{1}{2}\theta\{h^2u_0'' + \tfrac{1}{3}\theta(h^3u_0''' + \dots)\}], \quad (25\text{A})$$

the values of hu_0', h^2u_0'', h^3u_0''', ... being given by *

$$\left.\begin{aligned}
hu_0' &= (\mu\delta - \tfrac{1}{6}\mu\delta^3 + \tfrac{1}{30}\mu\delta^5 - \tfrac{1}{140}\mu\delta^7 + \dots)u_0 \\
h^2u_0'' &= (\delta^2 - \tfrac{1}{12}\delta^4 + \tfrac{1}{90}\delta^6 - \tfrac{1}{560}\delta^8 + \dots)u_0 \\
h^3u_0''' &= (\mu\delta^3 - \tfrac{1}{4}\mu\delta^5 + \tfrac{7}{120}\mu\delta^7 + \dots)u_0 \\
h^4u_0^{\text{iv}} &= (\delta^4 - \tfrac{1}{6}\delta^6 + \tfrac{7}{240}\delta^8 + \dots)u_0 \\
h^5u_0^{\text{v}} &= (\mu\delta^5 - \tfrac{1}{3}\mu\delta^7 + \dots)u_0 \\
h^6u_0^{\text{vi}} &= (\delta^6 - \tfrac{1}{4}\delta^8 + \dots)u_0
\end{aligned}\right\}. \quad (26)$$

It should be observed that (25) only agrees with (D) when in (26) we stop always at the same difference of an even order; and it will be found later that this is also necessary for purposes of greater accuracy.

* *Proceedings*, Vol. xxxi., p. 465, formulæ (74).

6. *Accuracy of Standard Formulæ.*—To determine the accuracy of the formulæ (16)–(24), or of (25) with the limitation mentioned at the end of the last paragraph, it is only necessary to consider (16), θ being supposed to be between 0 and 1. If θ is negative, so that the appropriate formula is (17), we must substitute its numerical (positive) value in the results we obtain.

If in (16) we express $\delta u_{\frac{1}{2}}, \delta^2 u_0, \ldots$ in terms of $\ldots, u_{-1}, u_0, u_1 \ldots$, it becomes

$$u_\theta = u_0 + \theta\delta u_{\frac{1}{2}} - \frac{\theta(1-\theta)}{2!}\delta^2 u_0 - \ldots$$

$$= u_0 + \theta(u_1 - u_0) - \frac{\theta(1-\theta)}{2!}(u_1 - 2u_0 + u_{-1}) - \ldots.$$

This gives, for the error in u_θ (see § 2),

$$a_\theta = a_0 + \theta\delta a_{\frac{1}{2}} - \frac{\theta(1-\theta)}{2!}\delta^2 a_0 - \frac{(1+\theta)\theta(1-\theta)}{3!}\delta^3 a_{\frac{1}{2}} + \ldots \tag{27}$$

$$= a_0 + \theta(a_1 - a_0) - \frac{\theta(1-\theta)}{2!}(a_1 - 2a_0 + a_{-1})$$

$$- \frac{(1+\theta)\theta(1-\theta)}{3!}(a_2 - 3a_1 + 3a_0 - a_{-1})$$

$$+ \frac{(1+\theta)\theta(1-\theta)(2-\theta)}{4!}(a_2 - 4a_1 + 6a_0 - 4a_{-1} + a_{-2}) + \ldots. \tag{27A}$$

If we collect coefficients of $\ldots, a_{-1}, a_0, a_1, \ldots$, it will be found that—

(1) the coefficient of a_0 is

$$1 - \frac{\theta}{1!\,0!} + \frac{\theta(1-\theta)}{1!\,1!} - \frac{(1+\theta)\theta(1-\theta)}{2!\,1!} + \frac{(1+\theta)\theta(1-\theta)(2-\theta)}{2!\,2!} - \ldots, \tag{28}$$

the successive values of which, according to the number of terms taken, are

$$1,$$

$$\frac{1-\theta}{1!},$$

$$\frac{(1+\theta)(1-\theta)}{1!\,1!},$$

$$\frac{(1+\theta)(1-\theta)(2-\theta)}{1!\,2!},$$

$$\ldots \quad \ldots;$$

so that, however many terms are taken, the coefficient of a_0 is positive;

(2) the first term in which a_r appears is that which contains $\delta^{2r-1}a_{\frac{1}{2}}$, and the coefficients of a_r are alternately of signs $(-)^{r-1}$ and $(-)^r$. The coefficients of a_r arising from the terms containing $\delta^{2n-1}a_{\frac{1}{2}}$ and $\delta^{2n}a_0$ are respectively

$$\left.\begin{array}{l} (-)^{r-1}\dfrac{(n-1+\theta)(n-2+\theta)\dots\theta(1-\theta)(2-\theta)\dots(n-1-\theta)}{(n-r)!\,(n+r-1)!} \\ \text{and} \\ (-)^r\dfrac{(n-1+\theta)(n-2+\theta)\dots\theta(1-\theta)(2-\theta)\dots(n-\theta)}{(n-r)!\,(n+r)!} \end{array}\right\}, \tag{29}$$

the sum of which is of sign $(-)^{r-1}$; so that, however many terms are taken, the coefficient of a_r is of sign $(-)^{r-1}$;

(3) the coefficient of a_{-r} may similarly be shown to be of sign $(-)^r$.

Hence the greatest possible value of a_θ is when each of the errors $\dots, a_{-1}, a_0, a_1, \dots$ is numerically equal to $\frac{1}{2}\rho$, but the signs (assuming a_0 positive) are

$$\left.\begin{array}{cccccccccc} \dots, & a_{-3}, & a_{-2}, & a_{-1}, & a_0, & a_1, & a_2, & a_3, & a_4, & \dots \\ \dots, & -, & +, & -, & +, & +, & -, & +, & -, & \dots \end{array}\right\}. \tag{30}$$

Taking these values, we find from (11) and (12) that

$$\left.\begin{array}{l} \delta a_{\frac{1}{2}} = \delta^3 a_{\frac{1}{2}} = \delta^5 a_{\frac{1}{2}} = \dots = 0 \\ \delta^2 a_0 = -{}_2C_1\cdot\tfrac{1}{2}\rho \\ \delta^4 a_0 = +{}_4C_2\cdot\tfrac{1}{2}\rho \\ \vdots \qquad\quad \vdots \\ \delta^{2n} a_0 = (-)^n{}_{2n}C_n\cdot\tfrac{1}{2}\rho \end{array}\right\}; \tag{31}$$

so that, substituting in (27), we have, as the limit of error when (16) is taken up to terms in $\delta^{2n}u_0$ or in $\delta^{2n+1}u_{\frac{1}{2}}$,

$$\tfrac{1}{2}\rho\left\{1+\frac{\theta(1-\theta)}{1!\,1!}+\frac{(1+\theta)\theta(1-\theta)(2-\theta)}{2!\,2!}+\dots + \frac{(n-1+\theta)\dots\theta(1-\theta)\dots(2-\theta)}{n!\,n!}\right\}. \tag{32}$$

This is therefore the limit of error in (18) or (22) when taken up to $\delta^{2n}u_0$ and $\delta^{2n}u_1$, in (19) or (23) when taken up to $\delta^{2n+1}u_{\frac{1}{2}}$, in (20) or (24) when taken up to $\delta^{2n-1}u_{-\frac{1}{2}}$ and $\delta^{2n-1}u_{\frac{1}{2}}$, and in (21) or (25) when taken up to $\delta^{2n}u_0$, provided that, in the last two cases, if θ is negative, its numerical magnitude is taken.

In (18) or (22) and in (19) or (23) θ lies between 0 and 1, while in (20) or (24) and in (21) or (25) it lies between $-\frac{1}{2}$ and $\frac{1}{2}$.

7. *Accuracy of Curtailed Formulæ.*—The expression (32) has only been shown to be the limit of error when the formula which we are using is the equivalent of (16) or (17): *e.g.*, when (19) or (23) is taken up to the term involving $\delta^{2n+1}u_{\frac{1}{2}}$, or when (21) or (25) is taken up to the term involving $\delta^{2n}u_0$. We might, however, in either of these cases, wish to stop at the next preceding term; and we have to see how this would affect the limit of error.

We must, in the first place, inquire into the signs of the a's. Assuming θ to be between 0 and $\frac{1}{2}$, so that ψ in (23) is negative, we have found that in (16) the coefficients of a_r, a_0, and a_{-r} are of signs $(-)^{r-1}$, $+$, and $(-)^r$ respectively. This will therefore hold if in (19) or (23) we stop at $\delta^{2n-1}u_{\frac{1}{2}}$, or in (21) or (25) at $\delta^{2n-2}u_0$. The single term which we take next is in the one case (see (14 A))

$$\tfrac{1}{2}(-)^n \frac{\{(\frac{1}{2})^2-\psi^2\}\dots\{(n-\frac{1}{2})^2-\psi^2\}}{(2n)!}$$
$$\left\{u_{n+1}-\dots+(-)^{n+1-r}\frac{2r-1}{2n+1}\,{}_{2n+1}C_{n+1-r}\,u_r\dots+u_{-n}\right\},$$

and in the other case (see (13 A))

$$\tfrac{1}{2}(-)^{n-1}\frac{\theta(1^2-\theta^2)\dots\{(n-1)^2-\theta^2\}}{(2n-1)!}\left\{u_n-\dots+(-)^{n-r}\frac{r}{n}\,{}_{2n}C_{n-r}\,u_r\dots-u_{-n}\right\};$$

and in each case the coefficients of a_r, a_0, and a_{-r} in the term added to a_θ will follow the above rule of signs. Hence, as in the last section, the limit of error will be found by putting each of the errors $\dots, a_{-1}, a_0, a_1, \dots$ equal to $\pm\frac{1}{2}\rho$, the signs being as in (30).

(i.) First consider (19) or (23). The expression (32) gives the limit of error when we take the formula up to $\delta^{2n+1}u_{\frac{1}{2}}$. If we only take it up to the next preceding term, we omit a term in $\delta^{2n+1}u_{\frac{1}{2}}$, and therefore we omit from a_θ a similar term involving $\delta^{2n+1}a_{\frac{1}{2}}$. But we see from (31) that, with the ascribed values, this last term is zero. Hence (32) gives the error in (19) or (23) when taken up to $\mu\delta^{2n}u_0$, as well as when taken up to $\delta^{2n+1}u_{\frac{1}{2}}$.

(ii.) Next consider (21) or (25). If we omit the term involving $\delta^{2n}u_0$, we omit from a_θ a term

$$(-)^{n-1}\frac{\theta^2(1^2-\theta^2)\dots\{(n-1)^2-\theta^2\}}{(2n)!}\,\delta^{2n}a_0,$$

the value of which, by (31), is

$$-\frac{\theta^2(1^2-\theta^2)\dots\{(n-1)^2-\theta^2\}}{n!\,n!}\,\tfrac{1}{2}\rho, \tag{33}$$

so that by omitting this term the limit of error is increased. By taking

$n = 1, 2, \ldots$ successively, we find that the limit of error in u_θ due to the use of (21) or (25) is

$$\tfrac{1}{2}\rho\left\{1+\theta-\frac{\theta^2}{1!\,1!}+\frac{\theta(1^2-\theta^2)}{1!\,2!}-\frac{\theta^2(1^2-\theta^2)}{2!\,2!}+\frac{\theta(1^2-\theta^2)(2^2-\theta^2)}{2!\,3!}-\ldots\right\}, \quad (34)$$

the series inside the brackets being taken to the same number of terms as are used in (25). This is on the assumption that θ is positive: if it is negative, we must, as before, use its numerical value.

It is therefore important, for the sake of accuracy, that in using (25) and (26) we should stop at a difference of even order, even though the portion of (25) due to this difference might be negligible if the values of u were exact.

C. *Comparison of Results.*

8. The limit of error as given by (32) for a formula going up to differences of any particular order is a good deal less than the corresponding limit as given by (9); and therefore, apart from the fact that the central-difference formula generally requires the use of fewer terms, it is more accurate as regards the tabular error. The following table* gives a comparison of the respective limits of error; the figures I. and II. denote the errors due to the advancing-difference and the central-difference formulæ, and the coefficient ρ is omitted throughout:—

θ		Error due to use of Differences up to and including						
		1st	2nd	3rd	4th	5th	6th	7th
·5	I.	·500	·625	·818	1·086	1·497	2·132	3·147
	II.	·500	·625	·625	·696	·696	·745	·745
·2	I.	·500	·580	·724	·960	1·343	1·976	3·042
	II.	·500	·580	·580	·624	·624	·653	·653
·4	I.	·500	·620	·812	1·104	1·553	2·265	3·422
	II.	·500	·620	·620	·688	·688	·734	·734
·6	I.	·500	·620	·788	1·024	1·366	1·886	2·700
	II.	·500	·620	·620	·688	·688	·734	·734
·8	I.	·500	·580	·676	·800	·969	1·213	1·582
	II.	·500	·580	·580	·624	·624	·653	·653

* Cf. Rice, *ubi sup.*, p. 51.

It has been assumed that the tabulated values of the differences are not the "corrected" values, but are obtained directly from the tabulated values of u. If the differences are corrected, *i.e.*, if each difference, like each value of u, is within $\pm\frac{1}{2}\rho$ of the true value, the formulæ for the limit of error will of course have to be altered.* These cases, however, need not be considered here, since, if reduction of the error is important, and the differences are corrected, the discrepancies between the successive columns in the table can be utilized for increasing the accuracy in the tabulated values of u.†

II. *Residual Error.*

9. We have so far assumed that the formula used is correct, and have examined the error due to the fact that the tabulated values of u are incorrect. We have now to consider the error due to the formula itself being incorrect, it being assumed that the tabulated values are correct.

Before doing this, however, it will be useful to inquire into the relation between the advancing-difference formula and the central-difference formula.

(i.) If the advancing-difference formula is taken up to the term involving $\Delta^m u_0$, we may write it in the form

$$u_\Theta = {}_mU_\Theta, \tag{35}$$

where

$$\begin{aligned} {}_mU_\Theta \equiv u_0 &+ \frac{\Theta}{1!}\Delta u_0 \\ &+ \frac{\Theta(\Theta-1)}{2!}\Delta^2 u_0 + \ldots + \frac{\Theta(\Theta-1)\ldots(\Theta-m+1)}{m!}\Delta^m u_0, \end{aligned} \tag{36}$$

the formula (35) being only approximately correct.

Now, if r is an integer,

$$u_r = (1+\Delta)^r u_0 = u_0 + \frac{r}{1!}\Delta u_0 + \frac{r(r-1)}{2!}\Delta^2 u_0 + \ldots + \frac{r(r-1)\ldots 1}{r!}\Delta^r u_0, \tag{37}$$

and therefore, if we substitute $\Theta = 0, 1, 2, \ldots, m$ in (36), we obtain

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* Cf. A. A. Markoff, *Differenzenrechnung* (1896), p. 30.

† Cf. *Biometrika*, Vol. II., p. 177, § 4.

$u_0, u_1, u_2, \ldots, u_m$. Hence, by Lagrange's theorem, since ${}_mU_\Theta$ is a rational integral function of Θ of degree m,

$$ {}_mU_\Theta = \frac{\Theta(\Theta-1)\ldots(\Theta-m)}{m!}$$

$$\times\left\{\frac{u_m}{\Theta-m} - {}_mC_1\frac{u_{m-1}}{\Theta-m+1} + {}_mC_2\frac{u_{m-2}}{\Theta-m+2} - \ldots + (-)^m\frac{u_0}{\Theta}\right\} \quad (38)$$

$$= \frac{\Theta(\Theta-1)\ldots(\Theta-m)}{m!}\Delta^m\frac{u_0}{\Theta-0}. \quad (38\text{A})$$

This may be verified by substituting u_1-u_0, $u_2-2u_1+u_0$, ... for Δu_0, $\Delta^2 u_0$, ... in (36), and collecting coefficients of $u_m, u_{m-1}, \ldots, u_0$ in the result.

(ii.) Similarly, if the central-difference formula (16), for values of θ from 0 to $\frac{1}{2}$, is taken up to the term involving $\delta^m u_0$ or $\delta^m u_{\frac{1}{2}}$, according as m is even or odd, we may write it in the form

$$u_\theta = {}_mV_\theta, \quad (39)$$

where

$$ {}_{2n}V_\theta \equiv u_0 + \frac{\theta}{1!}\delta u_{\frac{1}{2}} + \frac{\theta(\theta-1)}{2!}\delta^2 u_0 + \frac{(\theta+1)\theta(\theta-1)}{3!}\delta^3 u_{\frac{1}{2}} + \ldots$$

$$+ \frac{(\theta+n-1)(\theta+n-2)\ldots(\theta-n)}{(2n)!}\delta^{2n}u_0, \quad (40)$$

$$ {}_{2n+1}V_\theta \equiv u_0 + \frac{\theta}{1!}\delta u_{\frac{1}{2}} + \frac{\theta(\theta-1)}{2!}\delta^2 u_0 + \frac{(\theta+1)\theta(\theta-1)}{3!}\delta^3 u_{\frac{1}{2}} + \ldots$$

$$+ \frac{(\theta+n)(\theta+n-1)\ldots(\theta-n)}{(2n+1)!}\delta^{2n+1}u_{\frac{1}{2}}. \quad (41)$$

If we substitute u_1-u_0, $u_1-2u_0+u_{-1}$, ... for $\delta u_{\frac{1}{2}}$, $\delta^2 u_0$, ... in (40) and (41), we shall find that u_r first appears in $\delta^{2r-1}u_{\frac{1}{2}}$, and that u_{-r} first appears in $\delta^{2r}u_0$. By collecting coefficients it is not difficult to reduce (40) and (41) to the forms

$$ {}_{2n}V_\theta = \frac{(\theta+n)(\theta+n-1)\ldots(\theta-n)}{(2n)!}$$

$$\times\left\{\frac{u_n}{\theta-n} - {}_{2n}C_1\frac{u_{n-1}}{\theta-n+1} + {}_{2n}C_2\frac{u_{n-2}}{\theta-n+2} - \ldots + \frac{u_{-n}}{\theta+n}\right\} \quad (42)$$

$$ {}_{2n+1}V_\theta = \frac{(\theta+n)(\theta+n-1)\dots(\theta-n-1)}{(2n+1)!} $$

$$ \times \left\{ \frac{u_{n+1}}{\theta-n-1} - {}_{2n+1}C_1 \frac{u_n}{\theta-n} + {}_{2n+1}C_2 \frac{u_{n-1}}{\theta-n+1} - \dots - \frac{u_{-n}}{\theta+n} \right\}. \quad (43) $$

We see from (42) and (43) that, if we substitute $\theta = 0, 1, -1, 2, -2, \dots$ successively in (40) and (41), up to $-n$ in (40) or $n+1$ in (41), we obtain $u_0, u_1, u_{-1}, u_2, u_{-2}, \dots$ up to u_{-n} or u_{n+1}. This might have been proved directly,* and we should then, as in (i.), have deduced (42) and (43) at once by Lagrange's theorem.

(iii.) For values of θ from $-\frac{1}{2}$ to 0 the central-difference formula is (17). This, however, is really the same as (16), with the u's taken in the reverse order. We shall therefore find that for $m = 2n$ (17) will give u_θ in the form (42), while for $m = 2n+1$ it will give an expression exactly similar to (43), but with $u_n, \dots, u_{-n-1}$ substituted for $u_{n+1}, \dots, u_{-n}$, and $\theta-n, \dots, \theta+n+1$ for $\theta-n-1, \dots, \theta+n$. The expression (43) would therefore correspond to (17) adapted for values of θ from $\frac{1}{2}$ to 1.

(iv.) Hence, on the whole, the central-difference formulæ when taken up to a difference of order $2n$ will give (42) as the value of u_θ for values of θ from $-\frac{1}{2}$ to $\frac{1}{2}$, while when taken up to a difference of order $2n+1$ they will give (43) as the value of u_θ for values of θ from 0 to 1.

(v.) If now we compare (42) and (43) with (38), we see that ${}_{2n}V_{\theta-n}$ and ${}_{2n+1}V_{\theta-n}$ are of exactly the same form as ${}_{2n}U_\theta$ and ${}_{2n+1}U_\theta$, except that the latter contain the values $u_0, u_1, u_2, \dots$ up to u_{2n} or u_{2n+1}, while the former contain the values $u_{-n}, u_{-n+1}, u_{-n+2}, \dots$ up to u_n or u_{n+1}. In other words, the advancing-difference formula which we use for interpolating through the interval from x_p to x_{p+1}, Θ having values from 0 to 1, would become the central-difference formula if we used it for interpolating through the interval from $x_{p+n-\frac{1}{2}}$ to $x_{p+n+\frac{1}{2}}$, Θ having values from $n-\frac{1}{2}$ to $n+\frac{1}{2}$, when differences are taken up to those of the $(2n)$th order, or through the interval from x_{p+n} to x_{p+n+1}, Θ having values from n to $n+1$, when differences are taken up to those of the $(2n+1)$th order.

Geometrically, if we regard the given values of u as the ordinates of a curve, we interpolate by treating any other value of u as the ordinate of a parabola of degree $2n$ or $2n+1$ which passes through the extremities of $2n+1$ or $2n+2$ consecutive given ordinates; but in using the advancing-

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* See p. 339.

difference formula for the interval from x_0 to x_1 we take the ordinates $u_0, u_1, u_2, \ldots$, while in using the central-difference formula for the interval from $x_{-\frac{1}{2}}$ to $x_{\frac{1}{2}}$ or from x_0 to x_1 we commence with the nearest ordinate and continue with those ordinates which are successively the next nearest.

We can therefore simplify the consideration of the residual error by extending the range of values of θ or Θ. If we write

$$u_\Theta \equiv {}_mU_\Theta + {}_mR_\Theta, \tag{44}$$

where ${}_mU_\Theta$ has the value given by (38), we have to examine the value of ${}_mR_\Theta$, where Θ has any value between 0 and m. The error in the advancing-difference formula will then be found by treating Θ as between 0 and 1, while the error in the central-difference formula will be found by treating it as between $\frac{1}{2}m-\frac{1}{2}$ and $\frac{1}{2}m+\frac{1}{2}$.

There are various known expressions for ${}_mR_\Theta$. For our present purpose it will be sufficient to consider two of them.

10. We have seen that, if in ${}_mU_\Theta$ we substitute $\Theta = 0, 1, 2, \ldots, m$, we obtain $u_0, u_1, u_2, \ldots, u_m$. Hence, if in ${}_mU_\Theta$ we replace $u_0, u_1, u_2, \ldots, u_m$ by a constant k, the resulting expression will be equal to k when Θ has any of the $m+1$ values $0, 1, 2, \ldots, m$. The expression is, however, only of degree m in Θ; and therefore it must be identically equal to k. Hence, putting $k = 1$,

$$1 = \frac{\Theta(\Theta-1)\ldots(\Theta-m)}{m!}\left\{\frac{1}{\Theta-m} - {}_mC_1\frac{1}{\Theta-m+1} + \ldots + (-)^m\frac{1}{\Theta}\right\} \tag{45}$$

$$= \frac{\Theta(\Theta-1)\ldots(\Theta-m)}{m!}\Delta^m\frac{1}{\Theta-0}. \tag{45A}$$

Multiplying both sides of this by u_Θ, and substituting from this and from (38) in (44), we find*

$${}_mR_\Theta = \frac{\Theta(\Theta-1)\ldots(\Theta-m)}{m!}\left\{\frac{u_\Theta-u_m}{\Theta-m} - {}_mC_1\frac{u_\Theta-u_{m-1}}{\Theta-m+1} + \ldots + (-)^m\frac{u_\Theta-u_0}{\Theta}\right\} \tag{46}$$

$$= \frac{\Theta(\Theta-1)\ldots(\Theta-m)}{m!}\Delta^m\frac{u_\Theta-u_0}{\Theta-0} \tag{46A}$$

$$= \frac{\Theta(\Theta-1)\ldots(\Theta-m)}{m!}h\Delta^m T_0, \tag{46B}$$

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* Cf. Boole, p. 146.

where T_r is the tangent of the angle which the line joining (x_Θ, u_Θ) to (x_r, u_r) makes with the axis of x.

Expressed in terms of central differences, (21) and (22) being taken as the typical formulæ, this gives

$$u_\theta = u_0 + \theta\mu\delta u_0 + \frac{\theta^2}{2!}\delta^2 u_0 + \frac{\theta(\theta^2-1^2)}{3!}\mu\delta^3 u_0 + \frac{\theta^2(\theta^2-1^2)}{4!}\delta^4 u_0 + \dots$$
$$+\frac{\theta^2(\theta^2-1^2)\dots\{\theta^2-(n-1)^2\}}{(2n)!}\delta^{2n}u_0 + \frac{\theta(\theta^2-1^2)\dots(\theta^2-n^2)}{(2n)!}h\delta^{2n}t_0, \tag{47}$$

$$u_\theta = \phi u_0 + \frac{\phi(\phi^2-1^2)}{3!}\delta^2 u_0 + \dots + \frac{\phi(\phi^2-1^2)\dots(\phi^2-n^2)}{(2n+1)!}\delta^{2n}u_0$$
$$+\theta u_1 + \frac{\theta(\theta^2-1^2)}{3!}\delta^2 u_1 + \dots + \frac{\theta(\theta^2-1^2)\dots(\theta^2-n^2)}{(2n+1)!}\delta^{2n}u_1$$
$$+\frac{\theta(\theta^2-1^2)\dots(\theta^2-n^2)(\theta-n-1)}{(2n+1)!}h\delta^{2n+1}t_{\frac{1}{2}}, \tag{48}$$

where t_r is the tangent of the angle which the line joining (x_θ, u_θ) to (x_r, u_r) makes with the axis of x. The last term in (48) may also be written

$$+\frac{\phi(\phi^2-1^2)\dots(\phi^2-n^2)(\phi-n-1)}{(2n+1)!}h\delta^{2n+1}t_{\frac{1}{2}}. \tag{49}$$

The above formulæ are exact, but they do not give any very clear indication of the magnitude of the error ${}_mR_\Theta$, since $\Delta^m T_0$ might be very different according as Θ was between 0 and 1 or between $\frac{1}{2}m-\frac{1}{2}$ and $\frac{1}{2}m+\frac{1}{2}$. The expression found in the next section is more useful.

11. Let us write*

$$\phi(\lambda) \equiv u_\lambda - u_0 - \frac{\lambda}{1!}\Delta u_0 - \frac{\lambda(\lambda-1)}{2!}\Delta^2 u_0 - \dots - \frac{\lambda(\lambda-1)\dots(\lambda-m+1)}{m!}\Delta^m u_0$$
$$-\frac{\lambda(\lambda-1)\dots(\lambda-m)}{m!}\Delta^m\frac{u_\Theta-u_0}{\Theta-0}. \tag{50}$$

Then $\phi(\lambda)$ vanishes when λ has any of the $m+1$ values 0, 1, 2, ..., m, and also when $\lambda = \Theta$. Hence, provided Θ lies between 0 and m, the $(m+1)$th differential coefficient of $\phi(\lambda)$ with regard to λ vanishes for some value of λ between 0 and m; in other words, if

$$u_\Theta \equiv f(x_0+\Theta h), \tag{51}$$

* Markoff, p. 6.

so that
$$T_r \equiv \frac{f(x_0+\Theta h)-f(x_0+rh)}{(\Theta-r)h}, \tag{52}$$

then
$$\Delta^m T_0 = \frac{h^m}{m+1} f^{m+1}(\xi) \tag{53}$$

and
$${}_mR_\Theta = \frac{\Theta(\Theta-1)\dots(\Theta-m)}{(m+1)!} h^{m+1} f^{m+1}(\xi), \tag{54}$$

where ξ has some value between x_0 and x_m. This is for advancing differences; if we use central differences, we shall have a corresponding expression, where ξ has some value between those belonging to the extreme ordinates which enter into the formula.

This last result explains the general superiority of central-difference over advancing-difference formulæ, and it also indicates some of the limitations of interpolation by finite differences.

(i.) Suppose that throughout the range of values of x from x_0 to x_m $f^{m+1}(x)$ is approximately constant, or at any rate does not vary greatly. Then the magnitude of the residual error depends on the magnitude of $\Theta(\Theta-1)\dots(\Theta-m)$. This is obviously less when Θ is nearly equal to $\frac{1}{2}m$ than when it is nearly equal to 0; if, for instance, $m=5$, and we are interpolating at the middle of an interval, the above expression, when we use the advancing-difference formula, is

$$-\tfrac{1}{2}\cdot\tfrac{1}{2}\cdot\tfrac{3}{2}\cdot\tfrac{5}{2}\cdot\tfrac{7}{2}\cdot\tfrac{9}{2};$$

but, when we use the central-difference formula, it is

$$-\tfrac{5}{2}\cdot\tfrac{3}{2}\cdot\tfrac{1}{2}\cdot\tfrac{1}{2}\cdot\tfrac{3}{2}\cdot\tfrac{5}{2},$$

which is less than one-fourth of the former. Thus, by the use of central differences, we may be able to exclude a term which would otherwise have to be included, and thus to shorten our formulæ.

(ii.) It may, however, be the case that within a certain range of values $f^{m+1}(x)$ is small, while outside this range it becomes relatively great. In such a case we ought to adapt the formula to the circumstances, using central-difference formulæ towards the middle of the range, and advancing-difference or receding-difference formulæ towards the extremities.

(iii.) In such a case, also, we may obviously make our formula worse by introducing differences of a higher order.

These difficulties are usually solved by inspection of the differences themselves; their smallness along certain lines indicating the particular

formula that should be used. The formula may be neither a central-difference nor an advancing-difference formula: the differences taken into account may run first centrally and then diagonally. It is not necessary to obtain special formulæ for every such case; if the table indicates that we ought to stop at a particular difference $\Delta^m u_a$, any formula which is a proper formula for interpolation (*i.e.*, which would give u exactly if $\Delta^m u$ were constant), and which uses only values of $u, \Delta u, \ldots, \Delta^m u$ comprised within the triangle whose vertex is $\Delta^m u_a$, can be reduced to a central-difference formula by reconstructing, on the basis of $\Delta^m u_a$ being constant, the portion of the table which lies outside this triangle.

Suppose, for instance,* that we had to interpolate from a table in which values of $u \equiv \tan x$ are given by intervals of 10° in x from $-90°$ to $+90°$. The portion of the table from $x = 0°$ to $x = 90°$, taken up to 10th differences, would be as below; the portion from $x = -90°$ to $x = 0°$ being similar, except that the values of u and of the even differences would be negative.

x	u	Δu	$\Delta^2 u$	$\Delta^3 u$	$\Delta^4 u$	$\Delta^5 u$	$\Delta^6 u$	$\Delta^7 u$	$\Delta^8 u$	$\Delta^9 u$	$\Delta^{10} u$
		+	+	+	+	+	+	+	+	+	+
0°	·00000		0		0		0		0		0
		17633		1131		312		196		270	
10°	·17633		1131		312		196		270		594
		18764		1443		508		466		864	
20°	·36397		2574		820		662		1134		4359
		21338		2263		1170		1600		5223	
30°	·57735		4837		1990		2262		6357		52096
		26175		4253		3432		7957		57319	
40°	·83910		9090		5422		10219		63676		∞
		35265		9675		13651		71633		∞	
50°	1·19175		18765		19073		81852		∞		
		54030		28748		95503		∞			
60°	1·73205		47513		1 14576		∞				
		1 01543		1 43324		∞					
70°	2·74748		1 90837		∞						
		2 92380		∞							
80°	5·67128		∞								
		∞									
90°	∞										

It will be seen that all the differences first decrease and then increase; that for interpolating near $x = 0°$ we may use central differences up to about the 9th; that for values near $-50°$ and $+50°$ we must use advancing and receding differences respectively, also up to about the 9th; and that for intermediate values we require a special formula. We can

* Cf. *Journal of Royal Statistical Society*, Vol. LXIII., p. 446.

therefore frame a new table, as below, by taking the 9th difference as constant. The inserted differences are printed in dark type.

x	u	Δu	$\Delta^2 u$	$\Delta^3 u$	$\Delta^4 u$	$\Delta^5 u$	$\Delta^6 u$	$\Delta^7 u$	$\Delta^8 u$	$\Delta^9 u$
		+	+	+	+	+	+	+	+	+
0°	·00000		0		0		0		0	
		17633		1131		312		196		270
10	·17633		1131		312		196		270	
		18764		1443		508		466		270
20°	·36397		2574		820		662		540	
		21338		2263		1170		1006		270
30°	·57735		4837		1990		1668		810	
		26175		4253		2838		1816		270
40°	·83910		9090		4828		3484		1080	
		35265		9081		6322		2896		270
50°	1·19175		18171		11150		6380		1350	

It will be found that this table gives very good results. If, for instance, we calculate u for $x = 5°, 15°, 25°, 35°, 45°$, by the central-difference formula* for mid-way interpolation, the first four values will be correct to five places, while the fifth will only be wrong by ·00004.

APPENDIX.

1. The proposition in § 9 (ii.), that

$$\phi_r(u_0) \equiv u_0 + r\delta u_{\frac{1}{2}} + \frac{r(r-1)}{2!}\delta^2 u_0 + \frac{(r+1)r(r-1)}{3!}\delta^3 u_{\frac{1}{2}} + \dots \text{ (}m\text{ terms)}$$

is equal to u_r when r has any negative or positive integral value from $-n$ to n, if $m = 2n+1$, and from $-n$ to $n+1$, if $m = 2n+2$, may be proved by elementary methods as follows :—

(i.) Suppose the proposition true for $r = p$, so that

$$\begin{aligned} u_p &= \phi_p(u_0) \\ &= u_0 + p\,\delta u_{\frac{1}{2}} + \frac{p(p-1)}{2!}\delta^2 u_0 + \frac{(p+1)p(p-1)}{3!}\delta^3 u_{\frac{1}{2}} + \dots, \end{aligned} \tag{55}$$

the series continuing until the individual terms vanish, on account of p being within the limits mentioned above.

* *Proceedings*, Vol. xxxi., p. 462, formula (63).

The formula (55) being true identically, we can apply it to $\delta u_{\frac{1}{2}}$ and its differences, instead of to u_0, and we find

$$\delta u_{p-\frac{1}{2}} = \delta u_{\frac{1}{2}} + p\,\delta^2 u_1 + \frac{p(p-1)}{2!}\,\delta^3 u_{\frac{1}{2}} + \frac{(p+1)p(p-1)}{3!}\,\delta^4 u_1 + \ldots,$$

which, by taking terms after the first in pairs, can be reduced to

$$\delta u_{p-\frac{1}{2}} = \delta u_{\frac{1}{2}} + p\,\delta^2 u_0 + \frac{(p+1)p}{2!}\,\delta^3 u_{\frac{1}{2}} + \frac{(p+1)p(p-1)}{3!}\,\delta^4 u_0 + \ldots. \quad (56)$$

Adding (55) and (56),

$$\begin{aligned} u_{p+1} &= u_p + \delta u_{p+\frac{1}{2}} \\ &= u_0 + (p+1)\,\delta u_{\frac{1}{2}} + \frac{(p+1)p}{2!}\,\delta^2 u_0 + \frac{(p+2)(p+1)p}{3!}\,\delta^3 u_{\frac{1}{2}} + \ldots \quad (57) \\ &= \phi_{p+1}(u_0); \end{aligned}$$

so that the proposition is true for $r = p+1$.

But it is true for $r = 0$: therefore it is true for $r = 1, 2, \ldots$. The limit of these values is determined by the fact that the series in (56) goes up to a difference one degree higher than in (55); so that the next term in (55) must be one that contains $p-p$ as a factor.

(ii.) Again, (57) is true if we apply it to u_1, reading the u's in the other direction. This gives

$$\begin{aligned} u_{-p} &= u_{1-(p+1)} \\ &= u_1 - (p+1)\,\delta u_{\frac{1}{2}} + \frac{(p+1)p}{2!}\,\delta^2 u_1 - \frac{(p+2)(p+1)p}{3!}\,\delta^3 u_{\frac{1}{2}} + \ldots \\ &= u_0 - p\,\delta u_{\frac{1}{2}} + \frac{(p+1)p}{2!}\,\delta^2 u_0 - \frac{(p+1)p(p-1)}{3!}\,\delta^3 u_{\frac{1}{2}} + \ldots \\ &= u_0 + (-p)\,\delta u_{\frac{1}{2}} + \frac{(-p)(-p-1)}{2!}\,\delta^2 u_0 + \frac{(-p+1)(-p)(-p-1)}{3!}\,\delta^3 u^{\frac{1}{2}} + \ldots; \end{aligned}$$

so that, if the proposition is true for $r = p+1$, it is true for $r = -p$. The case of $r = -n$, when $2n+1$ terms are taken, can be proved specially.

2. The relation (55) may be written

$$(1+\Delta)^p u_0 = \left\{1 + \frac{p(p-1)}{2!}\,\frac{\Delta^2}{1+\Delta} + \frac{(p+1)p(p-1)(p-2)}{4!}\,\frac{\Delta^4}{(1+\Delta)^2} + \ldots\right\} u_0$$
$$+ \left\{\frac{p}{1!} + \frac{(p+1)p(p-1)}{3!}\,\frac{\Delta^2}{1+\Delta} + \ldots\right\} \Delta u_0.$$

If we write

$$\psi_p(x) \equiv 1 + \frac{p(p-1)}{2!}\frac{x^2}{1+x} + \frac{(p+1)p(p-1)(p-2)}{4!}\frac{x^4}{(1+x)^2} + \dots$$
$$+ \frac{(2p-2)\dots 1}{(2p-2)!}\frac{x^{2p-2}}{(1+x)^{p-1}},$$

$$\chi_p(x) \equiv \frac{p}{1!}x + \frac{(p+1)p(p-1)}{3!}\frac{x^3}{1+x} + \dots + \frac{(2p-1)\dots 1}{(2p-1)!}\frac{x^{2p-1}}{(1+x)^{p-1}},$$

it can be proved that

$$\psi_{p+1}(x) + \chi_{p+1}(x) = (1+x)\{\psi_p(x) + \chi_p(x)\},$$
$$\psi_{p+1}(x) - \chi_p(x) = \{\psi_p(x) - \chi_{p-1}(x)\}/(1+x),$$

and thence, by induction,

$$\psi_p(x) + \chi_p(x) = (1+x)^p,$$
$$\psi_{p+1}(x) - \chi_p(x) = (1+x)^{-p}.$$

These formulæ, by the substitution of Δ for x, would lead to the results in the last section.

ON THE GEOMETRICAL INTERPRETATION OF APOLAR BINARY FORMS

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1. Binary algebraic forms are usually represented geometrically by sets of points; of the three methods explained in Grace and Young's *Algebra of Invariants*, according to which the symbolical expression a_x^n is represented by n points (i) in a straight line, (ii) on a conic, (iii) in the Argand diagram, the first two only are employed in this paper. It will be assumed that the coefficients in the binary forms considered are real, so that imaginary roots of the equation $a_x^n = 0$ will occur, if at all, in conjugate pairs. The second method of interpretation is thus always possible; the first is possible only when there are no imaginary roots.

It is well known that, if two harmonic (*i.e.*, apolar) quadratics are thus represented, the construction for obtaining the fourth point when three are given is linear. In this paper, I prove a similar property for any two apolar forms of the same order.

2. If the form a_x^n be represented by the points $A_1, A_2, \ldots, A_n$, and (xy) be the point P, then the first polar form $a_x^{n-1}a_y$ will be represented by $n-1$ points which I shall call *the polar* $(n-1)$*-points of* P *for* $A_1, A_2, \ldots, A_n$; since the number of the points is more important than the order of the polar form (the first). Similarly, $a_x^{n-2}a_y^2$ will be represented by the polar $(n-2)$-points of P; and so on. This nomenclature serves also to avoid confusion with the polar lines, conics, &c., of a point with respect to curves, which will sometimes occur in the work.

The principle on which the method of this paper depends is most easily expressed symbolically: if a_x^n and $b_x^{n-1}c_x$ are apolar n-ics, then $(ac)\,a_x^{n-1}$ and b_x^{n-1} are apolar $(n-1)$-ics. This exhibits the connection between polar and apolar forms.

3. Let a range of n points, representing a form a_x^n according to the first method, be regarded as the intersections of the line with a plane n-ic curve. Then, if P, another point on the line, be defined by (xy),

the line will meet the first, second, ..., polar curves of P in the $(n-1)$-points, $(n-2)$-points, ..., of P for the range.

This furnishes a method of constructing P_1, the polar 1-point of a point P for a given triad A_1, A_2, A_3. For, if we draw through A_1, A_2, A_3, the special cubic curve consisting of three straight lines which form a triangle LMN, the polar line of P with respect to this triangle will pass through P_1.

The problem of finding the polar 2-points, Q_1, Q_2, of P for the triad is really the converse of this. For P is the polar 1-point for the triad of each of the points Q_1, Q_2. If therefore we draw two or more lines through P and find the poles $B_1, B_2, \ldots$, of each of these lines with respect to the triangle LMN, all these points B must lie on the polar conic of P for the triangle, and hence the required points Q_1, Q_2 are the intersections of the line $A_1A_2A_3$ with a conic circumscribing LMN, of which any number of points can be obtained.

Now let P, Q, R be a triad of points which is apolar to the triad A_1, A_2, A_3, and let P, Q be given while R is to be constructed. We know that Q and R will be harmonically conjugate with respect to Q_1, Q_2; *i.e.*, the polar line of Q for the conic-locus of the points B will pass through R. And this polar line of Q can be linearly constructed, although the points Q_1, Q_2 cannot themselves be obtained except by a construction of the second degree.

4. Throughout the rest of this paper I use the second method of interpretation; and by the polar line of a point, or the pole of a straight line, I shall always mean the polar or pole with regard to the fundamental conic; whenever I have occasion to refer to the polar line of a point with respect to a triangle this will be specially stated.

Any quadratic form a_x^2, interpreted by two points on the conic, determines a straight line, *i.e.*, the line joining these points; and hence determines also a point in the plane of the conic, *i.e.*, the pole of this line. This single point can be linearly constructed, and upon it depends the construction in the harmonic case of two quadratics.

In considering the cubic, I shall first suppose that the roots of the equation $a_x^3 = 0$ are $x_1/x_2 = 0$, 1, and ∞, so that

$$a_x^3 \equiv x_1^2x_2 - x_1x_2^2.$$

Let the points representing the form be P_1, P_2, P_3, and let the point Q represent any linear form (xy), for which the ratio $x_1/x_2 = \lambda$.

Then we may easily obtain the following results: for Q', the polar

1-point of Q for the triad, given by $a_x a_y^2 = 0$, the ratio

$$\frac{x_1}{x_2} = \frac{\lambda(2-\lambda)}{2\lambda-1};$$

for $\bar{Q}$, the harmonic conjugate of Q with respect to the Hessian points (for brevity, the H-points) of a_x^3, the ratio

$$\frac{x_1}{x_2} = \frac{\lambda-2}{2\lambda-1};$$

for q_2, the harmonic conjugate of Q with respect to P_1, P_3, the ratio

$$x_1/x_2 = -\lambda.$$

Hence we have

$$\{P_1 Q' P_3 \bar{Q}\} = \frac{\lambda(2-\lambda)}{2\lambda-1}\,\frac{2\lambda-1}{\lambda-2} = -\lambda = \{P_1 q_2 P_3 P_2\}; \tag{1}$$

and, since any three points can be projected into any other three points, this result must hold for the general cubic.

Applying this result to the present case, we are concerned first with the H-points.

Let all the roots of the cubic be real, and let $a_x^3 \equiv \alpha_x \beta_x \gamma_x$, where α_x is represented by P_1, β_x by P_2, and γ_x by P_3.

Then the H-points are imaginary, and the pole I of the line joining them is within the conic.

To construct the point I [see *Algebra of Invariants*, Ex. (i.), p. 240], join P_1 to the pole of the line P_2P_3, P_2 to the pole of P_3P_1, and P_3 to the pole of P_1P_2: the joining lines will be concurrent in I.

Next join QI, and produce to meet the conic, obtaining the point $\bar{Q}$, and join Q to the pole of P_1P_3, the line meeting the conic again in q_2.

Then the result (1) furnishes the following construction:—Let $\bar{Q}q_2$ meet P_1P_3 in M; then P_2M will meet the conic again in Q', the required polar 1-point of Q.

There is thus obtained an easy line construction for the point Q' (see Fig. 1).

The point arrived at must be the same whether we take, in our construction, the points P_1, P_3, or either of the two other pairs of the triad, P_1, P_2 and P_2, P_3. We have thus the geometrical theorem that the line MP_2 and the two other lines LP_1, NP_3, obtained similarly, are concurrent in a point on the conic.

In the figure the point Q' is obtained in the three possible ways, the same position being found each time. It is therefore to be noticed that the actual process of finding Q' is very much simpler than appears from

the complicated figure. It will be seen that for practical convenience it is desirable to select for the construction that pair of the points P_1, P_2, P_3 which are both on the same side of the line $QI\bar{Q}$.

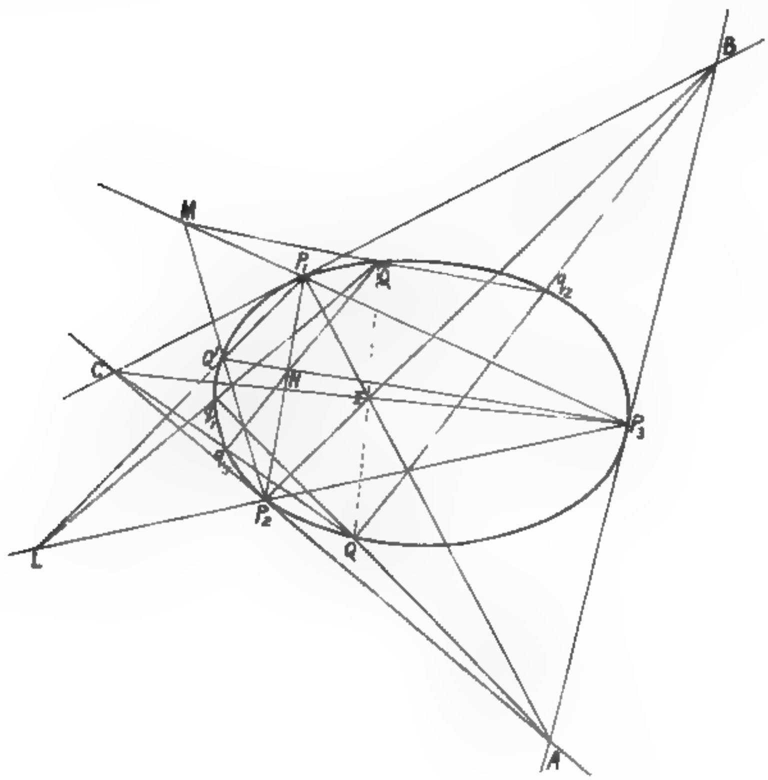

FIG. 1.

5. This geometrical theorem can be extended indefinitely. For the point q_1 is given by $\beta_y\gamma_x+\beta_x\gamma_y$, and therefore the triad q_1, q_2, q_3 is given by a cubic which is

$$= (\alpha_y\beta_x\gamma_x+\ldots+\ldots)(\alpha_x\beta_y\gamma_y+\ldots+\ldots)-\alpha_x\beta_x\gamma_x\alpha_y\beta_y\gamma_y$$

$$= 9a_x^2a_yb_xb_y^2-a_x^3b_y^3.$$

The polar 1-point of Q with respect to this triad is given by $a_x a_y^2$, and is therefore the point Q'.

Hence in the figure we should arrive at the same point Q' if, instead of the triad P_1, P_2, P_3, we took the triad q_1, q_2, q_3; and therefore also if we took the triad obtained from q_1, q_2, q_3 in the same way as the triad q_1, q_2, q_3 is obtained from P_1, P_2, P_3; and so on.

6. If Q_1, Q_2 are the polar 2-points of Q for the triad P_1, P_2, P_3, we know that Q_1, Q_2 are harmonically conjugate to Q, Q'; and they are so also with respect to the H-points.

Therefore the line TI, where T is the pole of QQ', will meet the conic in Q_1 and Q_2; and these points can therefore be constructed.

The pole K of the line Q_1Q_2 is the intersection of the line QQ' with the Hessian line; and, if any line whatever be drawn through K, meeting the conic in R and S, then the triad Q, R, S is apolar to the triad P_1, P_2, P_3.

If the points Q, R are given, and S is to be found, we therefore find K as above, and join KR: it will meet the conic in S. Another way of proceeding, which is really equivalent to the last, is this: Join QR, and produce to meet the Hessian line in K'. The polar of K' will meet the conic in the polar 2-points of S. It will be seen in § 7 that from this we are able to find S.

The solution of the problem is therefore complete.

This method of constructing Q', and then Q_1 and Q_2, is, however, somewhat artificial. I proceed to give two alternative methods of finding the points Q_1, Q_2, both of which are direct and do not depend on Q' being first obtained. The one is of interest chiefly on account of the simplicity of the result; the other because it can be extended to the general problem of two n-ics.

7. The first of these admits of very easy proof; it will therefore suffice to state the result, which is as follows:

The line joining Q_1, Q_2, the polar 2-points of Q, is the polar line of Q with respect to the triangle $P_1P_2P_3$.

By means of this result we can find the point whose polar 2-points are given (see § 6).

[We have seen that the line Q_1Q_2 always passes through I for all positions of Q; that is, for all points on the fundamental conic the polar line with respect to the triangle $P_1P_2P_3$ passes through I. Hence the fundamental conic is the polar conic of I with respect to the triangle.]

When the points Q_1, Q_2 have been constructed in this way, Q', the polar 1-point of Q, may be obtained by a much simpler process than that explained above, by finding the harmonic conjugate of Q with respect to Q_1, Q_2.

The problem of two apolar triads is then completed as before; and the result obtained may be thus stated:

If the triads P_1, P_2, P_3 and Q, R, S are apolar, then the polar triangle

of QRS with respect to the conic is inscribed in the polar triangle of QRS with respect to the triangle $P_1P_2P_3$.

8. To obtain the third construction for Q_1, Q_2 referred to above, we notice that $a_x^2 a_y$ can be written in the form

$$a_y\beta_x\gamma_x + a_x(\beta_y\gamma_x + \beta_x\gamma_y);$$

and therefore $a_x^2 a_y$ is apolar to the pair of points which is apolar both to β_x. γ_x and to a_x, $\beta_y\gamma_x + \beta_x\gamma_y$. That is, if q_1 is the harmonic conjugate of Q with respect to P_2, P_3 and P_1q_1 meets P_2P_3 in L', then the line Q_1Q_2 passes through L'.

Hence, if we construct similarly the points M' and N', these points L', M', N' must be collinear, the line of collinearity being the line Q_1Q_2 and so passing through I.

9. We have seen that if the points Q, R, S form a triad apolar to P_1, P_2, P_3, then R and S may be taken as the intersections of the conic with any line drawn through the pole of Q_1Q_2. If, therefore, the line chosen does not (geometrically) meet the conic at all, so that the points R and S are imaginary, the triad Q, R, S is still apolar to P_1, P_2, P_3.

Also, if Q, R, given points (while S is to be found), are imaginary points lying on a line which does not meet the conic, we can still construct S by the second method given in § 6.

But, if in the original triad P_1, P_2, P_3 two points, say P_2 and P_3, are imaginary, the constructions which have been given above are no longer immediately possible.

If K be the point of intersection of P_1A and P_2P_3 in Fig. 1, the range P_1, I, K, A is harmonic. From this we can, in the present case, construct I and afterwards the H-points, which are now real. Also q_1 and L', but not q_2, q_3, M', N', can be obtained; and then the line $L'I$ will meet the conic in Q_1 and Q_2, which may be either real or imaginary.

Or it is still possible to use the method of § 4; for L can be obtained, but not M and N, and then the line P_1L gives the point Q'.

10. If P_1, P_2, P_3 and Q, R, S be two apolar triads, and RI, SI meet the conic again in $\bar{R}$ and $\bar{S}$ respectively, it is known that the line $\bar{R}\bar{S}$ meets RS on the polar of I; therefore Q, $\bar{R}$, $\bar{S}$ is also an apolar triad to P_1, P_2, P_3.

There are thus three triads associated with Q, R, S, each of which is apolar to P_1, P_2, P_3; they are Q, $\bar{R}$, $\bar{S}$; $\bar{Q}$, R, $\bar{S}$; and $\bar{Q}$, $\bar{R}$, S.

And the members of the doubly-infinite system of triads apolar to P_1, P_2, P_3 are associated in sets of four, those in one such set being derivable from any one of the four.

11. Before passing on to consider the geometry of the quartic, I add a short note on the covariant points of two different cubics a_x^3, b_x^3 given by $(ab)^2 a_x b_x$. When the cubics coincide these points become, of course, the H-points; and when the cubics are apolar it will be seen that they are the double points of a certain involution.

Let F_1, F_2 be the points to be interpreted; then, since

$$((ab)^2 a_x b_x,\ a_x'^3) = \tfrac{1}{2}((aa')^2 a_x a_x',\ b_x^3),$$

we have that the point which with F_1, F_2 makes a triad apolar to a_x^3 is also the point which with the H-points of a_x^3 makes a triad apolar to b_x^3.

Hence the following construction:—Find the point which with the H-points of a_x^3 makes an apolar triad to b_x^3. Construct its polar 2-points for a_x^3 and find A_1, the pole of the line joining them. Similarly find B_1. Then $A_1 B_1$ meets the conic in the required points F_1, F_2.

Let us now take any point on the conic and find its polar 2-points with respect to the cubic a_x^3; then find the point which with these two points makes an apolar triad to b_x^3. This last point will be given by $(ab)^2 a_y b_x$.

Taking the triads in the reverse order, we get the point $(ab)^2 a_x b_y$. We have thus defined a (1, 1) correspondence of points on the conic of which the united points are F_1, F_2.

But
$$(ab)^2 a_y b_x - (ab)^2 a_x b_y = -(ab)^3 (xy),$$

and therefore when the triads are apolar the correspondence becomes an involution with F_1, F_2 as its double points.*

Another property of F_1 and F_2 is that the two lines joining the polar 2-points of either of them with respect to the two triads are conjugate.

The problem of finding a triad apolar to two given triads occurs above, and the method is obvious. That of finding the unique triad apolar to three given triads arises naturally in considering the case of the quartic. Its discussion is postponed to § 13.†

* Apolar forms of different orders are connected by a property somewhat similar to this. Let there be any two n-ics of points. Then a (1, 1) correspondence is defined by adding to each a single point in order to get two apolar $(n+1)$-ics. If the n-ics are themselves apolar, this correspondence becomes an involution. When n is 3 this involution is the same as that above, having F_1, F_2 for double points; but the correspondence (when the cubics are not apolar) is not the same as the above correspondence.

† Another method of interpreting binary cubics geometrically is by means of points on a rational twisted cubic curve (see *Algebra of Invariants*, § 194). If the curve is given by $\xi = a_x^3$, $\eta = b_x^3$, $\zeta = c_x^3$, $\varpi = d_x^3$, then every binary cubic form defines three points of the curve, and therefore corresponds uniquely to a plane. If two cubic forms are apolar, the point of intersection of the osculating planes of the three points lying in one of the corresponding apolar planes is itself in the other.

12. *The Quartic.*—Let P_1, P_2, P_3, P_4 represent the form

$$a_x^4, \quad \equiv a_x\beta_x\gamma_x\delta_x;$$

and let Q be any other point (xy).

Then the polar 1-point of Q for the four points may be thus obtained: Find q_1, the harmonic conjugate of Q for any two of the points, say P_1 and P_2; and find q_2, the harmonic conjugate of Q for the other two, P_3 and P_4.

Then find Q', the harmonic conjugate of Q for q_1 and q_2; it will be the point required, and is therefore the same in whatever way we divide the quartic into pairs of points.

To construct the polar 2-points of Q, choose any three of the four points, say P_2, P_3, P_4; and let q' be the polar 1-point and q_1', q_2' the polar 2-points of Q for this triad.

Let $q_1'q_2'$ and $q'P_1$ meet in Q_1'.

By selecting other sets of three points, we obtain similarly the points Q_2', Q_3', Q_4'.

Then these points Q_1', Q_2', Q_3', Q_4' are collinear, and their line of collinearity meets the conic in the required points Q_1, Q_2.

For Q_1 and Q_2 are given by $a_x^2a_y^2$, which can be written

$$a_x(\beta_x\gamma_y\delta_y+\beta_y\gamma_x\delta_y+\beta_y\gamma_y\delta_x)+a_y(\beta_y\gamma_x\delta_x+\beta_x\gamma_y\delta_x+\beta_x\gamma_x\delta_y),$$

and in this expression the coefficient of a_x is the form giving q', and the coefficient of a_y is the form giving q_1' and q_2'.

Therefore Q_1 and Q_2 are apolar to the pair of points which is apolar both to q', P_1 and to q_1', q_2'. Hence the line Q_1Q_2 passes through Q_1'; similarly, it passes through Q_2', Q_3', and Q_4'.

The polar 2-points of Q having been obtained in this way, the polar 1-point, Q', may be constructed by finding the harmonic conjugate of Q with respect to Q_1 and Q_2.

The polar 3-points of Q are given by

$$(a_y\beta_x+a_x\beta_y)\gamma_x\delta_x+a_x\beta_x(\gamma_y\delta_x+\gamma_x\delta_y),$$

and therefore the triad which they form is apolar to any triad which is itself apolar both to P_3, P_4, q_1 and to P_1, P_2, q_2. Such a triad can easily be constructed.

By dividing the quartic into pairs of points in different ways, we can obtain five other triads apolar to the triad required. Any three of these six triads are sufficient to define the polar 3-points of Q; and the problem is therefore reduced to that of finding the unique triad apolar to three given triads, referred to at the end of § 11.

13. I have been unable to obtain a solution by means of a linear construction; the following is the simplest solution which I have been able to find.

If the points X, Y, Z form the triad apolar to three given ones, then Y, Z are harmonically conjugate with the polar 2-points of X for each of the three triads.

Therefore the three lines joining these three pairs of polar 2-points of X must be concurrent, and the poles of these lines must be collinear.

Now, as a point P on the conic moves, the pole of the line joining its polar 2-points for the first triad moves along the Hessian line of that triad. And thus the three poles referred to above move along three straight lines, viz., the Hessian lines of the three triads.

The position of any one of these three poles determines P and the other two poles uniquely.

We have thus a (1, 1, 1) correspondence of points on three straight lines, and we require the three sets of corresponding points which are collinear.

Let the three Hessian lines be called A, B, C.

Then the lines joining corresponding points on A and B envelop a conic which touches A and B. And the lines joining corresponding points on B and C envelop a conic which touches B and C.

These two conics have four common tangents, real or imaginary. One of these is the real line B; the other three are the straight lines required.

Now, by taking different positions of P on the conic, we may linearly construct as many sets of corresponding points on A, B, and C as we desire.

Thus we can construct as many tangents to the above two conics as we please.

Hence the lines required are the three unknown common tangents to two conics, each of which touches a given straight line and is defined by its tangents.

If we obtain a conic similarly from the correspondence on A and C, it will also be touched by the three straight lines just found. Its fourth common tangent with the first of the above conics will be A; that with the second will be B.

These three straight lines are actually the lines joining two at a time the points X, Y, Z. They therefore meet in pairs on the fundamental conic, and form the triangle defined by the required apolar triad.

14. Hence we can obtain (though not by a line construction) the polar 3-points of a point Q with respect to a quartic P_1, P_2, P_3, P_4.

The solution of the problem : *Given a quartic and a triad of points, to construct the single point which with the triad forms an apolar quartic*, can now be stated thus :

Find the polar 3-points of one of the given triad of points for the quartic.

Find the polar 2-points of another point of the triad for these three points.

Find the polar 1-point of the third point of the triad for these two points. This will be the point required, whatever be the order in which the points of the given triad are taken.

In order to give a complete linear solution, I go on to shew that the second of the above three steps can be linearly performed without actually constructing the polar 3-points at all.

Let Q be the point (xy) and R the point (xz).

The polar 3-points of Q are given by

$$(a_y\beta_x+a_x\beta_y)\gamma_x\delta_x+(\gamma_y\delta_x+\gamma_x\delta_y)\,a_x\beta_x,$$

and therefore, as has been seen already, the polar 3-points of Q form a triad apolar to every triad which is itself apolar both to P_3, P_4, q_1 and to P_1, P_2, q_2. The two points which with R form such a triad are those on the line joining the poles of the polar lines of R with respect to the two triangles $P_3P_4q_1$, $P_1P_2q_2$.

Thus, by the six methods of dividing the quartic into pairs of points [and also from the four similar cases obtained by writing the polar 3-points of Q in the form

$$a_y\beta_x\gamma_x\delta_x+a_x(\beta_y\gamma_x\delta_x+\beta_x\gamma_y\delta_x+\beta_x\gamma_x\delta_y)],$$

we get six [and four] pairs of points which with R form apolar triads to the polar 3-points of Q.

The lines joining these different pairs of points must all be concurrent in the pole of the line which joins the polar 2-points of R for the triad consisting of the polar 3-points of Q.

These polar 2-points are given by

$$a_x\beta_x(\gamma_y\delta_z+\gamma_z\delta_y)+\text{five similar terms},$$

or $a_x^2a_ya_z$; and thus these points (which I shall call the mixed polar 2-points of y and z) can be linearly constructed (since the pole of the line joining them can be found) without first constructing the polar 3-points of Q.

This completes the linear solution of the problem for the case of the quartic.

Although we cannot construct linearly the polar 3-points of Q, we can at least construct the H-points of the triad which they form. For taking two positions, R_1 and R_2, of the point R, the lines obtained from them as above will intersect in the pole of the required Hessian line.

15. *General n-ic form.*—Let the form considered be $a_x a_x^{n-1}$.

The polar 1-point and 2-points of any point y with respect to a_x^{n-1} are given by $a_x a_y^{n-2}$ and $a_x^2 a_y^{n-3}$.

Then it is easy to prove that the intersection of the line joining the point $a_x a_y^{n-2}$ to the point a_x with the line joining the points $a_x^2 a_y^{n-3}$ lies on the line joining the polar 2-points of the point y for the n-ic.

If therefore we are able to construct linearly the polar 1-point and 2-points of y for an $(n-1)$-ic, we can obtain n different points all lying on the line joining the polar 2-points of y for any n-ic.

The polar 1-point of y for the n-ic is then obtained by constructing the harmonic conjugate of y with respect to the polar 2-points already found.

And it is then possible to proceed to the $(n+1)$-ic.

Hence starting from the cubic and quartic, we can construct the polar 1-point and 2-points of a point y for any binary form by this means.

16. There remains to be considered the problem: *Given n points on the conic, and n—1 other points, to find the point required to make two apolar n-ics.*

Here, as in the last paragraph, the method is that of proceeding from the case of a form of lower degree to that of one of degree higher by unity.

The solutions for the cubic and the quartic have been given: it will therefore be a sufficient explanation of the method for the general form if the case of the quintic is considered.

Let the quintic be $\alpha_x \beta_x \gamma_x \delta_x \epsilon_x$, and let the four other given points be Q, R, S, and T, Q being the point y.

The polar 4-points of Q for the quintic are given by

$$(\alpha_y \beta_x + \alpha_x \beta_y)\, \gamma_x \delta_x \epsilon_x + \alpha_x \beta_x\, (\gamma_y \delta_x \epsilon_x + \gamma_x \delta_y \epsilon_x + \gamma_x \delta_x \epsilon_y). \qquad (2)$$

This form is here written as the sum of two parts, and each part separately gives a quartic of points which can be easily constructed.

Now, by the known case of $n = 4$, we can construct linearly the lines joining the mixed polar 2-points of R and S for each of these quartics.

But it is obvious that the line joining the mixed polar 2-points of Q, R, and S for the quintic passes through the point of intersection of these lines.

And, by dividing the expression (2) in different ways, we can construct many other points, all lying on the line joining the mixed polar 2-points of Q, R, and S.

The line itself can therefore be constructed, and the mixed polar 2-points of Q, R, and S obtained. Then the harmonic conjugate of T with respect to these two points is the point required.

The most convenient way of dividing the expression (2), which gives the polar $(n-1)$-points of Q, is into 3 terms and $n-3$ terms. The first part then gives an $(n-1)$-ic of points which can be constructed. The second part gives an $(n-1)$-ic of points which cannot themselves be constructed if n is > 6, but the mixed polar 2-points of R, S, ... for them can be obtained by means of a process similar to that just used, viz., this second part must be sub-divided into 3 terms and $n-6$ terms, in two or more different ways, and the same applies to the sub-part of $n-6$ terms, and so on. Also at each point the results obtained for the previously considered case of the $(n-1)$-ic form are assumed.

The process is thus exceedingly complicated, and is, as might have been expected, of little practical use for forms of degree higher than the sixth or seventh. It is, nevertheless, theoretically complete.

It is scarcely necessary to point out that, since the forms (2), &c., may be divided into parts and sub-parts in different ways, this theory leads to an immense number of geometrical propositions concerning the collinearity of points and the concurrence of lines.

ON THE MOTION OF A SWARM OF PARTICLES WHOSE CENTRE OF GRAVITY DESCRIBES AN ELLIPTIC ORBIT OF SMALL ECCENTRICITY ROUND THE SUN

By E. J. ROUTH.

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THE problem of the disintegration of comets has been studied by Schiaparelli, Bessel, Charlier, and Luc Picart. A short account of the work done by each has been given by Tisserand in chap. xvi. of the fourth volume of his *Traité de Mécanique céleste*, 1896. The comet is regarded as a homogeneous spherical swarm of loose particles whose centre of gravity is describing a circle about the Sun with a constant angular velocity n.

The case in which the centre of gravity of the swarm describes an ellipse of small eccentricity has also been noticed briefly by Tisserand in Art. 123 of the volume already quoted. He refers the motion to moving axes ξ, η which turn round the Sun, so that the axis of ξ passes through the centre of the swarm, while the axis of η is measured at right angles to ξ in the plane of motion. He arrives at the result that the swarm is stable if $\mu > 3n^2 + He^2$ where μ is the constant of attraction of the sphere of particles at any internal point. He remarks that: "Il faudrait obtenir la valeur de H; mais ce calcul, que j'avais entrepris, est assez compliqué."

The chief object of this paper is to examine in greater detail the case in which the centre of gravity describes an ellipse of small eccentricity. The axes chosen are nearly the same as those of Tisserand, and the equations of motion (though obtained in a different way) agree with his, after the correction of a misprint. The process of solution would indeed have been extremely complicated if it had not happened that the equation to determine the periods took the Lagrangian determinantal form, and thus became more manageable. The consequence is that the conditions of stability, though still complicated, are very much simplified, and can be completely exhibited.

The problem has subsequently been discussed by M. O. Callandreau in the course of a valuable paper published in the *Annales de l'Observatoire de Paris*, Vol. XXIII., 1902. He arrives at a result different from that

mentioned above. The coordinates of any particle are found to contain periodic terms whose coefficients are multiples of e^2t. These results cannot represent in this form the motion when e^2t becomes large.

Another object of this paper is to consider the internal motion when the swarm of particles has unequal thicknesses in different directions. To effect this we have supposed the boundary to be an ellipsoid whose axes are not necessarily nearly equal. It is found that there are cases of motion in which the conditions for uniform density and for stability of form are sensibly satisfied, and in which there are few, if any, collisions between the particles.

In the most general case in which the initial motion is unrestricted these two conditions are not satisfied. At the same time the collisions between the particles become more numerous and important. As the algebra also becomes rather complex, it has been considered sufficient to compare the changes in length of the two diameters in the directions of ξ and η.

There is a short discussion on some of the subjects of this paper in the author's treatise on *Dynamics of a Particle*, 1898; see Art. 414 and the note on p. 406. These portions are more fully developed here, and many additions have been made.

1. We suppose the Sun, A, to be fixed in space. Let B be the centre of the swarm, C any particle. Let r, θ be the polar coordinates of B referred to A, and ξ, η the coordinates of C referred to B as origin, the axis of ξ being the prolongation of AB, and η being taken positively in the direction of motion of B. Let M be the mass of the Sun, m that of the swarm, ρ its density. Let the form of the swarm be an ellipsoid whose principal diameters are directed along the axes of ξ, η, ζ. The component attractions of the swarm at any internal point C are then $\mu_1\xi$, $\mu_2\eta$, $\mu_3\zeta$ where μ_1, μ_2, μ_3 are known functions of the ratios of the bounding ellipsoid, and their sum is $4\pi\rho$.

2. Taking the ordinary equations of motion for moving axes, we have for the particle C

$$\left.\begin{aligned} \frac{d^2(r+\xi)}{dt^2}-(r+\xi)\left(\frac{d\theta}{dt}\right)^2-\frac{1}{\eta}\,\frac{d}{dt}\left(\eta^2\frac{d\theta}{dt}\right) &= -\frac{M(r+\xi)}{(AC)^3}-\mu_1\xi \\ \frac{d^2\eta}{dt^2}-\eta\left(\frac{d\theta}{dt}\right)^2+\frac{1}{r+\xi}\,\frac{d}{dt}\left\{(r+\xi)^2\frac{d\theta}{dt}\right\} &= -\frac{M\eta}{(AC)^3}-\mu_2\eta \\ \frac{d^2\zeta}{dt^2} &= -\frac{M\zeta}{(AC)^3}-\mu_3\zeta \end{aligned}\right\}, \qquad (1)$$

where $$(AC)^2 = (r+\xi)^2+\eta^2+\zeta^2,$$

and ultimately, when the squares of ξ, η, ζ are rejected, $AC = r+\xi$. These equations also apply to the point B, where $\xi = 0$, $\eta = 0$. Hence when we expand in powers of $\xi\eta\zeta$ the terms independent of $\xi\eta\zeta$ cancel out. We thus have

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2}-2\frac{d\eta}{dt}\frac{d\theta}{dt}-\eta\frac{d^2\theta}{dt^2}-\xi\left(\frac{d\theta}{dt}\right)^2 &= 2\,\frac{M\xi}{r^3}-\mu_1\xi \\ \frac{d^2\eta}{dt^2}+2\frac{d\xi}{dt}\frac{d\theta}{dt}+\xi\frac{d^2\theta}{dt^2}-\eta\left(\frac{d\theta}{dt}\right)^2 &= -\frac{M\eta}{r^3}-\mu_2\eta \\ \frac{d^2\zeta}{dt^2} &= -\frac{M\zeta}{r^3}-\mu_3\zeta \end{aligned}\right\}. \qquad (2)$$

3. If the centre of gravity of the swarm describe a circle about the Sun, we write $r = a$, $d\theta/dt = n$. The equations then become, since

$$\frac{M}{a^3} = n^2,$$

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2}-2n\frac{d\eta}{dt}+(\mu_1-3n^2)\,\xi &= 0 \\ \frac{d^2\eta}{dt^2}+2n\frac{d\xi}{dt}+\quad \mu_2\eta \quad &= 0 \\ \frac{d^2\zeta}{dt^2} \qquad\qquad +\ (\mu_3+n^2)\,\zeta &= 0 \end{aligned}\right\}. \qquad (3)$$

We now put

$$\xi = A\cos(pt+\epsilon), \qquad \eta = B\sin(pt+\epsilon), \qquad \zeta = C\sin(qt+\epsilon'),$$

and we then find

$$\left.\begin{aligned} \frac{A}{B} = \frac{p^2-\mu_2}{-2np} &= \frac{-2np}{p^2-(\mu_1-3n^2)} \\ \{p^2-(\mu_1-3n^2)\}\{p^2-\mu_2\} = 4n^2p^2, &\quad q^2 = n^2+\mu_3 \end{aligned}\right\}. \qquad (4)$$

In order that the particles of the swarm should keep together, it is necessary that the roots of this quadratic should be real and positive. Since μ_1 and μ_2 are positive by definition, it is sufficient that $\mu_1 > 3n^2$. The two positive quantities μ_1-3n^2 and μ_2 clearly separate the roots.

The paths of the particles when the system is describing either of the two principal modes of motion are similar ellipses. *With the smaller value of p^2* the ratio A/B is positive and *the direction of rotation is the same as that of the swarm round the Sun. With the greater value of p^2,*

the ratio A/B is negative and *the rotation is opposite to that of the swarm.*

4. If the semi-axes a, b, c of the ellipsoid are in descending order, the quantities μ_1, μ_2, μ_3 are in ascending order (author's *Statics*, Vol. II., Art. 216). Hence, if that axis of the ellipsoid which is directed along the axis of ξ is the least of the three, μ_1 is greater than either μ_2 or μ_3, and, since their sum is $4\pi\rho_1$, we see that $\mu_1 > 4\pi\rho_1/3$, that is, μ_1 is greater for an ellipsoid so placed than for a sphere made of the same materials. *The stability is therefore greater for such an ellipsoid than for a sphere.* In the same way, if the greatest axis is placed along the axis of ξ, *the stability is less than for the sphere.*

5. When the centre of the swarm describes an ellipse we write in equations (2)

$$\left.\begin{aligned} \theta &= nt + 2e\sin nt + \tfrac{5}{4}e^2 \sin 2nt, \qquad \frac{h}{r^2} = \frac{d\theta}{dt} \\ \left(\frac{a}{r}\right)^3 &= 1 + 3e\cos nt + \tfrac{3}{2}e^2 + \tfrac{9}{2}e^2\cos 2nt \end{aligned}\right\}. \tag{5}$$

We thus obtain

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} - 2n\frac{d\eta}{dt} + \alpha^2\xi &= X \\ \frac{d^2\eta}{dt^2} + 2n\frac{d\xi}{dt} + \beta^2\eta &= Y \\ \frac{d^2\zeta}{dt^2} \qquad\quad + q^2\zeta &= Z \end{aligned}\right\} \tag{6}$$

where $\alpha^2 = \mu_1 - n^2(3+5e^2)$, $\beta^2 = \mu_2 - \tfrac{1}{2}n^2e^2$, $q^2 = \mu_3 + n^2$,

$$X = (4e\cos nt + 5e^2\cos 2nt)\, n\frac{d\eta}{dt}$$
$$- (2e\sin nt + 5e^2 \sin 2nt)\, n^2\eta + (10e\cos nt + 16e^2\cos 2nt)\, n^2\xi,$$

$$Y = -(4e\cos nt + 5e^2\cos 2nt)\, n\frac{d\xi}{dt}$$
$$+ (2e\sin nt + 5e^2\sin 2nt)\, n^2\xi + (e\cos nt + \tfrac{5}{2}e^2\cos 2nt)\, n^2\eta,$$

$$Z = -n^2(3e\cos nt + \tfrac{3}{2}e^2 + \tfrac{9}{2}e^2\cos 2nt).$$

As a first approximation we reject the terms X, Y, Z, since they contain powers of e, and thus fall back on the solution already obtained in Art. 3.

6. To find a second approximation we substitute

$$\xi = A\cos(pt+\epsilon), \qquad \eta = B\sin(pt+\epsilon) \tag{7}$$

in the expressions X and Y. These then take the form

$$X = \Sigma E\cos\{(p\pm n)t+\epsilon\}, \qquad Y = \Sigma E'\sin\{(p\pm n)t+\epsilon\} \tag{8}$$

where E, E' are constants and p is either root of the quadratic (4). After substituting in the differential equations (6), we arrive at a solution of the form

$$\left.\begin{aligned}\xi &= A\cos pt+Ge\cos(p+n)t+G'e\cos(p-n)t\\ \eta &= B\sin pt+He\sin(p+n)t+H'e\sin(p-n)t\end{aligned}\right\} \tag{9}$$

where the constant ϵ has been omitted for brevity and A, B, G, H, G', H' are constants. There will be also corresponding terms depending on the other root of the quadratic (4).

To arrive at a third approximation we substitute the values (9) of ξ, η in the expressions X, Y and search for terms which will rise into importance in the process of solution. Now we observe that the junction of any term of ξ, η of the form $\frac{\cos}{\sin}(p\pm n)t$ with a term $\frac{\cos}{\sin}(nt)$ in the expressions for X, Y will produce terms of the form $\cos pt$ or $\sin pt$. These, when the equations (6) are solved, will lead to terms in ξ, η of the form $t\cos pt$ and $t\sin pt$. They will also contain the small factor e^2. We must therefore, in our third approximation, add to the terms given in (9) such terms as $Ke^2t\cos pt$ and $K'e^2t\sin pt$.

The terms which contain the factor e^2t finally become large. This does not necessarily prove that the swarm must break up. *It may merely show that the divergence of the actual motion from that represented by* (7) *or* (9) *becomes large.* It may be possible to modify the approximations (7) or (9) by including in them these large terms and thus make the new terms introduced at the higher approximations wholly periodic. We shall therefore start with the equations (9) as our first approximation, where p is now supposed to be so far arbitrary that we may include in the terms $A\cos pt$, $B\sin pt$ all other terms of the same form which make their appearance in the third approximation.

7. The equations of motion (6) are satisfied when powers of e above the square are neglected by writing

$$\left.\begin{aligned}\xi &= A\cos pt+Ge\cos(p+n)t+G'e\cos(p-n)t\\ &\qquad +Je^2\cos(p+2n)t+J'e^2\cos(p-2n)t\\ \eta &= B\sin pt+He\sin(p+n)t+H'e\sin(p-n)t\\ &\qquad +Ke^2\sin(p+2n)t+K'e^2\sin(p-2n)t\end{aligned}\right\}. \tag{10}$$

To find the coefficients, we substitute these values of ξ, η in the differential equations (6) and equate the coefficients of like terms on each side. Those depending on $(p+n)t$ give

$$\left.\begin{aligned}\{-(p+n)^2+a^2\}G-2n(p+n)H &= 5n^2A+(2np+n^2)B\\ -2n(p+n)G+\{-(p+n)^2+\beta^2\}H &= (2np+n^2)A+\tfrac{1}{2}n^2B\end{aligned}\right\}. \tag{11}$$

Those depending on $(p-n)t$ give

$$\left.\begin{aligned}\{-(p-n)^2+a^2\}G'-2n(p-n)H' &= 5n^2A+(2np-n^2)B\\ -2n(p-n)G'+\{-(p-n)^2+\beta^2\}H' &= (2np-n^2)A+\tfrac{1}{2}n^2B\end{aligned}\right\}. \tag{12}$$

Those depending on pt give

$$\left.\begin{aligned}&\{-p^2+a^2\}A-2npB\\ &\qquad = e^2\{5n^2G+(2np+n^2)H\}+e^2\{5n^2G'+(2np-n^2)H'\}\\ &-2npA+\{-p^2+\beta^2\}B\\ &\qquad = e^2\{(2np+n^2)G+\tfrac{1}{2}n^2H\}+e^2\{(2np-n^2)G'+\tfrac{1}{2}n^2H'\}\end{aligned}\right\}. \tag{13}$$

Those depending on $(p+2n)t$ give

$$\left.\begin{aligned}\{-(p+2n)^2+a^2\}J-2n(p+2n)K &= \tfrac{1}{2}\{16n^2A+5n(p-n)B\}\\ -2n(p+2n)J+\{-(p+2n)^2+\beta^2\}K &= \tfrac{1}{2}\{-5n(p-n)A+\tfrac{5}{2}n^2B\}\end{aligned}\right\}. \tag{14}$$

Those depending on $(p-2n)t$ give

$$\left.\begin{aligned}\{-(p-2n)^2+a^2\}J'-2n(p-2n)K' &= \tfrac{1}{2}\{16n^2A+5n(p+n)B\}\\ -2n(p-2n)J'+\{-(p-2n)^2+\beta^2\}K' &= \tfrac{1}{2}\{-5n(p+n)A+\tfrac{5}{2}n^2B\}\end{aligned}\right\}. \tag{15}$$

The terms depending on $(p\pm 2n)t$ do not give rise to any terms of the form $\cos pt$ or $\sin pt$, and are therefore unimportant for our present purpose.

We find the values of G, H, G', H' from (11) and (12) in terms of A and B and substitute in (13). We thus arrive at two linear equations from which the ratio B/A and the values of p may be found.

We notice that the solution will contain four arbitrary constants for each particle, viz., A, ϵ, and the two others corresponding to the second root of the quadratic (4).

8. Let the two linear equations just mentioned be

$$\left.\begin{aligned}\left\{p^2-a^2+e^2\left(\frac{L}{\Delta}+\frac{L'}{\Delta'}\right)\right\}A+\left\{2np+e^2\left(\frac{M}{\Delta}+\frac{M'}{\Delta'}\right)\right\}B &= 0\\ \left\{2np+e^2\left(\frac{N}{\Delta}+\frac{N'}{\Delta'}\right)\right\}A+\left\{p^2-\beta^2+e^2\left(\frac{P}{\Delta}+\frac{P'}{\Delta'}\right)\right\}B &= 0\end{aligned}\right\} \tag{16}$$

where L, M, N, P are derived from (11) and L', M', N', P' from (12).

It will be found, after a short calculation, that

$$M = N, \qquad M' = N'. \tag{17}$$

After some algebraical substitutions, which it does not seem necessary to reproduce, we find

$$\left.\begin{aligned} L &= -n^2(p+n)(p+3n)\{(2p-n)^2+n^2\}+n^2(2p+n)^2\alpha^2+25n^4\beta^2 \\ M &= -n^3(p+n)(p+3n)(3p-\tfrac{1}{2}n)+n^3(2p+n)(\tfrac{1}{2}\alpha^2+5\beta^2) \\ P &= -n^2(p+n)(4p^3+8np^2+\tfrac{5}{4}n^2p-\tfrac{3}{4}n^3)+\tfrac{1}{4}n^4\alpha^2+(2p+n)^2n^2\beta^2 \end{aligned}\right\}; \tag{18}$$

and, if Δ is the eliminant of the left-hand side of (11),

$$\Delta = (p+n)^2(p-n)(p+3n)-(p+n)^2(\alpha^2+\beta^2)+\alpha^2\beta^2.$$

We also find

$$\left.\begin{aligned} L' &= -n^2(p-n)(p-3n)\{(2p+n)^2+n^2\}+n^2(2p-n)^2\alpha^2+25n^4\beta^2 \\ M' &= -n^3(p-n)(p-3n)(3p+\tfrac{1}{2}n)+n^3(2p-n)(\tfrac{1}{2}\alpha^2+5\beta^2) \\ P' &= -n^2(p-n)(4p^3-8np^2+\tfrac{5}{4}n^2p+\tfrac{3}{4}n^3)+\tfrac{1}{4}n^4\alpha^2+(2p-n)^2n^2\beta^2 \end{aligned}\right\}; \tag{19}$$

and, if Δ' is the eliminant of the left-hand side of (12),

$$\Delta' = (p-n)^2(p+n)(p-3n)-(p-n)^2(\alpha^2+\beta^2)+\alpha^2\beta^2.$$

We see by comparing these expressions that L', P', Δ' may be derived from L, P, Δ by changing the sign of either n or p; while M' may be derived from M by changing the sign of either n or p and also changing the sign of the whole expression.

It follows that L/Δ, L'/Δ' take the forms

$$\frac{L}{\Delta} = \frac{f_1(p^2)+pf_2(p^2)}{F_1(p^2)+pF_2(p^2)}, \qquad \frac{L'}{\Delta'} = \frac{f_1(p^2)-pf_2(p^2)}{F_1(p^2)-pF_2(p^2)}$$

where the highest power of p in any numerator or denominator is p^4. Thus $L/\Delta+L'/\Delta'$ and, in the same way, $P/\Delta+P'/\Delta'$ are functions which contain only even powers of p, while $M/\Delta+M'/\Delta'$ is a fraction which contains only odd powers of p. For the sake of brevity we write

$$5n^2+\frac{L}{\Delta}+\frac{L'}{\Delta'} = \phi(p^2), \qquad \tfrac{1}{2}n^2+\frac{P}{\Delta}+\frac{P'}{\Delta'} = \psi(p^2), \qquad \frac{M}{\Delta}+\frac{M'}{\Delta'} = p\chi(p^2).$$

When written at length, we have

$$\phi(p^2) = 5n^2+\frac{2n^2e^2R}{\Delta\Delta'},$$

$$R = -2(p^2-n^2)^2(p^2-9n^2)(2p^2+n^2)+\{8p^6-29n^2p^4+18n^4p^2+3n^6\}\alpha^2$$
$$+\{4p^6+3n^2p^4+62n^4p^2-69n^6\}\beta^2-\{8p^4+26n^2p^2+32n^4\}\alpha^2\beta^2$$
$$-\{4p^4-3n^2p^2+n^4\}\alpha^4-\{25n^2p^2+25n^4\}\beta^4+\{4p^2+n^2\}\alpha^4\beta^2+25n^2\alpha^2\beta^4$$

where $$\alpha^2 = \mu_1 - n^2(3+5e^2), \quad \beta^2 = \mu_2 - \tfrac{1}{2}n^2e^2.$$

9. The equation to find p is obtained by eliminating the ratio B/A from (16). Since $M = N$, $M' = N'$, this becomes

$$\left[p^2-\alpha^2+e^2\left(\frac{L}{\Delta}+\frac{L'}{\Delta'}\right)\right]\left[p^2-\beta^2+e^2\left(\frac{P}{\Delta}+\frac{P'}{\Delta'}\right)\right]$$
$$-\left[2np+e^2\left(\frac{M}{\Delta}+\frac{M'}{\Delta'}\right)\right]^2 = 0. \quad (20)$$

We substitute for α^2, β^2 from (6) and introduce the functional symbols, and this reduces to

$$[p^2-\mu'+e^2\phi(p^2)][p^2-\mu_2+e^2\psi(p^2)]-p^2[2n+e^2\chi(p^2)]^2 = 0 \quad (21)$$

where $\mu' = \mu_1 - 3n^2$.

We notice that this equation is not altered by changing the sign of p, so that the roots run in pairs of equal quantities, but with opposite signs. In other words, the equation when expanded will contain only even powers of p. The values of ξ, η will therefore take the form $\Sigma A(e^{pt} \pm e^{-pt})$ where p is real or imaginary.

10. When the small quantity e^2 is zero the equation (21) takes the form

$$D = (p^2-\mu')(p^2-\mu_2)-(2np)^2 = 0, \quad (22)$$

which, of course, is identical with (4), since $\mu' = \mu_1 - 3n^2$. Let the roots of (22) be p_1^2, p_2^2. Since the left-hand side of (22) takes the signs $+$, $-$, $-$, $+$ when $p^2 = \infty$, μ', μ_2, $-\infty$ respectively and is equal to $\mu'\mu_2$ when $p^2 = 0$, we see (1) that, if μ' and μ_2 are positive, both p_1^2 and p_2^2 are positive and one is greater than both μ', μ_2 and the other less; (2) if either μ' or μ_2 is negative, one of the roots p_1^2, p_2^2 is negative and the other positive; (3) if both μ', μ_2 are negative, then both p_1^2, p_2^2 are negative. It follows from this reasoning, or because D is a Lagrangian determinant, that μ', μ_2 separate (that is, lie between) the values of p_1^2, p_2^2.

If we apply the same reasoning to (21), we arrive at somewhat similar results. Since the two values of $L/\Delta + L'/\Delta'$ are finite and equal, and also those of $P/\Delta + P'/\Delta'$, when $p^2 = +\infty$ and $p^2 = -\infty$ it follows that, if q^2 be any real root of either of the equations

$$p^2-\mu'+e^2\phi(p^2) = 0, \qquad p^2-\mu_2+e^2\psi(p^2) = 0,$$

the equation (21) has at least one root greater and one root less than q^2.

11. We do not require for our present purpose all the roots of (21): we only want those two roots which differ from p_1^2, p_2^2 [the roots of (22)] by quantities of the order e^2. To approximate to these, we substitute in succession for p^2 in the small terms the two values p_1^2, p_2^2 given by (22). The equation (21) is then resolved into the two quadratics

$$[p^2-\mu'+e^2\phi(p_1^2)][p^2-\mu_2+e^2\psi(p_1^2)]-p^2[2n+e^2\chi(p_1^2)]^2=0, \quad (23)$$

$$[p^2-\mu'+e^2\phi(p_2^2)][p^2-\mu_2+e^2\psi(p_2^2)]-p^2[2n+e^2\chi(p_2^2)]^2=0, \quad (24)$$

where, as before, $\mu'=\mu_1-3n^2$. In each quadratic we take the root which differs from p_1^2, p_2^2 respectively by quantities of the order e^2. The swarm is stable to a first approximation if both these roots are positive.

We see at once, by looking at the equation (23), that a *stable oscillation* with a period $2\pi/p_1$ *in a circular orbit remains stable in an elliptic orbit if both* $\mu'-e^2\phi(p_1^2)$, $\mu_2-e^2\psi(p_1^2)$ *are positive, but becomes unstable in the elliptic orbit if both these quantities are negative.* In fact, by Art. 10, both the roots of the quadratic (23) are real and positive in the first case, and both are real and negative in the second case. We are here concerned with only one of the roots, but in each case the conclusion is the same whichever is taken. If, however, the quantities $\mu'-e^2\phi(p_1^2)$ and $\mu_2-e^2\psi(p_1^2)$ have different signs, it is necessary to determine which of the two roots is nearest in value to p_1^2.

To examine the case when one of these quantities is negative and the other positive, we write the equation (23) in the abbreviated form

$$(p^2-R_1^2)(p^2-S_1^2)-p^2(2n_1)^2=0. \quad (23)'$$

The oscillation being stable with a circular orbit, μ' and μ_2 are positive; but, if μ' is of the order e^2, we may have R_1^2 negative and of the same order. If r_1^2, s_1^2 are the roots of the quadratic (23)′, we have

$$(r_1^2-s_1^2)^2=(R_1^2-S_1^2)^2+8n_1^2(R_1^2+S_1^2)+16n_1^4.$$

Hence $r_1^2-s_1^2$ cannot be of the order e^2 unless S_1^2 and n_1^2 (as well as R_1^2) are also of that order, and therefore all the three quantities μ', μ_2, n^2 must be of that order. But in forming the equations of motion some at least of these quantities are regarded as being of the order of the principal terms of motion, while the eccentricity e of the orbit is so small that such terms as $e^2\phi(p^2)$ are much smaller.

With this limit to the magnitude of the eccentricity e, we see that $r_1^2-s_1^2$ cannot be so small as to be of the order e^2. There is therefore only one root of the quadratic (23) which differs from p_1^2 by quantities of the order e^2, and it is this root which must be positive if the swarm is to be stable.

In the same way, when μ' in equation (22) is of the order e^2, $p_1^2-p_2^2$ cannot be of that order if e^2 is of an order higher than μ_2 or n^2.

It follows that *when p_1^2 is greater than both the positive quantities* μ', μ_2 the corresponding root of (23) is greater than the positive quantity S_1^2, and *the oscillation remains stable when the circular orbit becomes slightly elliptical. If p^2, though positive, is less than both* μ', μ_2, the corresponding root of (23) is less than the negative quantity R_1^2 and *the oscillation becomes unstable when the orbit is elliptical.*

12. *The other Roots of the Fundamental Equation.*—The equation (21) has other roots besides those two which we have been considering. Though they are not important for our present purpose, it may be interesting to examine briefly what their meaning may be.

The following numerical identities may be established by substitution from Art. 8, or otherwise:—

$$LP-M^2 = (16p^4+32p^3n+4p^2n^2-12pn^3+\tfrac{9}{4}n^4)\Delta,$$
$$L'P'-M'^2 = (16p^4-32p^3n+4p^2n^2+12pn^3+\tfrac{9}{4}n^4)\Delta'.$$

These we shall write briefly

$$LP-M^2 = K\Delta, \qquad L'P'-M'^2 = K'\Delta'.$$

We observe also that, by Art. 8, each of the functions $\phi(p^2)$, ... is the sum of fractions whose numerators are L, M, P, ..., and whose denominators are Δ or Δ'. After making these substitutions, the equation (21) takes the form

$$(p^2-\alpha^2)(p^2-\beta^2)-4n^2p^2 = e^2\frac{X}{\Delta\Delta'}+e^4\frac{Y}{\Delta\Delta'}$$

where α^2, β^2 are as defined in Art. 5, and

$$X = [(p^2-\beta^2)L'+(p^2-\alpha^2)P'-4npM']\Delta$$
$$+[(p^2-\beta^2)L+(p^2-\alpha^2)P-4npM]\Delta',$$
$$Y = K\Delta'+K'\Delta+PL'+P'L-2MM'.$$

Since L, L', P, P', Δ, Δ' are quartic functions of p, the functions X, Y are integral rational functions of p, X being of the 10th degree and Y of the 8th degree.

By multiplying the equation (21) by $\Delta\Delta'$, we see that it is of the 12th degree. If also e^2 were zero, the twelve roots would be given by equating what is then the left-hand side to zero. Thus, when e^2 is not zero, but small, four of the values of p differ slightly from the roots of

$$D = (p^2-\mu')(p^2-\mu_2)-4n^2p^2 = 0,$$

while the eight remaining roots differ but slightly from those of $\Delta\Delta' = 0$; that is, differ by quantities of the order e^2. Now

$$\Delta = \{(p+n)^2-\mu'\}\{(p+n)^2-\mu_2\}-4n^2(p+n)^2,$$

$$\Delta' = \{(p-n)^2-\mu'\}\{(p-n)^2-\mu_2\}-4n^2(p-n)^2.$$

If, then, p_1^2, p_2^2 are taken, as before, to represent the roots of the quadratic $D=0$, the roots of $\Delta=0$ are $(p_1-n)^2$, $(p_2-n)^2$, while those of Δ' are $(p_1+n)^2$, $(p_2+n)^2$.

These correspond to the terms $Ge \frac{\cos}{\sin}(p\pm n)t$, ..., which occur in the expansions for ξ, η in Art. 7. Since the roots which originate in $\Delta=0$ or $\Delta'=0$ only make their appearance in the equation (21), when e is not zero, we infer that the coefficients Ge, $G'e$ in the expressions for ξ, η must contain some power of e as a factor. We therefore again arrive at the result that the preliminary assumption (Art. 7) should be of the form

$$\xi = A\cos pt + Ge\cos(p+n)t + G'e\cos(p-n)t,$$

with a similar expression for η, where p has two values which differ slightly from p_1, p_2. The third terms in Art. 7 only make their appearance in equation (21) when that equation is carried to a higher degree of approximation.

13. *Principal Oscillations.*—Let us compare the motions of the system according as the diameter of the swarm in the direction of the axis of ξ is longer or shorter than that along the axis of η. To simplify matters we shall suppose that the centre of the swarm describes a circle round the Sun.

If the system is describing a principal oscillation or motion, we have, by Art. 3,

$$\xi = A\cos(pt+\epsilon), \qquad \eta = B\sin(pt+\epsilon),$$

$$\left(\frac{A}{B}\right)^2 = \frac{p^2-\mu_2}{p^2-(\mu_1-3n^2)} = 1-\frac{\mu_2-(\mu_1-3n^2)}{p^2-(\mu_1-3n^2)}.$$

Thus the projections of the particles on the plane $\xi\eta$ describe similar ellipses in the same periodic time, but the ellipses of one principal motion are not similar to those of the other. It also appears that for the same values of μ_2, μ_1-3n^2 the ratio $(A/B)^2$ is greater than unity in one principal motion and less than unity in the other. Thus the major axes of the elliptic paths which correspond to one value of p^2 are perpendicular to those which correspond to the other.

We know by the theory of the attractions of ellipsoids that μ_1 is

greater or less than μ_2 according as the major axis of the attracting ellipsoid is directed along the axis of η or ξ (see Art. 4).

Let us first suppose that the diameter of the swarm directed along η is longer than that along ξ; then $\mu_2 < \mu_1$. The principal oscillation in which the major axes of the paths also lie along η is distinguished by having the larger or smaller value of p^2 according as μ_2 is greater or less than $\mu_1 - 3n^2$. In either case the elliptic paths resemble that of the boundary of the swarm, and should be similar to it. The particles then keep within the boundary and the shape of the boundary is not altered. In the other oscillation the major axes of the paths are directed along the minor axis of the swarm, and the particles not close to the centre will have portions of their elliptic paths outside the swarm. If all the particles are performing this principal oscillation, the effect would be that the major axes of the swarm would follow the particles, and the swarm would take up a new position relatively to the axes of ξ, η.

Similar remarks apply if the longer diameter of the swarm is directed along the axis of ξ. In this case $\mu_2 > \mu_1$, and the principal oscillation in which the major axes of the paths are also along the axis of ξ has the smaller value of p^2. In the other oscillation the major axis is directed along the axis of η.

In all these cases one principal motion has an element of instability which does not exist in the other: one oscillation is always more stable than the other. It is only when the particles on the boundary move along the boundary that the swarm has assumed a stable form.

14. *Does the swarm remain homogeneous when performing a principal oscillation* in which the elliptic paths of the external particles lie on the boundary of the swarm?

We see from the values of ξ, η (Art. 13) that the particles initially on any ordinate parallel to the axis of ζ remain on the same ordinate while it moves round the axis of ζ with an acceleration tending to that axis in a periodic time $2\pi/p$.

Construct, then, a cylinder with a triangular section having for its edges three of these moving ordinates, and let them cut the plane of $\xi\eta$ in P, Q, R. It is not difficult to prove that the area of the triangle PQR is constant. If the area PQR is indefinitely small, the length of the column, being bounded by the external ellipsoid, is also constant.

Since the same particles continue to fill the column during the motion, the average density of each elementary column will be constant throughout the motion. Now the particles also move up and down the column in a periodic time $2\pi/q$ (Art. 3) and resume their original positions at

these intervals. If the column is originally homogeneous, it will be strictly homogeneous at these intervals. If we also suppose that there are equal large numbers of particles running each way, the column will continue to be sensibly homogeneous throughout its length.

It follows from the argument that the particles in one column cannot impinge on those in another. The only collisions which can occur are those between particles in the same column. When two perfectly elastic particles of equal mass impinge directly on each other they merely exchange velocities, and such collisions do not affect the density. Even if these conditions are not satisfied, yet, if the particles are of very small size, some at least in the column will pass each other in their up and down motion without touching. Thus the number of collisions will be very much less than if the directions of motion of the particles were unrestricted.*

15. It has already been remarked that two assumptions were made in forming the equations of motion in Art. 3. First, the swarm was treated as sensibly homogeneous; and, secondly, the boundary of the swarm was supposed to remain sensibly of the same form, so that μ_1, μ_2, μ_3 could be treated as constants. We have now shown that *this hypothesis is satisfied whenever the swarm is performing a principal oscillation.*

The initial conditions that the system should describe either oscillation strictly are that

$$\frac{d\xi}{dt} = -\frac{A\omega}{B}\eta, \qquad \frac{d\eta}{dt} = \frac{B\omega}{A}\xi,$$

where ω has either of the values p_1, p_2, and B/A has the value given in Art. 3. Though these initial conditions may not be exactly satisfied, yet they may be nearly satisfied in some swarms. In these cases we may regard the swarm as sensibly stable.

16. *Other Positions of the Ellipsoid.*—We have hitherto supposed that the external boundary of the swarm is approximately an ellipsoid having its axes directed along the axes ξ, η, ζ. Let us now consider how far this limitation is necessary.

Let one axis of the swarm, as before, be perpendicular to the plane of motion ξ, η. Let the other two make constant angles α, $\alpha+\frac{1}{2}\pi$ with the moving axis of ξ. Let the attractions of the swarm at any internal point be represented (as before) by $\mu_1 x$, $\mu_2 y$, $\mu_3 z$ directed along the principal

* One of the referees has suggested that when the swarm is very thin in the direction of the axis of ζ the motion of the particles up and down the columns becomes insignificant. This disc, being the limiting case of the ellipsoid when the axis of c is zero, is heterogeneous, but will retain the same law of density throughout the motion of the swarm.

diameters. The equations of Art. 3 become

$$\frac{d^2\xi}{dt^2}-2n\frac{d\eta}{dt}+a\xi+b\eta=0,$$

$$\frac{d^2\eta}{dt^2}+2n\frac{d\xi}{dt}+b\xi+c\eta=0,$$

where
$$a=\mu_1\cos^2\alpha+\mu_2\sin^2\alpha-3n^2,$$
$$b=(\mu_1-\mu_2)\sin\alpha\cos\alpha,$$
$$c=\mu_1\sin^2\alpha+\mu_2\cos^2\alpha.$$

After solving these equations we find for a principal oscillation

$$\xi=A\left\{2np\cos(pt+\epsilon)-b\sin(pt+\epsilon)\right\},$$
$$\eta=A(-p^2+a)\sin(pt+\epsilon),$$

where p^2 has either of the two values given by the quadratic

$$(p^2-a)(p^2-c)=(2np)^2+b^2,$$

and A, ϵ are two of the four arbitrary constants included in this solution.

The values of p^2 are positive if a and c are positive and are separated by the values $p^2=a$, $p^2=c$. Since c is necessarily positive, the test of stability to a first approximation is that a should be positive; see Art. 3.

To find the directions of rotation of the particles for a principal motion, we notice that $2n\,(d\eta/dt)=(-p^2+a)\,\xi$ when $\eta=0$. If, then, p^2 has its lesser value, $(d\eta/dt)$ is positive or negative according as ξ is positive or negative. It follows that the direction of rotation of the particles of the swarm in either principal oscillation is the same as, or opposite to, that of the swarm round the Sun according as p^2 has its lesser or greater value. The particles therefore move opposite ways in the two oscillations.

The projections of the paths of the particles on the plane of $\xi\eta$ are the ellipses

$$\left\{\xi(-p^2+a)+b\eta\right\}^2+\left\{2np\eta\right\}^2=K^2,$$

where K is any constant. For either principal oscillation these conics are all similar to each other, but are not the same for both the oscillations.

If β be the angle either principal diameter makes with the axis of ξ, we find after using the quadratic

$$\tan 2\beta=\frac{2b}{a-c}=\frac{\tan 2\alpha\,(\mu_1-\mu_2)}{\mu_1-\mu_2-3n^2\sec 2\alpha}.$$

It may also be shown that the major axes of the elliptic paths in the two oscillations are at right angles to each other.

It follows from these results that the principal diameters of the elliptic paths do not coincide with those of the swarm, except when these lie along the axes of ξ and η. As before, this fact will introduce an element of instability into both the principal oscillations, though that instability is greater for one oscillation than for the other. In either oscillation the paths most remote from the centre of the swarm are occupied only in part by the particles, and as these move on the protuberant parts of the swarm must follow them. For proper stability the particles on the boundary of the swarm must be moving along the boundary.

17. We are thus led to inquire what motions are possible in which the axes of the swarm rotate round the axis of ζ, relatively to those of ξ, η, with an angular velocity p.

We now put $\alpha = pt+\epsilon$ in the equations of motion (Art. 16). These become

$$\frac{d^2\xi}{dt^2} - 2n\frac{d\eta}{dt} + a^2\xi + \gamma\xi\cos 2(pt+\epsilon) + \gamma\eta\sin 2(pt+\epsilon),$$

$$\frac{d^2\eta}{dt^2} + 2n\frac{d\xi}{dt} + \beta^2\eta + \gamma\xi\sin 2(pt+\epsilon) - \gamma\eta\cos 2(pt+\epsilon),$$

where $a^2 = \frac{1}{2}(\mu_1+\mu_2) - 3n^2, \quad \beta^2 = \frac{1}{2}(\mu_1+\mu_2), \quad \gamma = \frac{1}{2}(\mu_1-\mu_2).$

An easy solution may be found when the swarm is so nearly spherical that the square of γ can be neglected; we have then

$$\xi = A\cos(pt+\epsilon) + G\gamma\cos 3(pt+\epsilon),$$

$$\eta = B\sin(pt+\epsilon) + H\gamma\sin 3(pt+\epsilon),$$

where p^2 has either of the two values given by

$$(-p^2+a^2+\tfrac{1}{2}\gamma)(-p^2+\beta^2+\tfrac{1}{2}\gamma) = (-2np+\tfrac{1}{2}\gamma)^2.$$

The conditions of stability are that $a^2+\frac{1}{2}\gamma$ and $\beta^2+\frac{1}{2}\gamma$ are both positive.

We also find that

$$G\Delta = -\tfrac{1}{2}(A-B)(-9p^2+\beta^2+6np),$$

$$H\Delta = -\tfrac{1}{2}(A-B)(-9p^2+a^2+6np),$$

where $$\Delta = (-9p^2+a^2)(-9p^2+\beta^2) - (6np)^2.$$

If we roughly trace the path, we find that it is of an oval form having the longest and shortest diameters directed along the axes of ξ and η.

These cannot therefore coincide with the longest and shortest diameters of a swarm which has a rotatory motion relatively to those axes.

18. *Compound Oscillations.*—When both the principal oscillations are described simultaneously the motion becomes very complicated. The motion of any one particle is then given by

$$\xi = A_1 \cos(p_1 t+\epsilon_1)+A_2 \cos(p_2 t+\epsilon_2),$$

$$\eta = A_1 m_1 \sin(p_1 t+\epsilon_1)+A_2 m_2 \sin(p_2 t+\epsilon_2),$$

which contain the four undetermined constants $A_1, A_2, \epsilon_1, \epsilon_2$ (Art. 3). Here

$$m_1 = \frac{B_1}{A_1} = \frac{-2np_1}{p_1^2-\mu_2} = \frac{p_1^2-\mu'}{-2np_1};$$

therefore
$$m_1^2-1 = \frac{\mu_2-\mu'}{p_1^2-\mu_2};$$

$$m_2 = \frac{B_2}{A_2} = \frac{-2np_2}{p_2^2-\mu_2} = \frac{p_2^2-\mu'}{-2np_2};$$

therefore
$$m_2^2-1 = \frac{\mu_2-\mu'}{p_2^2-\mu_2};$$

and
$$\mu' = \mu_1-3n^2.$$

By using the properties of the fundamental quadratic (Art. 3) we find that

$$m_1 m_2 = -(\mu'/\mu_2)^{\frac{1}{2}},$$

which, we notice, is independent of the motion of the swarm, and depends only on its form and structure. If we represent the lesser and greater roots of the fundamental quadratic by p_1^2 and p_2^2, then m_1 is positive and m_2 negative (Art. 3).

We may picture the motion by constructing the two ellipses which are adapted to the two principal oscillations (Art. 13). Let a point P_1 describe an ellipse whose principal diameters are directed along the axes of ξ, η and have lengths A_1 and $A_1 m_1$ respectively. Let a point P_2 describe a second ellipse whose centre is P_1 and whose axes are parallel to those of the first ellipse and have lengths A_2 and $-A_2 m_2$. Let these points move with velocities equal to $p_1 D_1$ and $p_2 D_2$ respectively, where D_1, D_2 are the semi-conjugates of P_1, P_2. Let the direction of rotation of P_1 be from the positive side of the axis of ξ to that of η, while that of P_2 is in the opposite direction. Then the motion of P_2 represents that of any one particle of the swarm. The magnitudes of these ellipses are fixed for any one particle, but may be different for different particles.

We see that, since m_1 is positive and m_2 negative, $(p_1 t+\epsilon_1)$ and $-(p_2 t+\epsilon_2)$ may be regarded as the eccentric angles of P_1, P_2, each measured from the axis of ξ in the positive direction.

When both the principal oscillations are simultaneously described $\mu_2-\mu'$ must be the same for both; and therefore one of the two ratios m_1^2, m_2^2 is greater than unity and the other less. Thus one of the ellipses has its major axis along ξ and the other along η.* Since p_1^2 has been chosen to represent the lesser root of the quadratic, $m_1^2 > 1$ or < 1 according as $\mu_2 >$ or $< \mu'$.

19. To fix our ideas, let us suppose that, initially, all the particles of the swarm have an angular velocity Ω round the axis of ζ. To abbreviate the algebra, let us restrict our comparison of the subsequent motions of the particles to any two Q_1, Q_2 initially situated on the axes of ξ, η at distances ξ_0, η_0 from the centre of the swarm. These particles have initial velocities $\Omega\xi_0$ and $-\Omega\eta_0$ respectively parallel to the axes of η, ξ. We then find for the motion of Q_1

$$A_1 = \frac{(-p_2 m_2+\Omega)\,\xi_0}{p_1 m_1-p_2 m_2}, \qquad A_2 = \frac{(p_1 m_1-\Omega)\,\xi_0}{p_1 m_1-p_2 m_2} \qquad (\epsilon_1 = 0,\ \epsilon_2 = 0),$$

and for the motion of Q_2

$$A_1 m_1 = \frac{(-p_2/m_2+\Omega)\,\eta_0}{p_1/m_1-p_2/m_2}, \qquad -A_2 m_2 = \frac{(p_1/m_1-\Omega)\,\eta_0}{p_1/m_1-p_2/m_2}$$
$$(\epsilon_1 = \tfrac{1}{2}\pi,\ \epsilon_2 = -\tfrac{1}{2}\pi).$$

It follows that for all particles *initially on the axis of* ξ the ratios ξ/ξ_0 and η/ξ_0 are the same; so that *the paths of these particles are similar and similarly situated.* In the same way, for all particles *initially on the axis of* η, ξ/η_0 and η/η_0 are the same, and *their paths also are similar to each other.* We also infer that these paths will be the principal paths (Art. 13), if in the first case $\Omega = p_1 m_1$ or $p_2 m_2$, and in the second case $\Omega = p_1/m_1$ or p_2/m_2.

20. To simplify the argument further, let T be a time such that $p_1 T$ is an even multiple of π and $p_2 T$ an odd multiple of π. Strictly, these cannot always be made exact multiples, but they may be nearly so, and then what follows will repeat itself for many intervals each equal to T.

At the end of each interval the ellipses will be very nearly in their

* There is an error in the seventh line of p. 407 of the author's treatise on *Dynamics of a Particle*. The theorem from the author's *Statics* is wrongly quoted, and the result as to the direction of the major axis should be in accordance with that given in Art. 414.

original positions, and each particle will be on the same axis of ξ or η as before, but at the end of the diameter opposite to that which it occupied at the beginning of the interval. Let ξ_1, η_1 be the distances of Q_1, Q_2 from the origin in these new positions. Then, by writing

$$p_1 t = 2i\pi, \qquad p_2 t = (2j+1)\pi,$$

we find
$$\frac{\xi_1}{\xi_0} = \frac{2\Omega - p_1 m_1 - p_2 m_2}{p_1 m_1 - p_2 m_2} = \frac{4n(\Omega+n)+(\mu_2-\mu')}{p_2^2-p_1^2},$$

$$\frac{\eta_1}{\eta_0} = \frac{2\Omega - p_1/m_1 - p_2/m_2}{p_1/m_1 - p_2/m_2} = \frac{4n(\Omega+n)-(\mu_2-\mu')}{p_2^2-p_1^2},$$

the second value in each case being derived from the first by using some properties of the fundamental quadratic (Art. 3).

We notice that when Ω has either of the values $p_1 m_1$, $p_2 m_2$ the ratio ξ_1/ξ_0 is numerically equal to unity, for values of Ω between these limits the ratio ξ_1/ξ_0 is numerically less than unity, beyond these limits ξ_1/ξ_0 is numerically greater than unity. In the same way, η_1/η_0 is less or greater than unity according as Ω lies between or outside the limits p_1/m_1, p_2/m_2.

The initial lengths ξ_0, η_0 of the semi-diameters of the swarm become ξ_1, η_1 after the first interval T, and then repeat these values at the times $2T$, $3T$, ... Thus *the form of the swarm is continually changing, enlarging and contracting in these two directions alternately, according to the rules just stated.* With the same value of Ω for all the particles the system could not describe a principal oscillation.

21. The relation between the changes of the whole swarm and those of the two diameters placed along the axes of ξ, η will be made clear if we briefly consider the motion of any point Q which is *not initially* on the axes of ξ or η. If x_0, y_0 are the coordinates of its projection on the plane $\xi\eta$ in its initial position, the coordinates at any time t are (as in Arts. 18, 19)

$$\xi = x_0 C_2 \cos p_1 t + y_0 D_2 \sin p_1 t - x_0 C_1 \cos p_2 t - y_0 D_1 \sin p_2 t,$$

$$\eta = m_1 x_0 C_2 \sin p_1 t - m_1 y_0 D_2 \cos p_1 t - m_2 x_0 C_1 \sin p_2 t + m_2 y_0 D_1 \cos p_2 t,$$

where

$$C_2 = \frac{\Omega - p_2 m_2}{p_1 m_1 - p_2 m_2}, \quad C_1 = \frac{\Omega - p_1 m_1}{p_1 m_1 - p_2 m_2}, \quad m_1 D_2 = \frac{-\Omega + p_2/m_2}{p_1/m_1 - p_2/m_2}, \quad m_2 D_1 = \frac{-\Omega + p_1/m_1}{p_1/m_1 - p_2/m_2}.$$

If x_1, y_1 are the coordinates of the projection of Q on the plane $\xi\eta$ at the time $t = T$, we find, by writing $p_1 t = 2i\pi$, $p_2 t = (2j+1)\pi$,

$$x_1 = (C_1 + C_2)\, x_0, \qquad y_1 = -(m_2 D_1 + m_1 D_2)\, y_0.$$

It follows by Art. 20 that $x_1/x_0 = \xi_1/\xi_0$ and $y_1/y_0 = \eta_1/\eta_0$.

Let us describe two subsidiary ellipses, one having two arbitrary lengths ξ_0, η_0 (Art. 20) for the semi-axes, and the other having ξ_1, η_1 for the corresponding lengths. *Then, if the projection of any particle Q lie initially on the first ellipse, its projection at the time T will lie on the second.*

22. Let us now return to the consideration of particles initially placed on the axes of ξ, η. Since the equality

$$\frac{\eta_1}{\xi_1} = \frac{4n(\Omega+n)-(\mu_2-\mu')}{4n(\Omega+n)+(\mu_2-\mu')}\,\frac{\eta_0}{\xi_0}$$

holds for all particles in the axes ξ, η, *the two diameters of the swarm in these directions are unequally changed.* Suppose, for example, that the swarm is initially spherical, so that we may put $\xi_0 = \eta_0$. The swarm will assume a shape resembling an ellipsoid after an interval T. The major axis of the section $\xi\eta$ will be placed along the axis of ξ if η_1/ξ_1 is numerically less than unity, that is, if its square is less than unity. This requires that $(\Omega+n)(\mu_2-\mu')$ should be a positive quantity, where $\mu' = \mu_1 - 3n^2$ (Art. 18).

The result is that a spherical swarm in which all the particles have initially a common angular velocity Ω about the axis of ζ cannot remain spherical; the section $\xi\eta$ must at intervals assume an oval form. The dynamics of the problem requires that the major axis of this section should lie along the axis of ξ or that of η, according as $(\Omega+n)(\mu_2-\mu')$ is positive or negative. The theory of attraction requires that the major axis should lie along the axis of ξ or that of η, according as $(\mu_2-\mu_1)$ is positive or negative.

23. Here we notice that, if the swarm starts as a spherical mass in which $\mu_1 = \mu_2$, and presently assumes an ellipsoidal form, we cannot strictly assume that μ_1 and μ_2 are constants during the motion. The changes of density and form should both be allowed for. When this is done the equations of motion in Art. 3 will be greatly modified. A proper investigation would require us to write for μ_1, μ_2, μ_3 their known values as definite integrals in terms of the axes of the instantaneous ellipsoid.

We may, however, obtain a roughly approximate solution by equating μ_1, μ_2, μ_3 to the means of their values for the swarm in its initial shape when its semi-axes are ξ_0, η_0, ζ_0 and its shape after the first interval T when its semi-axes are ξ_1, η_1, ζ_0. Treating these as constants, we may use the results already obtained to deduce some conclusions which are important, though not all we could desire.

There may also be impacts between the particles. If, however, the particles are very small and not too numerous, they may be able to pass close to each other without touching. It may then happen that the number of impacts will not be so great as to affect the general character of the motion.

With these limitations, we deduce from Art. 22 that, if $\Omega+n$ is positive, the dynamical and statical conditions require that either the greater diameter should lie along the axis of ξ or that, if it lie on the axis of η, the difference of the diameters should be sufficiently great to make $\mu_1-\mu_2 > 3n^2$.

If $\Omega+n$ is negative, the longer diameter cannot lie along the axis of ξ; for this would require $\mu_2-\mu'$ and $\mu_1-\mu_2$ to be both negative. Since $\mu' = \mu_1-3n^2$, this is impossible. The longer diameter could lie along the axis of η, if $\mu_2-\mu'$ and $\mu_1-\mu_2$ are both positive, and this requires that the difference in lengths of the two diameters should be sufficiently small to make $\mu_1-\mu_3 > 3n^2$.

[Here Ω is the initial angular velocity of the swarm about an axis drawn through the centre of gravity perpendicular to the plane of motion, n the angular velocity of the centre of gravity about the Sun, and μ_1, μ_2, μ_3 are the coefficients of the component attractions inside the ellipsoid; see Arts. 1, 3, 19.]

ON LINEAR DIFFERENTIAL EQUATIONS OF RANK UNITY

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THE present paper is concerned with linear ordinary differential equations with rational coefficients. If such an equation be

$$y^{(n)}+p_1 y^{(n-1)}+\dots+p_n y = 0,$$

and p_r has a pole of order ϖ_r at infinity (ϖ_r being negative if p_r is zero at infinity), and if p is the least integer greater than the greatest of the quantities ϖ_r/r ($r = 1, \dots, n$), the equation may be written

$$y^{(n)}+P_1 x^p y^{n-1}+\dots+P_n x^{np} y = 0$$

where P_r is finite and developable in powers of $1/x$ near $x = \infty$.

If $p = -1$, the integrals are regular in the neighbourhood of $x = \infty$, and the nature of their singularity is determined.

If $p \geqslant 0$, expansions of the integrals in the neighbourhood of $x = \infty$ are possible in the form of normal and subnormal series, but these expansions are, in general, divergent, and give little information about their nature.

For $p = 0$ an explicit solution is known in the form of Laplace's definite integral; but for $p > 0$ this, too, fails. An extension of this definite integral solution is here obtained in the case of $p = 1$, which appears to be capable of extension to greater values of p, though the analysis would be cumbrous.

The form of solution suggested by the known normal series is the following:—

$$\iint e^{tx+\frac{1}{2}ux^2} U\, du\, dt$$

where u is a function of t and u, and an appropriate contour is to be determined for each integration.

If this expression be substituted for y in the given equation, and the double integral obtained as the left-hand member be transformed by integration by parts, a partial differential equation is obtained which takes the place of the ordinary differential equation known as the Laplace transformation for $p = 0$.

The complete integration of this equation is not required, but particular integrals are obtained which lead to the divergent normal series already known to exist.

The equation considered will be

$$P(y) \equiv \phi_\lambda(x)\,y^{(n)}+\phi_{\lambda+1}(x)\,y^{(n-1)}+\ldots+\phi_{\lambda+n}(x)\,y = 0$$

where the affix of each coefficient denotes its degree in x, each being supposed a polynomial.

Let λ be an even integer—(if not, the equation may be multiplied throughout by x)—and let $\lambda+2n = 2m$. Then, if

$$y = \iint e^{tx+\frac{1}{2}ux^2}\,U\,du\,dt,$$

the contours being independent of x,

$$y' = \iint e^{tx+\frac{1}{2}ux^2}(t+ux)\,U\,du\,dt,$$

$$y'' = \iint e^{tx+\frac{1}{2}ux^2}\{(t+ux)^2+u\}\,U\,du\,dt,$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

In general,
$$y^{(r)} = \iint e^{tx+\frac{1}{2}ux^2}\,\omega_r U\,du\,dt$$

where the quantities ω_r are given by

$$\omega_r = \frac{\partial\omega_{r-1}}{\partial x}+\omega_{r-1}\,\omega_1, \qquad \omega_1 = t+ux.$$

It is important in the sequel to notice that

(i.) ω_r is a polynomial of degree r in u and t combined, and of degree r in x alone;

(ii.) that the *two* highest powers of x in ω_r arise only from the term $(t+ux)^r$ in ω_r.

Multiply the expression $P(y)$ by x^n, and then substitute the values of $y^{(r)}$ $(r = 1, \ldots, n)$.

$$x^n P(y) = \iint e^{tx+\frac{1}{2}ux^2}(\phi_\lambda x^n\,\omega_n+\phi_{\lambda+1}x^n\,\omega_{n-1}+\ldots+x^n\,\phi_{\lambda+n})\,U\,du\,dt$$

$$= \iint e^{tx+\frac{1}{2}ux^2}\,\Pi\,(t,\,u,\,x)\,U\,du\,dt.$$

Π is now a polynomial of degree $\lambda+2n$ in x and of degree n in t and u combined.

If ϕ_{sr} denotes the coefficient of x^s in ϕ_r, and π_λ the coefficient of x^λ in Π,

$$\pi_{\lambda+2n} = u^n \phi_{\lambda\lambda} + u^{n-1} \phi_{\lambda+1, \lambda+1} + \dots + \phi_{\lambda+n, \lambda+n};$$

$$\pi_{\lambda+2n-1} = \{nu^{n-1} \phi_{\lambda\lambda} + (n-1) u^{n-2} \phi_{\lambda+1, \lambda+1} + \dots\}$$
$$+ \{\phi_{\lambda-1, \lambda} u^n + \dots + \phi_{\lambda+n-1, \lambda+n}\}$$
$$= t \frac{\partial \pi_{\lambda+n}}{\partial u} + \psi_{\lambda+n-1}.$$

Similarly, $$\pi^{\lambda+2n-2} = t^2 \frac{\partial^2 \pi_{\lambda+n}}{\partial u^2} + t\psi_{\lambda+n-2} + \chi_{\lambda+n-2},$$

and so on.

Now

$$\iint e^{tx+\frac{1}{2}ux^2} U x^r \, du \, dt = \int e^{tx} dt \left[2e^{\frac{1}{2}ux^2} x^{r-2} U\right]_u - 2 \iint e^{tx+\frac{1}{2}ux^2} x^{r-2} \frac{\partial U}{\partial u} du \, dt$$

where the square brackets in the first integral on the right-hand side denote that the difference of the values of the contained function at the extremities of the contour is to be the subject of integration with respect to t.

In the same way a single integration with respect to t gives

$$\iint e^{tx+\frac{1}{2}ux^2} U x^r \, du \, dt = \int e^{\frac{1}{2}ux^2} du \left[e^{tx} U x^{r-1}\right]_t - \iint e^{tx+\frac{1}{2}ux^2} x^{r-1} \frac{\partial U}{\partial t} du \, dt.$$

By a repeated application of these two equations the expression $x^n P(y)$ is reducible to the form

$$\iint e^{tx+\frac{1}{2}ux^2} f(U, u, t) \, du \, dt + \bar{P}$$

where $\bar{P}$ represents an aggregate of semi-integrated terms, and $f(U, u, t)$ contains derivates of U with respect to u and t. If, then, U be chosen to be a solution of the partial differential equation $f(U, u, t) = 0$, and the contours of the integrations can be assigned so that $\bar{P}$ vanishes identically, the integral, if existent, will be a true solution of the equation $P(y) = 0$.

The highest power of x in $\Pi(t, u, x)$ being $x^{\lambda+2n}$, by integrating m times with respect to u, the corresponding term in $f(U, u, t)$ is

$$(-2)^m \frac{\partial^m}{\partial u^m} \{U \pi_{\lambda+2n}\}.$$

Similarly, the term arising from the term involving $x^{\lambda+2n-1}$ in Π is

$$(-1)^m 2^{m-1} \frac{\partial^m}{\partial u^{m-1} \partial t} \{U \pi_{\lambda+2n-1}\}.$$

All the remaining terms may be integrated after the same manner, no derivatives of more than the $(m-1)$-th order in respect to u occurring; but there is a certain amount of arbitrariness in dealing with the later terms which requires some explanation. The equation $f(U, u, t) = 0$ is, however, in any case of the form

$$\pi_{\lambda+2n} \frac{\partial^m U}{\partial u^m} = \Sigma a_{rs} \frac{\partial^{r+s} U}{\partial u^r \partial t^s} \qquad (r = 0, 1, \ldots, m-1),$$

and the coefficients a_{rs} are polynomials in u and t. It appears, therefore, that in the neighbourhood of any pair of values $u = \alpha$, $t = \beta$ a function can be found satisfying this equation, and developable in a double power series in u and t, but that the region of convergence is always limited by the points at which $\pi_{\lambda+2n}$ vanishes.

As in Laplace's solution, however, it is the special integrals related to these singular points which lead to the required result.

Let the roots of the equation $\pi_{\lambda+2n} = 0$ be assumed all different. Let a be a root of this equation, and let $v = u - a$. Further, if

$$\left(\frac{\partial \pi_{\lambda+2n}}{\partial u}\right)_{v=0} = \beta \qquad \text{and} \qquad (\psi_{\lambda+2n-1})_{v=0} = \gamma,$$

let $s = t + \gamma/\beta$. Then the term independent of v in $\pi_{\lambda+2n-1}$ is simply βs. We have then

$$P(y) = e^{\frac{1}{2}ax^2 - (\gamma x/\beta)} \iint e^{\frac{1}{2}vx^2 + sx}\, U \Pi'(s, v, x)\, dv\, ds,$$

Π' being the result of changing the variables in Π.

A part of the last integral arising from a term $x^k v^r s^{r'}$ in Π' will be reduced to a double integral independent of x by means of r' or $r'+1$ integrations in respect to s, together with $\frac{1}{2}(k-r')$ or $\frac{1}{2}(k-r'-1)$ in respect to v, according as $k-r'$ is an even or odd integer. This is always possible, since, the equation having been multiplied by n, the least value of k is n, and the greatest value of r' is m.

The result of this is that the function within the double integral becomes of the form

$$\left[(-2)^m \frac{\partial^m}{\partial v^m} \{\pi_{\lambda+2n}(v)\, U\} - (-2)^{m-1} \frac{\partial^m}{\partial v^{m-1} \partial s} \{U\beta s + v(s, v)_{1, n-1}\}\right]$$

$$+ (-2)^{m-2} \frac{\partial^m}{\partial v^{m-2} \partial s^2} \{U(s, v)_{2, n}\}$$

$$- (-2)^{m-3} \frac{\partial^m}{\partial v^{m-3} \partial s^3} \{U(s, v)_{3, n}\}$$

$$\cdots \qquad \cdots \qquad \cdots \qquad \cdots$$

where $(s, v)_{r, n}$ denotes a polynomial of degree r in s and n in v.

On expansion this equation is of the form

$$2^m \pi_{\lambda+2n} \frac{\partial^m U}{\partial v^m} + 2^{m-1}\{\beta s + v(s, v)_{1, n-1}\} \frac{\partial^m U}{\partial v^{m-1}\partial s}$$

$$+ \sum_{r=0}^{n+1} \sum_{k=2}^{m} \{s^r(1, v)_n + s^{r-1}(1, v)_n\} \frac{\partial^{m-k+r} U}{\partial v^{m-k}\partial s^r} = 0.$$

In connection with the singular point $v = 0$, a particular solution of this equation is sought in the form

$$v^\rho \{f_0(t) + v f_1(t) + v^2 f_2(t) + \dots\}.$$

Substituting in the equation, and equating to zero the coefficients of the successive powers of v, beginning from the lowest, the following equations are obtained :—

(1) $$\rho(\rho-1)\dots(\rho-m+1)\, 2^m \beta f_0 + \rho \dots (\rho-m+2)\, 2^{m-1} \beta s \frac{\partial f_0}{\partial s}$$
$$+\rho(\rho-1)\dots(\rho-m+2)\, a_0 f_0 = 0,$$

(2) $$(\rho+1)\rho\dots(\rho-m+2)\, 2^m \beta f_1 + (\rho+1)\dots(\rho-m+3)\, 2^{m-1}\beta . s \frac{\partial f_1}{\partial s}$$
$$+(\rho+1)\dots(\rho-2m+3)\, a^0 f_1$$
$$= (a_1 s^2 + b_1 s)\frac{\partial^2 f_0}{\partial s^2} + (c_0 s + d_0)\frac{\partial f_0}{\partial s} + e_0 f_0,$$

... ...

The first equation gives

$$s \frac{\partial f_0}{\partial s} + \left\{2(\rho-m+1) + \frac{a_0}{\beta}\right\} f_0 = 0$$

or $$f_0 = A s^{-\sigma}$$

where $$\sigma = 2(\rho-m+1) + \frac{a_0}{\beta}.$$

Substituting this in the right-hand side of equation (2), this equation becomes

$$(\rho+1)\dots(\rho-m+3)\, 2^{m-1}\beta \left\{s \frac{\partial f_1}{\partial s} + (\sigma+2) f_1\right\} = B s^{-\sigma} + C s^{-\sigma-1},$$

giving $$s^{\sigma+2} f_1 = \frac{Bs^2}{2} + Cs$$

or $$f_1 = s^{-\sigma}\left(\frac{C}{s} + \frac{B}{2}\right).$$

The next equation gives f_2 of the form

$$s^{-\sigma}\left(1, \frac{1}{s}\right)_2,$$

and, in general, $$f_r = s^{-\sigma}\left(1, \frac{1}{s}\right)_r.$$

Thus a formal solution of the equation is obtained in the form

$$v^\rho s^{-\sigma}\left\{1+v\left(1, \frac{1}{s}\right)_1+v^2\left(1, \frac{1}{s}\right)_2+\ldots+v^r\left(1, \frac{1}{s}\right)_r+\ldots\right\}$$

where $$\sigma = 2(\rho-m+1)+\frac{a_0}{\beta}.$$

Of this solution it will now be proved that it is absolutely convergent within a finite circle about $v=0$ and for all values of s greater in modulus than any given finite quantity s_0.

For the proof of this we may, without loss of generality, assume that $k=0$ and $\rho=0$. If otherwise, let $U = u^\rho s^{-\sigma} U'$; then U' will satisfy a precisely similar equation.

If the variable $t = \dfrac{1}{s}$ be used in place of s, the equation becomes

$$2^m \pi^{\lambda+2n}\frac{\partial^m U}{\partial v^m} - 2^{m-1}\{\beta t + vt(t, v)_{1, n-1}\}\frac{\partial^m U}{\partial v^{m-1}\partial t}$$
$$+\Sigma\Sigma\{t^r(1, v)_n + t^{r+1}(1, v)_n\}\frac{\partial^{m-k+r} U}{\partial v^{m+r}\partial t^r} = 0,$$

and the solution becomes

$$1+v(1, t)_1+v^2(1, t)_2+\ldots.$$

Consider now the equations giving the successive polynomials $(1, t)_r$. They are of the form

$$\text{(A)} \qquad -t\frac{\partial f_r}{\partial t} 2rf_r = \sum_h \sum_{k=0}^{r-1} \frac{\partial^h f_k}{\partial t^h}(a_{hk}t^h + b_{hk}t^{h+1})$$

and give f_r as a polynomial of degree r in t.

Compare with these the equations obtained by making the coefficients all positive.

$$\text{(B)} \qquad -t\frac{\partial \phi_r}{\partial t} + 2r\phi_r = \Sigma\sum_{k=0}^{r-1}\frac{\partial^h \phi^k}{\partial t^h}(|a_{hk}|t^h + |b_{hk}|t^{h+1}).$$

Then ϕ_r will be a polynomial in t whose coefficients are the moduli of those of f_r, and therefore $$|f_r^{(h)}(t)| < \phi_r^{(h)}|t|$$

for all values of t. Again, if $t > 1$, $t^h < t^{h+1}$.

Now compare with equations (B) the equations

$$\text{(C)} \qquad -t\frac{\partial \psi_r}{\partial t} + 2r\psi_r = \Sigma\Sigma\frac{\partial^h \psi_k}{\partial t^h}(|a_{hk}|+|b_{hk}|)t^{h+1}.$$

Then, if t is a positive quantity greater than 1, and if

$$\left|\frac{\partial^h\psi_k}{\partial t^h}\right| \geqslant \left|\frac{\partial^h\phi_k}{\partial t^h}\right| \quad (k = 0, \ldots, r-1),$$

the equations shew that $\quad |t^{-2r}\psi_r| \geqslant |t^{-2r}\phi_r|,$

and therefore that $\quad |\psi_r| \geqslant |\phi_r|.$

But also the solution of equations (C), taking

$$\psi_0 = \phi_0 = f_0 = 1,$$

gives $\psi_r = c_r t^r$, c_r being positive, while $\phi_r = (1, t)_r$, the coefficients being positive. Hence, if $\psi_k \geqslant \phi_k$ for all values of $t > 1$,

$$\frac{\partial^h\psi_k}{\partial t^h} \geqslant \frac{\partial^h\phi^k}{\partial t^k};$$

so that we deduce successively that

$$\frac{\partial^h\psi_1}{\partial t^h} \geqslant \frac{\partial^h\phi_1}{\partial t^h}, \quad \frac{\partial^h\psi_2}{\partial t^h} \geqslant \frac{\partial^h\phi_2}{\partial t^h}, \quad \ldots,$$

and therefore that $\quad \dfrac{\partial^h\psi_r}{\partial t^h} \geqslant \left|\dfrac{\partial^h f_r}{\partial t^h}\right|.$

Now the expression $\quad V = 1 + v\psi_1 + v^2\psi_2 + \ldots$

becomes $\quad 1 + c_1 vt + c_2 v^2 t^2 + \ldots,$

and satisfies an equation of the form

$$\frac{\partial^m V}{\partial v^m} - 2t\frac{\partial^m V}{\partial v^{m-1}\partial t} = \sum_h \sum_k t^{h+1} P_{hk}(v) \frac{\partial^{m-k+h}}{\partial v^{m-k}\partial t^h} V$$

where P_{hk} is a power series converging up to the root of $\pi_{\lambda+2n}$ nearest to the origin, say for $|v| < a$.

If the variable v be replaced by $w = vt$, a partial differential equation is obtained which is satisfied by a function of w alone and whose coefficients are developable in power series converging for $|w| < at$. Thus V satisfies an ordinary differential equation in w whose coefficients converge if $|w| < at$ and are finite for $w = \infty$, and hence the series V converges provided $|w| < at$, *i.e.*, provided $|v| < a$; and this is proved only under the assumption that $t > 1$. Hence the series

$$U = 1 + vf_1(t) + v_2 f_2(t) + \ldots$$

converges absolutely if $|v| < a$ and $|t|$ is finite and greater than unity.

Clearly, moreover, since $f_r(t)$ is a polynomial in t, the restriction that $|t| > 1$ may be removed. Hence U converges for all finite values of t,

that is, for all non-zero values of s, including $s = \infty$, and for $|v| < a$. Also, if the real part of sz is negative, $\underset{s=\infty}{\text{Lt}} |e^{sz} U| = 0$ if $|v| < a$.

We have now to consider the continuation of this function to values of v outside the circle $|v| = a$, in order that we may know its behaviour for large values of v.

We may, without loss of generality, suppose that no root of the equation $\pi_{\lambda+n}(v) = 0$ is a real positive quantity. Let c be a real positive quantity less than a, and let $v = v_1+c$. Then the expansion of U in powers of v_1 is

$$U_c+v_1\left(\frac{\partial U}{\partial v}\right)_c+\tfrac{1}{2}v_1^2\left(\frac{\partial^2 U}{\partial v^2}\right)_c+\dots,$$

while that of V is $$V_c+v_1\left(\frac{\partial V}{\partial v}\right)_c+\tfrac{1}{2}v_1^2\left(\frac{\partial^2 V}{\partial v^2}\right)_c+\dots,$$

and it has been shown that, for any value of $t > 1$, the sum of the absolute values of the terms of the power series in t which constitutes the expression $\dfrac{\partial^{h+k} U}{\partial v^h \partial t^k}$ is less than the sum of the absolute values of the terms of $\dfrac{\partial^{h+k} V}{\partial v^h \partial t^k}$ which are powers of t with positive coefficients.

Now, since V satisfies a certain ordinary differential equation, its development in powers of v_1 will converge within a finite circle extending to the nearest root of $\pi_{\lambda+n} = 0$, and so, too, will its differential coefficients with respect to v and t.

It follows, therefore, that the expansion

$$U_c+v_1\left(\frac{\partial U}{\partial v}\right)_c+\dots$$

will converge within the same circle, and likewise its differential coefficients.

Again, taking a point $v_1 = d$ within this circle, V and its differential coefficients are at this point power series in t with positive coefficients, and the above reasoning applies again; so that, in general, we find that U is developable within a finite region containing the real axis, and that it and its differential coefficients are at all points on the real axis less in modulus than V or its corresponding differential coefficient.

Again, since, as remarked above, the differential equation in w $(= vt)$ satisfied by V has its coefficients finite for $w = \infty$, a finite quantity λ exists such that

$$\underset{w=\infty}{\text{Lt}} \{e^{-\lambda w} V\} = 0;$$

or, if t is other than zero, $\underset{v=\infty}{\text{Lt}} \{e^{-\lambda vt} V\} = 0.$

Under the same conditions, also,

$$\underset{v=\infty}{\mathrm{Lt}} \{e^{-\lambda vt}|U|\} = 0,$$

and therefore, if t be restricted to be less than a finite quantity τ, a finite quantity z can always be found so that

$$\underset{v=\infty}{\mathrm{Lt}} \{e^{-\frac{1}{2}vz^2}|U|\} = 0$$

for all values of t greater than unity.

But, since U is absolutely convergent and contains only positive powers of t, the restriction that $t < 1$ may be removed, and the same is true for all values of $|t| < \tau$, including zero.

Similarly, also, it is found that

$$\underset{v=\infty}{\mathrm{Lt}} \left\{e^{-\frac{1}{2}vz^2}\left|\frac{\partial^{h+k} U}{\partial v^h \partial t^k}\right|\right\} = 0.$$

If z^2 is not real and positive, the same result will be obtained if we proceed to infinity in a direction such that the real part of vz^2 is negative, provided that direction does not pass through a root of $\pi_{\lambda+2n} = 0$.

Reverting to the original variable s, therefore, we see that, provided $|s|$ exceeds any definite finite quantity, s_0 (that is, τ^{-1}),

$$\underset{v=\infty}{\mathrm{Lt}} \left[e^{-\frac{1}{2}vx^2}\frac{\partial^{h+k} U}{\partial v^h \partial s^k}\right] = 0,$$

and also that

$$\underset{s=\infty}{\mathrm{Lt}} \left[e^{-sx}\frac{\partial^{h+k} U}{\partial v^h \partial s^k}\right] = 0,$$

it being assumed in both cases that $|x|$ is sufficiently large and that v and s are made infinite with such arguments that the real parts of vx^2 and sx are both positive. This is always possible. Hence it is always possible to assign contours of integration, for v and s respectively, consisting of loops encircling the points $v = 0$ and $s = 0$ and extending to infinity in appropriate directions, so that $\iint e^{sx+\frac{1}{2}vx^2}\, U dv\, ds$ exists, and so that when this is substituted in the equation, and the integration by parts indicated on p. 376 is carried out, the integrated part vanishes at the infinite limit. The double integral is therefore an integral of the given equation.

Recalling the form of U, the integral is

$$e^{\frac{1}{2}\alpha x^2-\beta x/\gamma}\iint e^{sx+\frac{1}{2}vx^2} v^{\rho} s^{-k}\left\{1+v\left(1,\ \frac{1}{s}\right)_1+v^2\left(1,\ \frac{1}{s}\right)_2+\dots\right\} dv\, ds.$$

Consider in particular

$$\iint e^{sx+\frac{1}{2}vx^2} v^\rho s^{-k} \left\{\frac{v^h}{s^l}\right\} dv\, ds.$$

With the contours found this is equal to

$$x^{(l+k-1)-2(\rho+h+1)} \int_{-\infty}^{(0+)} \int_{-\infty}^{(0+)} e^{\xi+\eta} \xi^{\rho+h} \eta^{-(k+l)} d\xi\, d\eta$$

$$= \pm x^{l-2h+k-2\rho-3} \Gamma(\rho+h+1) \Gamma(1-k-l)$$

$$= \pm x^{2m-a_0/\beta-1-(2h-l)} \Gamma(\rho+h+1) \Gamma(1-k-l).$$

Now the coefficient of $v^{\rho+h}$ in U is $s^{-k}\left(1, \frac{1}{s}\right)_h$; so that l varies from 0 to h.

Hence the expansion of the integral is of the form

$$e^{\frac{1}{2}ax^2-\beta x/\gamma} x^{2m-a_0/\beta-1} P\left(\frac{1}{x}\right),$$

$P\left(\frac{1}{x}\right)$ being a series of ascending positive integral powers of $\frac{1}{x}$, which will in general be divergent, but which, exactly as in Poincaré's development of equations of rank 1, can be shewn to represent the value of the integral asymptotically.

If the expansion is to be a valid representation of the function, it must either converge or terminate. The latter will be the case if, and only if, the function u terminates.

The necessary modification of the foregoing in the case of equal roots of the equation $\pi_{\lambda+2n} = 0$ will not be carried out here. There appears to be no essential difficulty, but the analysis is cumbrous and does not illustrate any new fact of importance.

The extension to equations of rank p greater than 1 consists in the adoption of a trial solution in the form

$$\iiint \ldots e^{tx+\frac{1}{2}ux^2+\ldots+vx^{p+1}/(p+1)} U\, dv \ldots du\, dt,$$

U being a function of $t, u, \ldots, v$; and, again, the difficulty is in the writing rather than the reasoning.

Particular integrals of linear partial differential equations with rational coefficients can be investigated by means of a similar analysis in the form

$$\iint \ldots e^{tx+uy+\ldots+vz} U\, dv \ldots du\, dt,$$

there being many points of similarity with the foregoing analysis (*v.* Picard, *Rend. del Circ. Mat. di Palermo*, v.).

ON TWO CUBIC CURVES IN TRIANGULAR RELATION

By F. MORLEY.

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Two plane cubic curves—one of points and one of lines—can be so related that there is an infinity of triangles with points on the former and lines on the latter. I shall say that two such cubics are *in triangular relation.*

This problem, which is investigated here, stands, in a way, between Poncelet's problem, with its various developments, and Klein's tetrahedra whose points and planes are on a Kummer quartic.

The case when the two cubics are in apolarity occurs in the memoir by Prof. White, "On Twisted Cubic Curves that have a Directrix," *Trans. Amer. Math. Soc.*, Vol. IV., p. 136.

The problem belongs to the theory of two connexes, but I shall first give its genesis from a single cubic. The identification of the pairs of cubics arrived at in the two ways is made in § 8.

1. Given a pencil of conics and a point cubic ϕ^3, each conic determines six points on the cubic. Consider the locus f of the joins of these points, and in particular its class.

Let x and y be two of the six points and let them be on a line with a point p. Then x and y, being both on a line of a pencil and a conic of a pencil, are in a Cremona involution. Writing

$$y_i = p_i + \lambda x_i \quad (i = 1, 2, 3)$$

and (the base points of the conic being 1, ± 1, ± 1)

$$| y_i^2, \; x_i^2, \; 1 | = 0,$$

we get at once cubic expressions for y_i in terms of x_i, whence, when x is is on the cubic ϕ, y is on a curve of order 9, ψ^9.

But the locus of the double points of the involution is a cubic, its intersections with ϕ are also on ψ; there are then $9 \times 3 - 9$ or 18 other intersections, and, as these pair off in the involution, there are nine lines on p which cut out, from the cubic ϕ, pairs of points; or the class of f is 9.

If, however, a base-point b of the pencil of conics is on the cubic, the line pb counts twice among the nine lines and the class reduces to 7.

Thus, when n base-points are not on the cubic, the number of free intersections is $6-(4-n)$ or $n+2$, and the locus of their joins is a curve of class $9-2\,(4-n)$ or $2n+1$.

The case $n=0$ is familiar (see Salmon, *Higher Plane Curves*, § 154 of the Third Edition).

The case $n=1$ is the one under consideration.

2. Such a pair of cubics presents itself in the theory of two connexes* of class and order 1. With a single connex is bound up the principal pencil

$$k_1\,(ax)(a\xi)+k_0(x\xi) = 0,$$

where $(x\xi)=0$ is identity, or rather the incidence condition.

There are three singular connexes in the pencil; their singular points s and singular lines σ fit into its singular triangle (or, from the standpoint of collineations, the fixed triangle).

With two connexes is bound up the "principal net"

$$k_1(ax)(a\xi)+k_2\,(\beta x)(b\xi)+k_0(x\xi) = 0.$$

The singular connexes of the net have their singular points s on the cubic

$$\phi \equiv (ax)(\beta x)\,|\,abx\,| = 0,$$

and their singular lines σ on the cubic

$$f \equiv (a\xi)(b\xi)\,|\,ab\xi\,| = 0.$$

These cubics are then in triangular relation. They are in one-to-one correspondence.

3. If we regard k_0, k_1, k_2 as coordinates of a point k in the plane, then for singular connexes k is on an auxiliary cubic; to the point I whose coordinates are 1, 0, 0 corresponds identity, and to any line on this point corresponds a principal pencil of connexes; so that to the intersections of the line with the auxiliary cubic correspond three singular connexes whose singular points and lines fit into a triangle. Thus the one-to-one series of triangles is put into correspondence with a pencil of lines, and can accordingly be numbered. To the tangents from I to the auxiliary cubic correspond triangles with two coincident lines, and therefore two coincident points; these may be called *parabolic* triangles, the triangle of three real distinct non-collinear points being hyperbolic. Each such parabolic triangle affords a common point and a common line of ϕ and f,

* This theory is best set out in Clebsch's lectures (see Vol. III., chap. ii., of the French translation). Here (ax) is written in place of Clebsch's symbol a_x for a row product.

paired off in this way: the line of f on a common point, and the point of ϕ on a common line, are incident.

This accounts for six common points or lines; there are also six contacts, as will be seen later.

4. The points and lines common to a connex $(ax)(a\xi)$ and to the identical connex $(x\xi)$ form a "principal coincidence." To each point is assigned a line on it. Calling incident point and line an element, we have ∞^2 elements. The connex, referred to its singular triangle, being

$$(\mu x\xi) = 0,$$

the line ξ associated with a point x is the join of x and

$$(\mu x\xi) = 0.$$

Hence $$\xi_1 = (\mu_2 - \mu_3)\, x_2 x_3, \qquad \text{or} \qquad x_\iota \xi_\iota = \lambda_\iota,$$

where

(1) $$\lambda_1 + \lambda_2 + \lambda_3 = 0.$$

A principal coincidence is determined by its singular triangle and a double ratio—the double ratio made on ξ by x and the points on the three singular lines, and equally made on x by ξ and the lines on the singular points.

The pair of cubics arising from a principal net are formed by the elements common to two principal coincidences.*

The pair of cubics is fully determined by two singular triangles and an element; but, as the two triangles may be picked from ∞^1 singular triangles, and the element from ∞^1 elements, the coordinates of the pair are $12+3-2-1$ or 12.

Thus, *a pair of cubics in triangular relation has twelve coordinates; on every point x of ϕ are three lines of f; two of these belong to a triangle, and the third makes with x and any selected triangle a double ratio which is the same for all points x.*

5. Hence a figure may be constructed. Take a cubic, say of lines, and mark three of its lines. On a variable line ξ of the cubic mark the point x, making a constant double ratio with the points on ξ and the three lines. The locus of x is the other cubic.

This, applied to the deltoid by Mr. J. F. Messick, gave the figure annexed.

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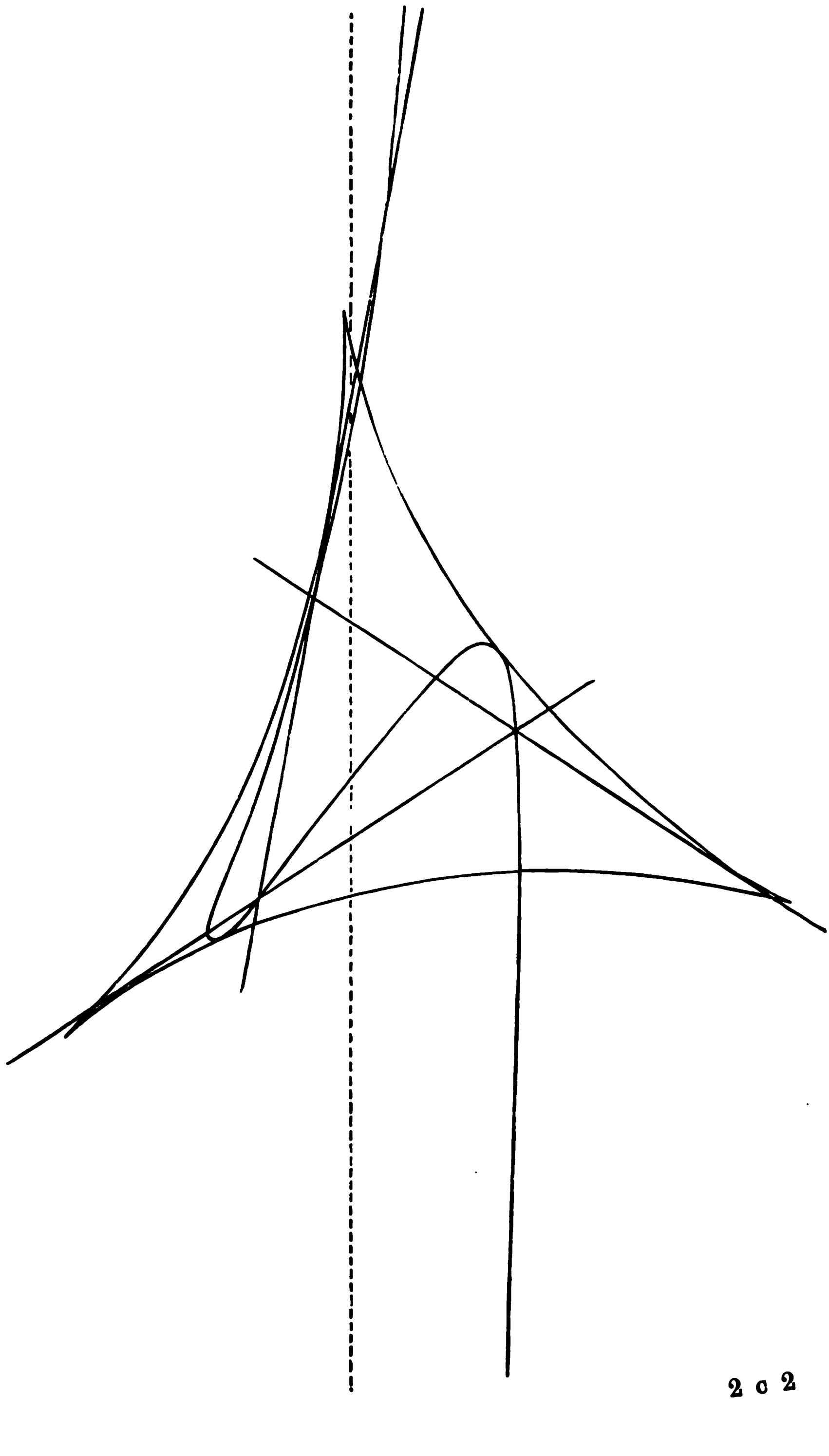
* If in a principal coincidence each element is continued so as to obtain the W-curves of Klein and Lie, then, given two families of W-curves, their elements of contact generate the pair of cubics.



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A convenient pair of equations is obtained from the equations

$$x_\iota \xi_\iota = \lambda_\iota.$$

If namely the cubic ϕ is $(ax)^3 = 0,$

then the cubic f is $(a\lambda/\xi)^3 = 0$

where $a_\iota^3 = 0, \qquad \Sigma\lambda_\iota = 0.$

To examine the intersections not arising from the parabolic triangles, we have only to make these cubics intersect at (1, 0, 0). When f is on (1, 0, 0) it must touch at this point a line of reference, say the line (0, 1, 0). This requires

$$a_1^2 a_3 = 0,$$

and this is the condition that ϕ has the same line at the same point. Hence *there are six contacts or common elements.*

It is verified at once from the developed expressions of the invariants S and T that

$$S \text{ of } f = \lambda_1^2\lambda_2^2\lambda_3^2 \, S \text{ of } \phi,$$

$$T \text{ of } f = \lambda_1^3\lambda_2^3\lambda_3^3 \, T \text{ of } \phi,$$

whence *the absolute invariants are equal.*

6. Let us now discuss the matter parametrically. Let the cubic ϕ, referred as before to a singular triangle, be

$$x_\iota = \frac{\sigma(u-a_\iota+\kappa)}{\sigma(u-a_\iota)},$$

where a_ι is the value of the parameter u at a point of reference and

$$\kappa = a_1+a_2+a_3.$$

Then for f
$$\xi_\iota = \lambda_\iota \frac{\sigma(u-a_\iota)}{\sigma(u-a_\iota+\kappa)}.$$

The line ξ meets ϕ where, if

$$V \equiv \Sigma\lambda_1 \frac{\sigma(v-a_1+\kappa)\,\sigma(v-a_2)\,\sigma(v-a_3)\,\sigma(u-a_1)}{\sigma(u-a_1+\kappa)},$$

V vanishes; that is where $v = u$, and say u_1 and u_2, where

(2) $$u+u_1+u_2 = 0.$$

Thus $$V \equiv A\sigma(v-u)\,\sigma(v-u_1)\,\sigma(v-u_2)$$

where A is independent of v.

We eliminate A by differentiation and division, observing that

$$D\sigma u = \sigma u\,\zeta u.$$

We thus get on eliminating A

$$(3)\qquad \Sigma\lambda_1 \frac{\sigma(v-a_1+\kappa)\,\sigma(u-a_1)}{\sigma(u-a_1+\kappa)\,\sigma(v-a_1)}\,\{\zeta(v-a+\kappa)+\zeta(v-a_2)+\zeta(v-a_3)\}$$

$$\equiv \Sigma\lambda_1 \frac{\sigma(v-a_1+\kappa)\,\sigma(u-a_1)}{\sigma(u-a_1+\kappa)\,\sigma(v-a_1)}\,\{\zeta(v-u)+\zeta(v-u_1)+\zeta(v-u_2)\}.$$

Letting $v = a_i$, we have three equations of the type

$$(4)\qquad \lambda_1\sigma\kappa\,\frac{\sigma(u-a_1)}{\sigma(u-a_1+\kappa)}\,\{\zeta\kappa+\zeta(a_1-a_2)+\zeta(a_1-a_3)\}$$

$$+\lambda_2\frac{\sigma(a_1-a_2+\kappa)\,\sigma(u-a_2)}{\sigma(a_1-a_2)\,\sigma(u-a_2+\kappa)}+\lambda_3\frac{\sigma(a_1-a_3+\kappa)\,\sigma(u-a_3)}{\sigma(a_1-a_3)\,\sigma(u-a_3+\kappa)}$$

$$= \lambda_1\sigma\kappa\,\frac{\sigma(u-a_1)}{\sigma(u-a_1+\kappa)}\,\{\zeta(a_1-u)+\zeta(a_1-u_1)+\zeta(a_1-u_2)\}.$$

7. Now always

$$(5)\qquad \frac{\sigma(b+c)\,\sigma(c+a)\,\sigma(a+b)}{\sigma a\,\sigma b\,\sigma c\,\sigma(a+b+c)} = \zeta a+\zeta b+\zeta c-\zeta(a+b+c),$$

as is proved immediately.

Multiply (4) by $\sigma(u-a_1+\kappa)/\sigma(u-a_1)$, and observe that, from (5),

$$\frac{\sigma(a_1-a+\kappa)}{\sigma\kappa\,\sigma(a_1-a)}\,\frac{\sigma(u-a)\,\sigma(u-a_1+\kappa)}{\sigma(u-a_1)\,\sigma(u-a+\kappa)} = \zeta(u-a_1)-\zeta(u-a+\kappa)+\zeta(a_1-a)+\zeta\kappa.$$

Then (4) becomes

$$\lambda_1\{\zeta\kappa+\zeta(a_1-a_2)+\zeta(a_1-a_3)-\zeta(a_1-u)-\zeta(a_1-u_1)-\zeta(a_1-u_2)\}$$

$$+\lambda_2\{\zeta(u-a_1)-\zeta(u-a_2+\kappa)+\zeta(a_1-a_2)+\zeta\kappa\}$$

$$+\lambda_3\{\zeta(u-a_1)-\zeta(u-a_3+\kappa)+\zeta(a_1-a_3)+\zeta\kappa\},$$

or $$\lambda_2\{\zeta(u-a_2+\kappa)-\zeta(a_1-u_1)-\zeta(a_1-u_2)+\zeta(a_1-a_3)\}$$

$$+\lambda_3\{\zeta(u-a_3+\kappa)-\zeta(a_1-u_1)-\zeta(a_1-u_2)+\zeta(a_1-a_2)\} = 0,$$

or, again using (5),

$$\lambda_2 \frac{\sigma(u-a_3)\,\sigma(u_2-a_3)\,\sigma(u_1+u_2-2a_1)}{\sigma(u_1-a_1)\,\sigma(u_2-a_1)\,\sigma(a_1-a_3)\,\sigma(u-a_2+\kappa)}$$

$$+\lambda_3 \frac{\sigma(u_1-a_2)\,\sigma(u_2-a_2)\,\sigma(u_1+u_2-2a_1)}{\sigma(u_1-a_1)\,\sigma(u_2-a_1)\,\sigma(a_1-a_2)\,\sigma(u+a_3-\kappa)} = 0$$

or

(6) $$\frac{\lambda_2}{\lambda_3} = \frac{\sigma(a_3-a_1)\,\sigma(u_1-a_2)\,\sigma(u_2-a_2)\,\sigma(u_1+u_2-a_3-a_1)}{\sigma(a_1-a_2)\,\sigma(u_1-a_3)\,\sigma(u_2-a_3)\,\sigma(u_1+u_2-a_1-a_2)}.$$

This is the relation between u_1 and u_2; for a given u_1 there are two values of u_2, say u_2 and u_3, connected by

$$u_2+u_3 = a_3+a_1+a_2-u_1$$

or

$$u_1+u_2+u_3 = \kappa,$$

which establishes again the existence of the triangles.

The relations connecting the vertices u_ι of a singular triangle are then

(7) $$u_1+u_2+u_3 = \kappa,$$

and three equations of the type

(8) $$\rho\lambda_1 = \sigma(a_2-a_3)\,\sigma(u_1-a_1)\,\sigma(u_2-a_1)\,\sigma(u_3-a_1),$$

where we get, by adding, Weierstrass's "equation with three terms."

8. The first genesis (§ 1) leads at once to the first of these equations; for, if a conic on three fixed points of the cubic cuts again at u_ι, then

$$u_1+u_2+u_3 = \text{const.}$$

But also, if a conic U meets the cubic at u_ι ($\iota = 1, 2, \ldots, 6$), and a conic V meets at v_ι, then

$$U \equiv A\Pi\sigma(u-u_\iota), \qquad V \equiv B\Pi\sigma(u-v_\iota),$$

where A, B are independent of u. And the conic $U-\mu V$ meets ϕ at points given by

$$A\Pi\sigma(u-u_\iota) = \mu B\Pi\sigma(u-v_\iota).$$

Thus the points on conics of a pencil are zeros of

$$\Pi\sigma(u-u_\iota)-z\Pi\sigma(u-v_\iota)$$

for a variable z.

If, then, three base points are on ϕ, say $u_4 = v_4$, $u_5 = v_5$, $u_6 = v_6$, the other intersections of conics of a pencil are zeros of

$$\frac{\sigma(u-u_1)\,\sigma(u-u_2)\,\sigma(u-u_3)}{\sigma(u-v_1)\,\sigma(u-v_2)\,\sigma(u-v_3)} = z.$$

But, from (8), replacing u_ι by v_ι and ρ by ρ', and dividing,

$$\frac{\sigma(a_1-u_1)\,\sigma(a_1-u_2)\,\sigma(a_1-u_3)}{\sigma(a_1-v_1)\,\sigma(a_1-v_2)\,\sigma(a_1-v_3)}=\frac{\rho}{\rho'}.$$

Thus the same pair of cubics can be generated in either way. Given the singular triangles, we obtain a suitable pencil by taking any three points a_ι of ϕ on a conic with one and therefore with every singular triangle. All the conics are in a pencil, meeting again at a point b. For different selections of a_ι, we can move b over the whole plane, the points of ϕ excepted.

In the first genesis, then, the apparent number of coordinates (9 for ϕ, 3 for the points on it, 2 for the point off it) must be diminished by 2 inasmuch as ∞^2 pencils give the same pair. That is, the coordinates of the pair are 12, as before.

SUPPLEMENTARY NOTE ON THE REPRESENTATION OF CERTAIN ASYMPTOTIC SERIES AS CONVERGENT CONTINUED FRACTIONS

By L. J. ROGERS.

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THE methods employed *supra* pp. 83 to 87 for representing certain definite integrals as continued fractions may be used for obtaining more general results.

Consider the integral

$$\int_0^\infty \frac{\sinh at}{\sinh t} e^{-t/x}\, dt.$$

It obviously has a finite value if $1/x > a-1$, and, if expanded in powers of x, will give an asymptotic series of odd powers of x.

Calling it ϕx, we shall assume

$$\phi x = \frac{ax}{1-}\,\frac{e_1 e_2 x^2}{1-}\,\frac{e_3 e_4 x^2}{1-}, \ \ldots.$$

Now
$$\phi\frac{x}{1-x} - \phi\frac{x}{1+x} = \frac{2ax^2}{1-x^2} = \frac{x}{1-ax} - \frac{x}{1+ax};$$

therefore

$$\phi\frac{x}{1-x} + \frac{x}{1+ax} = \phi\frac{x}{1+x} + \frac{x}{1-ax} = -\left\{\phi\frac{-x}{1+x} - \frac{x}{1-ax}\right\}.$$

Hence $\phi\dfrac{x}{1-x} + \dfrac{x}{1+ax}$ is odd, and we shall assume for it the form

$$\frac{(1+a)\,x}{1-}\,\frac{f_1 f_2 x^2}{1-}\,\frac{f_3 f_4 x^2}{1-},$$

so that

$$\frac{ax}{1-x-}\,\frac{e_1 e_2 x^2}{1-x-}\,\frac{e_3 e_4 x^2}{1-x-}\cdots + \frac{x}{1+ax} = \frac{(1+a)x}{1-}\,\frac{f_1 f_2 x^2}{1-}\,\frac{f_3 f_4 x^2}{1-}\cdots.$$

Changing x into $\dfrac{x}{1-ax}$, and dividing by x, we get

$$1 + \frac{a}{1-(a+1)x-}\,\frac{e_1 e_2 x^2}{1-(a+1)\,x-}\cdots = \frac{1+a}{1-ax-}\,\frac{f_1 f_2 x^2}{1-ax-}\cdots.$$

By Lemma I. on p. 74, we may write this

$$1+\frac{e_0}{1-}\,\frac{e_1x}{1-}\,\frac{e_2x}{1-}\cdots=\frac{f_0}{1-}\,\frac{f_1x}{1-}\,\frac{f_2x}{1-}\cdots,$$

where $$e_1=e_2+e_3=e_4+e_5=\ldots=a+1,$$

and $$f_1=f_2+f_3=\ldots=a,$$

so that $$e_{2n+1}=f_{2n+1}+n+1,$$

$$e_{2n}=f_{2n}-n.$$

But, by Lemma II., p. 74, we have

$$f_0=1+e_0,\quad f_0f_1=e_0e_1,\quad f_1+f_2=e_1+e_2,\quad f_2f_3=e_2e_3,\quad f_3+f_4=e_3+e_4,\quad\ldots,$$

so that $$e_{2n}e_{2n+1}=(e_{2n}+n)(e_{2n+1}-n-1);$$

and therefore $$(2n+1)\,e_{2n}=n(a-n),$$

$$(2n+1)\,e_{2n+1}=(n+1)(a+1+n).$$

Hence, finally,

$$\int_0^\infty\frac{\sinh at}{\sinh t}e^{-t/x}\,dt=\frac{ax}{1+}\,\frac{(1^2-a^2)\,1^2x^2}{3+}\,\frac{(2^2-a^2)\,2^2x^2}{5+}\,\frac{(3^2-a^2)\,3^2x^2}{7+}\cdots.$$

If a is a positive integer, the integral evidently reduces to an algebraic fraction, whose value is that identical with the terminating chain-fraction in the above identity.

Moreover we evidently obtain

$$\int_0^\infty\frac{\sin at}{\sinh t}e^{-t/x}\,dt=\frac{ax}{1+}\,\frac{(1^2+a^2)\,1^2x^2}{3+}\,\frac{(2^2+a^2)\,2^2x^2}{5+}\cdots.$$

Other identities may be obtained by similar processes, into which it is scarcely necessary to enter in detail.

I find that

$$\int_0^\infty\frac{\cosh at}{\cosh t}e^{-t/x}\,dt=\frac{x}{1+}\,\frac{(1^2-a^2)\,x^2}{1+}\,\frac{2^2x^2}{1+}\,\frac{(3^2-a^2)\,x^2}{1+}\,\frac{4^2x^2}{1+}\cdots,$$

$$\int_0^\infty\frac{\sinh at}{\cosh t}e^{-t/x}\,dt=\frac{ay}{1+}\,\frac{(2^2-a^2)\,y}{1+}\,\frac{2^2y}{1+}\,\frac{(4^2-a^2)\,y}{1+}\,\frac{4^2y}{1+}\cdots,$$

where $y=x^2/(1-x^2)$, as on p. 87.

$$\exp \int_0^\infty \frac{1}{t}\left(1-\frac{\cosh 2at}{\cosh 2t}\right) e^{-t/x}\, dt = 1+\frac{2(1-a^2)x^2}{1+}\,\frac{(3^2-a^2)x^2}{1+}\,\frac{(5^2-a^2)x^2}{1+}\cdots,$$

$$\tanh\left(\tfrac{1}{2}\int_0^\infty \frac{\sinh 2at}{t\cosh t} e^{-t/x}\, dt\right) = \frac{ax}{1+}\,\frac{(1^2-a^2)x^2}{1+}\,\frac{(2^2-a^2)x^2}{1+}\,\frac{(3^2-a^2)x^2}{1+}\cdots,$$

$$\tanh\left(\int_0^\infty \frac{\sinh at}{t\cosh t} e^{-t/x}\, dt\right) = \frac{ax}{1+}\,\frac{1^2x^2}{1+}\,\frac{(2^2-a^2)x^2}{1+}\,\frac{3^2x^2}{1+}\,\frac{(4^2-a^2)x^2}{1+}\cdots.$$

The convergency of these fractions is easily established by the criterion quoted on p. 75 as Lemma IV.

ON THE EXPANSION OF POLYNOMIALS IN SERIES OF FUNCTIONS

By L. N. G. FILON.

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1. *Introduction and Summary.*

The problem of expanding a given function $f(x)$ in a series of functions of given form—thus:

$$f(x) = a_1\phi(\kappa_1, x)+a_2\phi(\kappa_2, x)+\ldots+a_n\phi(\kappa_n, x)+\ldots, \tag{1}$$

where $\kappa_1, \kappa_2, \ldots, \kappa_n, \ldots$ are the roots of a transcendental equation

$$\psi(z) = 0 \tag{2}$$

—is one which has been familiar to mathematicians since the days of Fourier. This problem, in most cases which occur in mathematical physics, is usually solved by the method of normal functions; that is, functions $\chi(\kappa, x)$ are determined such that

$$\int_a^\beta \chi(\kappa_r, x)\,\phi(\kappa_s, x)\,dx = 0 \tag{3}$$

when r, s are different, but has some definite value when $r = s$. Thus, multiplying (1) by $\chi(\kappa_n, x)$ and integrating from a to β, the coefficient a_n is readily determined.

The great disadvantages of this method are that it gives no clue for the discovery of the functions χ when the form of the latter is not obvious from other considerations, and that it gives no means of predicting, given the functions ϕ and the transcendental equation (2), whether the required expansion is possible or not.

Another method has been given by Cauchy, and is described in Picard's *Cours d'Analyse* (pp. 169 *et seq.*). This method depends on the calculus of residues. Cauchy (and Picard after him) restricted himself to the case of trigonometrical series (see Cauchy, *Œuvres Complètes*, t. VII, 2^e Série: "Sur les Résidus des Fonctions exprimées par des Intégrales définies," p. 393), but the process by which the result is arrived at seems artificial. The function [denoted below by $F(z)$] on which the whole expansion hinges is selected from an *a priori* knowledge of the coefficients in

Fourier's expansion, and no method is given for finding it in the general case.

Dini, in his book on Fourier series (*Serie di Fourier e altri rappresentazione analitiche delle funzioni di una variabile reale*, Pisa, 1880), has employed a mixed method, depending partly on normal functions, partly on Cauchy's residue theorem. He gives a determination of the function $F(z)$ of the present paper, but in order to do so seems to assume (*loc. cit.*, pp. 131, 132) that the conjugate functions are practically already known, and that $\chi(\kappa, x) = \phi(\kappa, x)\,\theta(x)$, $\theta(x)$ being a function of x independent of κ. This restricts very considerably the generality of his results.

Dini's analysis seems to be directed rather to giving exact proofs of expansions already known than to developing methods for obtaining new expansions.

The object of the present paper is to extend and generalize the application of Cauchy's method of residues to expansions, and to give a rule for finding the *form* of the expansion in certain large classes of cases.

In what follows the functions to be expanded are supposed finite polynomials. This enables us to dispense at present with troublesome considerations of convergence.

The paper begins by establishing a general theorem for expanding a polynomial in a series of functions of the form $\phi(\kappa x)$, κ being a root of $\psi(z) = 0$. The theorem is practically contained in equations (6) and (15). Exceptional cases, when $z = 0$ is a zero of $\psi(z)$, are next dealt with. An example of the method is then given, showing how to expand a function $f(x)$ in the form $\Sigma\left\{A_n \cos(\kappa_n x) + B_n \sin(\kappa_n x)\right\}$, the κ_n's being roots of the transcendental equation $J_0(\kappa a) = 0$.

It is also verified that the method will give the expansions of Fourier, Schlömilch's expansion, and expansions in Bessel functions of order zero which occur in physical examples. New forms are obtained for the coefficients in the expansions in Bessel functions of order zero.

Also, in each case, the method enables us to find the range of validity of the expansion and the values of the series at the extremities of the range of validity. Thus the results (34), (35), which give the values at the ends of the range for Fourier's second trigonometrical series, I have not been able to find anywhere.

The latter part of the paper, after a brief consideration of the possibility of extending the results to functions other than polynomials, is devoted to applying the method to series of functions $\phi(\kappa, x)$ where κ, x do not appear exclusively as a product κx. The results are applied to an expansion in functions occurring in the theory of elasticity, which expan-

sion I believe to be new. Mr. John Dougall, in a paper on "An Analytical Theory of the Equilibrium of an Isotropic Elastic Plate" (*Trans. Roy. Soc. Edin.*, Vol. XLI., Part I., 8, 1904), has used functions of this type and obtained the sum of a few series involving them by the method of residues, but he has not attempted the converse problem of determining the coefficients, given the sum.

2. *Determination of the Coefficients.*

Let $\phi(z, x)$ be a function of two variables, z being complex and x real. Consider the function $F(z)\phi(z, x)/\psi(z)$, and integrate it round any closed contour C in the z-plane, enclosing the origin. Then

$$\frac{1}{2\pi i}\int_C \frac{F(z)\phi(z, x)}{\psi(z)}\,dz = \text{sum of the residues of the function inside } C.$$

Suppose now that $\phi(z, x)$ is a function without poles, and that $F(z)$ has poles only at the origin. Then the poles of the function which contribute to the residues are the zeros of $\psi(z)$ inside C and the origin. To begin with, we shall assume that $\psi(z)$ has no zero at the origin.

Let $\kappa_1, \kappa_2, \ldots, \kappa_n$ be the zeros of $\psi(z)$ inside C, arranged in the order of magnitude of their moduli. Then

$$\frac{1}{2\pi i}\int_C \frac{F(z)\phi(z, x)}{\psi(z)}\,dz = \sum_{r=1}^{n} \frac{F(\kappa_r)\phi(\kappa_r, x)}{\psi'(\kappa_r)} + \text{residue at the origin.}$$

The residue at the origin will be some function of x. Denote it by $f(x)$. We have

$$f(x) = -\sum_{r=1}^{n} \frac{F(\kappa_r)\phi(\kappa_r, x)}{\psi'(\kappa_r)} + \frac{1}{2\pi i}\int_C \frac{F(z)\phi(z, x)}{\psi(z)}\,dz. \tag{4}$$

If, now, as the contour C becomes larger and larger,

$$\mathrm{L}\,\frac{1}{2\pi i}\int_C \frac{F(z)\phi(z, x)}{\psi(z)}\,dz = 0, \tag{5}$$

we obtain, on proceeding to the limit

$$f(x) = -\sum_{r=1}^{r=\infty} \frac{F(\kappa_r)\phi(\kappa_r, x)}{\psi'(\kappa_r)}, \tag{6}$$

the roots κ_r occurring in the order of their moduli. The problem is so to determine $F(z)$ that $f(x)$ shall be identical with a given polynomial and that (5) shall hold.

Consider
$$\frac{1}{2\pi i}\int \frac{F(z)\phi(z, x)}{\psi(z)}\,dz$$

taken round a small circle enclosing the origin. Since $\phi(z, x)$ is without poles, its expansion in powers of z is absolutely and uniformly convergent over this circle. Also, if the radius of this circle be $< |\kappa_1|$, $1/\psi(z)$ may

be replaced by the equivalent Taylor series. The product of the two series is absolutely and uniformly convergent over the path of integration. Thus $\phi(z, x)/\psi(z)$ may be replaced by the two series and the result of multiplying these out integrated term by term. We shall further suppose that $\psi(z)$ itself has no pole infinitely close to $z = 0$, and take the radius of the circle of integration less than the modulus of its nearest pole (if any), so that $\psi(z)$ itself can be replaced by a power series.

Take, as a particular case,

$$F(z) = z^{-(n+1)}\psi_n(z) \tag{7}$$

where

$$\psi_n(z) = a_0 + a_1 z + \ldots + a_n z^n, \tag{8}$$

i.e., $\psi_n(z)$ denotes the first $(n+1)$ terms of the denominator $\psi(z)$ of the above integral.

$$\psi(z) = a_0 + a_1 z + \ldots + a_n z^n + a_{n+1} z^{n+1} + \ldots; \tag{9}$$

$$\frac{F(z)}{\psi(z)} = z^{-(n+1)} \frac{[\psi(z) - a_{n+1} z^{n+1} - a_{n+2} z^{n+2} - \ldots]}{\psi(z)}$$

$$= z^{-(n+1)} - \frac{(a_{n+1} + a_{n+2} z + \ldots)}{\psi(z)}$$

$$= z^{-(n+1)} + \text{power series in } z.$$

Thus

$$\frac{1}{2\pi i} \int \frac{\psi_n(z)}{z^{n+1}} \frac{\phi(z, x)}{\psi(z)}\, dz = \frac{1}{2\pi i} \int \frac{\phi(z, x)}{z^{n+1}}\, dz.$$

Let

$$\phi(z, x) = f_0(x) + z f_1(x) + \ldots + z^n f_n(x) + \ldots.$$

Then required residue $= f_n(x)$.

If we take

$$F(z) = \frac{p_0 \psi_0(z)}{z} + \frac{p_1 \psi_1(z)}{z^2} + \ldots + \frac{p_n \psi_n(z)}{z^{n+1}} \tag{10}$$

where $p_0, p_1, \ldots, p_n$ are constants, then

$$f(x) = p_0 f_0(x) + p_1 f_1(x) + \ldots + p_n f_n(x). \tag{11}$$

We will now consider more specially the important particular case where

$$f_n(x) = q_n x^n$$

or $\phi(z, x) =$ a function of zx only: thus

$$\phi(z, x) = \phi(zx) = q_0 + q_1 zx + \ldots + q_n (zx)^n + \ldots. \tag{12}$$

Hence

$$q_n = \frac{\phi^n(0)}{n!}. \tag{13}$$

(11) gives

$$f(x) = p_0 q_0 + p_1 q_1 x + \ldots + p_n q_n x^n$$

or

$$p_n q_n = \frac{f^n(0)}{n!}, \tag{14}$$

that is

$$p_n = f^n(0)/\phi^n(0).$$

We shall suppose $\phi^n(0)$ is not zero; so that no term is missing in the expansion of $\phi(z)$.

The required form for $F(z)$ is therefore

$$F(z) = \frac{f(0)}{\phi(0)}\,\frac{\psi_0(z)}{z} + \frac{f'(0)}{\phi'(0)}\,\frac{\psi_1(z)}{z^2} + \ldots + \frac{f^n(0)}{\phi^n(0)}\,\frac{\psi_n(z)}{z^{n+1}}, \tag{15}$$

It will be convenient in what follows to conceive the series (15) as extending to infinity. This will prove in many cases to be really a simplification, and, if we remember that, after a certain term, all the terms of the series vanish, no difficulties of convergency will be introduced.

3. *Case where $\psi(z) = 0$ at the origin.*

Suppose that $z = 0$ is a zero of $\psi(z)$ of ρ-th order. Then

$$\psi(z) = a_\rho z^\rho + a_{\rho+1} z^{\rho+1} + \ldots + a_{\rho+s} z^{\rho+s} + \ldots.$$

In this case $\psi_0(z)$, $\psi_1(z)$, ..., $\psi_{\rho-1}(z)$ all vanish. Consider

$$\begin{aligned}\frac{\psi_{\rho+s-1}(z)}{z^{\rho+s}\psi(z)} &= \frac{1}{z^{\rho+s}}\left\{\frac{a_\rho z^\rho + \ldots + a_{\rho+s-1} z^{\rho+s-1}}{a_\rho z^\rho + \ldots + a_{\rho+s} z^{\rho+s} + \ldots}\right\} \\ &= \frac{1}{z^{\rho+s}}\left\{1 - \frac{a_{\rho+s} z^{\rho+s} + a_{\rho+s+1} z^{\rho+s+1} + \ldots}{a_\rho z^\rho + \ldots + a_{\rho+s} z^{\rho+s} + \ldots}\right\} \\ &= \frac{1}{z^{\rho+s}}\left\{1 - \frac{(a_{\rho+s} z^s + a_{\rho+s+1} z^{s+1} + \ldots)}{a_\rho + a_{\rho+1} z + \ldots + a_{\rho+s} z^s + \ldots}\right\}.\end{aligned} \tag{16}$$

In the expansion of the above in ascending powers of z only the negative powers are required. Therefore it is sufficient to expand

$$\frac{1}{a_\rho + a_{\rho+1} z + \ldots + a_{\rho+s} z^s + \ldots}$$

as far as $z^{\rho-1}$.

Let

$$(a_\rho + a_{\rho+1} z + \ldots + a_{\rho+s} z^s + \ldots)^{-1} = b_0 + b_1 z + \ldots + b_{\rho-1} z^{\rho-1} + \ldots; \tag{17}$$

then we have the equations, to find the b's,

$$\left.\begin{aligned} 1 &= b_0 a_\rho \\ 0 &= b_0 a_{\rho+1} + b_1 a_\rho \\ &\ldots \quad \ldots \quad \ldots \quad \ldots \\ 0 &= b_0 a_{\rho+t} + b_1 a_{\rho+t-1} + \ldots + b_t a_\rho \\ &\ldots \quad \ldots \quad \ldots \quad \ldots \\ 0 &= b_0 a_{2\rho-1} + b_1 a_{2\rho-2} + \ldots + b_{\rho-1} a_\rho \end{aligned}\right\}. \tag{18}$$

Using (17), (16) becomes

$$\frac{\psi_{\rho+s-1}(z)}{z^{\rho+s}\psi(z)} = \frac{1}{z^{\rho+s}}\{1-z^s[b_0 a_{\rho+s}+z(b_0 a_{\rho+s+1}+b_1 a_{\rho+s})$$
$$+z^2(b_0 a_{\rho+s+2}+b_1 a_{\rho+s+1}+b_2 a_{\rho+s})+\dots$$
$$\dots$$
$$+z^{\rho-1}(b_0 a_{2\rho+s-1}+b_1 a_{2\rho+s-2}+\dots+b_{\rho-1} a_{\rho+s})]\}$$
$$+ \text{ positive powers of } z. \tag{19}$$

(19) has been established for positive values of s. But it is easy to see that it will still hold for $s = 0$ or $s =$ a negative integer numerically less than ρ.

If s be negative (or zero), then, using (18) and remembering $a_t = 0$ if $t < \rho$, we see that the only term in the square bracket in (19) which does not vanish is $z^{-s} b_0 a_\rho$, *i.e.*, z^{-s}. Thus, if $s \leqslant 0$, (19) gives

$$\psi_{\rho+s-1}(z)/z^{\rho+s}\psi(z) = 0,$$

as it should.

We may therefore write (19)

$$\frac{\psi_s(z)}{z^{s+1}\psi(z)} = \frac{1}{z^{s+1}} - \frac{1}{z^\rho}[b_0 a_{s+1}+z(b_0 a_{s+2}+b_1 a_{s+1})+z^2(b_0 a_{s+3}+b_1 a_{s+2}+b_2 a_{s+1})$$
$$+\dots+z^{\rho-1}(b_0 a_{s+\rho}+b_1 a_{s+\rho-1}+\dots+b_{\rho-1} a_{s+1})]$$
$$+ \text{ positive powers of } z, \tag{20}$$

and (20) holds for $s = 0$ or any positive integer. If we take, as before,

$$F(z) = \sum_{s=0}^{s=\infty} p_s z^{-(s+1)}\psi_s(z)$$

and

$$\phi(z, x) = \sum_{s=0}^{s=\infty} z^s f_s(x),$$

then the residue at the origin of $F(z)\phi(z, x)/\psi(z)$ is

$$\sum_{s=0}^{s=\infty} p_s f_s(x) - f_{\rho-1}(x)\sum_{s=0}^{s=\infty} b_0 a_{s+1} p_s - f_{\rho-2}(x)\sum_{s=0}^{s=\infty}(b_0 a_{s+2}+b_1 a_{s+1})p_s - \dots$$
$$-f_0(x)\sum_{s=0}^{s=\infty}(b_0 a_{s+\rho}+\dots+b_{\rho-1}a_{s+1})p_s.$$

If, now, we have, as before, been able to expand $f(x)$ in a series of functions $f_s(x)$ so that

$$f(x) = \sum_{s=0}^{s=\infty} p_s f_s(x),$$

it follows that $f(x)$ is not expansible *solely* in functions $\phi(z, x)$, but

contains a finite number of terms of a different form. We have

$$f(x) = f_0(x) \sum_{s=0}^{s=\infty} (b_0 a_{s+\rho} + \ldots + b_{\rho-1} a_{s+1}) p_s$$

$$+ f_1(x) \sum_{s=0}^{s=\infty} (b_0 a_{s+\rho-1} + \ldots + b_{\rho-2} a_{s+1}) p_s + \ldots + f_{\rho-1}(x) \sum_{s=0}^{s=\infty} b_0 a_{s+1} p_s$$

$$- \sum_{r=1}^{r=\infty} \frac{F(\kappa_r)}{\psi'(\kappa_r)} \phi(\kappa_r, x). \tag{21}$$

In the important particular case where $\phi(z, x)$ is given by (12)

$$f_n(x) = \frac{\phi^n(0)}{n!} x^n, \qquad p_s = \frac{f^s(0)}{\phi^s(0)}.$$

Then

$$f(x) = \phi(0) \sum_{s=0}^{s=\infty} (b_0 a_{s+\rho} + \ldots + b_{\rho-1} a_{s+1}) \frac{f^s(0)}{\phi^s(0)}$$

$$+ \frac{x\phi'(0)}{1!} \sum_{s=0}^{s=\infty} (b_0 a_{s+\rho-1} + \ldots + b_{\rho-2} a_{s+1}) \frac{f^s(0)}{\phi^s(0)} + \ldots$$

$$+ \frac{x^{\rho-1}\phi^{\rho-1}(0)}{(\rho-1)!} \sum_{s=0}^{s=\infty} (b_0 a_{s+1}) \frac{f^s(0)}{\phi^s(0)} - \sum_{r=1}^{r=\infty} \frac{F(\kappa_r)}{\psi'(\kappa_r)} \phi(\kappa_r, x) \tag{22}$$

where $F(z)$ is given by (15).

4. *Application to a New Trigonometrical Expansion.*

As an example of the application of this method to trigonometrical expansions in general, let it be proposed to expand $f(x)$ in the form

$$f(x) = \Sigma (a_{n_s} \cos n_s x + b_{n_s} \sin n_s x)$$

where n_s is a root of the transcendental equation $J_0(z) = 0$.

We take $\qquad \phi(xz) = e^{xz}, \qquad \psi(z) = I_0(z) = J_0(iz).$

We proceed to calculate $F(z)$. We have here

$$\phi(0) = \phi'(0) = \ldots = \phi^n(0) = \ldots = 1.$$

Then (15) gives

$$F(z) = \left[\left(\frac{\psi_0(z)}{z} + \frac{D\psi_1(z)}{z^2} + \ldots + \frac{D^n \psi_n(z)}{z^{n+1}} \right) f(u) \right]_{u=0} \tag{23}$$

where $D \equiv d/du$. Or, writing out $\psi_n(z)$,

$$F(z) = \left[\left\{ a_0 \frac{1}{z} + a_0 \frac{D}{z^2} + a_1 \frac{D}{z} + a_0 \frac{D^2}{z^3} + a_1 \frac{D^2}{z^2} + a_2 \frac{D^2}{z} + \ldots \right\} f(u) \right]_{u=0}. \tag{24}$$

The series (24) contains only a finite number of terms. It is, therefore, permissible to change the order of the terms. Collecting terms in $z^{-1}, z^{-2}, \ldots$, we find

$$F(z) = \left[\left\{\frac{1}{z}(a_0+a_1D+a_2D^2+\ldots)+\frac{D}{z^2}(a_0+a_1D+a_2D^2+\ldots)\right.\right.$$
$$\left.\left.+\frac{D^2}{z^3}(a_0+a_1D+a_2D^2+\ldots)\right\} f(u)\right]_{u=0}$$
$$= \left[\frac{\psi(D)}{z-D} f(u)\right]_{u=0}. \tag{25}$$

We may note here that (25) is the symbolic value of $F(z)$ for *all* trigonometric series, since it has been obtained without reference to the form of $\psi(z)$.

In the present case $\psi(z)$ has no zero $z=0$. But in those cases [*e.g.*, that of a Fourier series, where $\psi(z) = \sinh(bz)$] where $\psi(z)$ has a single zero $z=0$ the additional terms in (22) take a simple form. For they then reduce to the first term, namely,

$$b_0 \sum_{s=0}^{s=\infty} a_{s+1} f^s(0) = \left[\frac{b_0\psi(D)}{D} f(u)\right]_{u=0}. \tag{25A}$$

We proceed to evaluate $\dfrac{\psi(D)}{z-D} f(u)$.

By a known transformation in the theory of differential operators, this is

$$\psi(D)e^{zu}\left(-\frac{1}{D}\right)e^{-zu}f(u) = e^{zu}\psi(D+z)\int_u^a e^{-zu}f(u)\,du,$$

a being some upper limit.

It is not necessary to evaluate a. In fact, a may be given any convenient value.

For, if we change a to β, the difference between the two values of

$$\frac{\psi(D)}{z-D} f(u)$$

is
$$e^{zu}\psi(D+z)\int_a^\beta e^{-zu}f(u)\,du,$$

that is, since the limits of the integral are now constants with regard to u,

$$e^{zu}\psi(z)\int_a^\beta e^{-zu}f(u)\,du.$$

But we are going to compute the value of $F(z)$ only for such values of z as are roots of $\psi(z) = 0$. The above difference, which contains $\psi(z)$ as a factor, is therefore irrelevant.

We have then

$$F(z) = \Big[\psi(D+z)\int_u^a e^{-zu} f(u)\,du\Big]_{u=0} + \psi(z)\,\{\text{some factor}\}. \quad (2$$

So far our results are independent of the form of $\psi(z)$; that is, they ho for all trigonometric series.

Take now the value for $\psi(z)$ assumed at the beginning of the prese section. We have (see Gray and Mathews' *Bessel Functions*, p. 89)

$$J_0(q) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-iq\sin\theta}\,d\theta.$$

Thus
$$\psi(z) = J_0(iz) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{z\sin\theta}\,d\theta,$$

$$\psi(D+z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{z\sin\theta}\,e^{D\sin\theta}\,d\theta.$$

Therefore, using the symbolic form of Taylor's theorem,

$$\Big[\psi(D+z)\int_u^a e^{-uz} f(u)\,du\Big]_{u=0} = \Big[-\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{z\sin\theta}\Big\{\int_a^{u+\sin\theta} e^{-uz} f(u)\,du\Big\}\,d\theta\Big]_u$$

$$= -\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{z\sin\theta}\Big\{\int_a^{\sin\theta} e^{-uz} f(u)\,du\Big\}\,d\theta.$$

Since a is arbitrary, we may take it equal to 0. Hence

$$F(z) = -\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^{\sin\theta} e^{z(\sin\theta-u)} f(u)\,d\theta\,du.$$

The terms in the expansion corresponding to the roots $z = \pm in_s$ are

$$+\frac{\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^{\sin\theta} e^{in_s(\sin\theta-u)} f(u)\,d\theta\,du}{iJ'(-n_s)}\,e^{in_s}$$

$$+\frac{\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^{\sin\theta} e^{-in(\sin\theta-u)}\;(u)\,d\theta\,du}{iJ'\;n_s)}\,e^{-in_s x}$$

$$= \frac{1}{\pi J'(n_s)}\int_{-\pi}^{\pi}\int_0^{\sin\theta} \sin n_s(u-x-\sin\theta) f(u)\,d\theta\,du,$$

whence we get

$$f(x) = \sum_{s=1}^{s=\infty}\Big[\frac{1}{\pi J'(n)}\cos n_s x\int_{-\pi}^{\pi}\int_0^{\sin\theta}\sin\{n_s(u-\sin\theta)\} f(u)\,d\theta\,du$$

$$-\frac{1}{\pi J_0'(n_s)}\sin n_s x\int_{-\pi}^{\pi}\int_0^{\sin\theta}\cos\{n_s(u-\sin\theta)\} f(u)\,d\theta\,du\Big]. \quad (27$$

5. *Validity and Limits of the last Expansion.*

To obtain the conditions under which the expansion (27) is valid we have to consider the integral

$$\int_C \frac{F(z)\,e^{xz}}{I_0(z)}\,dz,$$

taken round a very large contour C in the z-plane.

The form (25) for $F(z)$ shows that when z becomes infinite in any manner

$$\mathop{\mathrm{L}}_{z=\infty} zF(z) = [\psi(D)\,f(u)]_{u=0} = [J_0(iD)\,f(u)]_{u=0} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\sin\theta)\,d\theta,$$

which is finite.

With regard to the values of $\dfrac{\phi(xz)}{\psi(z)}$, that is of $\dfrac{e^{xz}}{I_0(z)}$, we have, when z is large,

$$J_0(z) = \sqrt{\left(\frac{2}{\pi z}\right)}\cos\left(z-\frac{\pi}{4}\right) \qquad \text{(real part of } z>0\text{)},$$

$$J_0(z) = \sqrt{\left(\frac{2}{-\pi z}\right)}\cos\left(z+\frac{\pi}{4}\right) \qquad \text{(real part of } z<0\text{)},$$

$$J_0(iy) = J_0(-iy) = (2\pi y)^{-\frac{1}{2}}e^{y},$$

where the square roots are so taken that their real part is positive (see Hankel, *Math. Annalen*, Vol. I., pp. 500, 501).

From these we deduce

$$I_0(z) = (2\pi z)^{-\frac{1}{2}}(e^{z}+e^{-z+\frac{1}{2}i\pi}) \qquad \text{(imaginary part of } z>0\text{)},$$

$$I_0(z) = (2\pi z)^{-\frac{1}{2}}(e^{z}+e^{-z-\frac{1}{2}i\pi}) \qquad \text{(imaginary part of } z<0\text{)},$$

$$I_0(z) = (2\pi z)^{-\frac{1}{2}}e^{z} \qquad \text{(} z \text{ real and positive)},$$

$$I_0(z) = (-2\pi z)^{-\frac{1}{2}}e^{-z} \qquad \text{(} z \text{ real and negative)},$$

the values of the roots above being determined by taking $\sqrt{z}$ to be the positive real root of z when z is real and positive.

It is clear that, if $x=\pm 1$, $e^{xz}/I_0(z)$ is in general comparable with $(2\pi z)^{\frac{1}{2}}$, and

$$\int_C \frac{F(z)\,e^{xz}}{I_0(z)}\,dz$$

need not be finite. Accordingly the expansion is not valid for the end values $x=\pm 1$. Still less is it valid if $|x|>1$.

Consider $|x| < 1$. Let the path of integration be a circle of radius R passing through $z = i(n\pi + \frac{1}{4}\pi)$;

$$\frac{e^{xz}}{I_0(z)} = \frac{(2\pi z)^{\frac{1}{2}}}{e^{z(1-x)} + e^{-z(1+x) \pm \frac{1}{2}\pi i}},$$

when $|z|$ is large.

Let $z = p + iq$ be any point on the path of integration.

Consider first those parts of the contour which lie inside an angle 2ϵ enclosing the imaginary axis: let us restrict for the present our attention to that arc which is bisected by the positive half of the above axis.

Here $$I_0(z) = (2\pi z)^{-\frac{1}{2}}(e^z + e^{-z+\frac{1}{2}i\pi}),$$

$$|I_0(z)| = (2\pi R)^{-\frac{1}{2}}|e^z|\,|1+e^{-2z+\frac{1}{2}i\pi}| = (2\pi R)^{-\frac{1}{2}} e^p\,|1+e^{-2p}\,e^{i(-2q+\frac{1}{2}\pi)}|$$
$$= (2\pi R)^{-\frac{1}{2}} e^{-p}\,|1+e^{2p}\,e^{i(2q-\frac{1}{2}\pi)}|.$$

Thus $$|I_0(z)| > (2\pi R)^{-\frac{1}{2}} e^{|p|}\left[1+e^{-2|p|}\cos(2q-\tfrac{1}{2}\pi)\right],$$

since $$|a+ib| > |a|,$$

and $$\left|\frac{e^{xz}}{I_0(z)}\right| < \frac{(2\pi R)^{\frac{1}{2}} e^{-|p|(1-|x|)}}{1+e^{-2|p|}\cos(2q-\frac{1}{2}\pi)}.$$

Let p_0 be given by the equation

$$e^{p_0(1-|x|)} = \lambda(2\pi R)^{\frac{1}{2}},$$

λ being any constant.

Then, if $p > p_0$,

$$\left|\frac{e^{xz}}{I_0(z)}\right| < \frac{1}{\lambda[1+e^{-2|p|}\cos(2q-\frac{1}{2}\pi)]} < \frac{1}{\lambda(1-e^{-2p_0})}$$

$<$ a fixed finite quantity Q if R exceeds a fixed value, since p_0 becomes infinite with R.

Let the arc bounded by $\pm p_0 + iq_0$ subtend an angle $2\epsilon'$ at the origin.

Then $$\sin\epsilon' = \frac{p_0}{R} = \frac{1}{2}\,\frac{\log R}{R[1-|x|]} + \frac{\log\{\lambda\sqrt{(2\pi)}\}}{R[1-|x|]}.$$

Thus ϵ' tends to zero as R increases, and the arc $2\epsilon'$ ultimately lies inside the arc 2ϵ.

The integral over the arc $2\epsilon - 2\epsilon'$

$$< Q(2\epsilon - 2\epsilon') \mathop{\mathrm{L}}_{z=\infty} zF(z)$$

in the limit.

Over the arc $2\epsilon'$ $\qquad R > q > R - p_0^2/R.$

Now p_0^2/R is of order $\dfrac{1}{(1-|x|)^2}\left(\dfrac{\log\sqrt{R}}{\sqrt{R}}\right)^2$ and tends to zero as R in-

creases. Hence, if R is large enough,

$$n\pi+\tfrac{1}{4}\pi > q > n\pi+\tfrac{1}{4}\pi-\tfrac{1}{2}\theta,$$

θ being any assigned positive quantity which may be taken as small as we please, and

$$\cos(2q-\tfrac{1}{2}\pi) > \cos\theta.$$

Hence over the arc $2\epsilon'$

$$\left|\frac{e^{zx}}{I_0(z)}\right| \leqslant \frac{(2\pi R)^{\frac{1}{2}}\, e^{-|p|(1-|x|)}}{1+e^{-2|p|}\cos\theta} < \frac{(2\pi R)^{\frac{1}{2}}}{1+e^{-2|p|}\cos\theta} < (2\pi R)^{\frac{1}{2}}.$$

The integral over the arc $2\epsilon'$ is therefore less than

$$2\epsilon'\,(2\pi R)^{\frac{1}{2}} \underset{z=\infty}{\mathrm{L}}\; zF(z)$$

in the limit.

But
$$\underset{R=\infty}{\mathrm{L}}\; 2\epsilon'\,(2\pi R)^{\frac{1}{2}} = \underset{R=\infty}{\mathrm{L}}\; \text{const.}\,\frac{\log R}{\sqrt{R}} = 0.$$

Thus the whole integral over the arc 2ϵ tends to a quantity less than

$$2Q\epsilon \underset{z=\infty}{\mathrm{L}}\; zF(z),$$

when R becomes indefinitely great.

Thus, making ϵ small, we see that the arc bisected by the positive half of the imaginary axis ultimately contributes nothing to the contour integral. By symmetry the arc bisected by the negative half of the imaginary axis also contributes nothing.

Now, over the parts of the contour lying outside the angle 2ϵ, it is obvious that $\mathrm{L}\left(\dfrac{e^{zx}}{I_0(z)}\right) = 0$. Hence these parts also contribute nothing. Therefore, if $|x| < 1$,

$$\int_C \frac{F(z)\,\phi(zx)}{\psi(z)}\,dx = 0.$$

When the radius of C becomes infinitely great the expansion then holds.

6. *The Expansions of Fourier.*

The same method may be applied to deduce from the general theorem of Art. 2 the well known series of Fourier

$$f(x) = a_0+a_1\cos\frac{\pi x}{b}+a_2\cos\frac{2\pi x}{b}+\ldots+b_1\sin\frac{\pi x}{b}+b_2\sin\frac{2\pi x}{b}+\ldots, \tag{28}$$

and another series, also given by Fourier,

$$f(x) = a_1\cos n_1x+a_2\cos n_2x+\ldots+b_1\sin n_1x+b_2\sin n_2x+\ldots, \tag{29}$$

where $n_1, n_2, \ldots$ are the roots of the equation

$$\cos nr - \frac{\lambda \sin nr}{nr} = 0, \tag{30}$$

λ being < 1.

In the case of Fourier's series, we find, in a manner similar to that employed on p. 403, using the symbolic form of Taylor's theorem,

$$F(z) = \tfrac{1}{2}\int_b^a e^{-zu} e^{zb} f(u)\,du - \tfrac{1}{2}\int_{-b}^a e^{-zu} e^{-zb} f(u)\,du + \psi(z)\{\text{some factor}\},$$

where a, as before, is arbitrary and $\psi(z) = \sinh zb$. Taking $a = -b$ and putting $z =$ any root κ of $\psi(z) = 0$, we have

$$F(\kappa) = -\tfrac{1}{2}\int_{-b}^{+b} e^{\kappa(b-u)} f(u)\,du. \tag{31}$$

The constant term is obtained from the additional term due to the single root $z = 0$ of $\psi(z)$. It is

$$\left[\frac{\psi(D)}{D} f(u)\right]_{u=0} = \frac{1}{2b}\int_{-b}^{b} f(u)\,du. \tag{32}$$

It is easily verified that (31) and (32) lead to the well known expansion. An investigation similar to that of Art. 5 will then show that the expansion is valid if $-b < x < b$, $\underset{z=\infty}{\mathrm{L}}\, zF(z)$ being here equal to

$$\tfrac{1}{2}[f(b) - f(-b)].$$

In the case where $x = \pm b$ it is easily shown (see also Picard, *Cours d'Analyse*, pp. 167–177) that

$$\int_C \frac{F(z)\, e^{zb}}{\sinh zb}\,dz = \pi i [f(b) - f(-b)]$$

and

$$\int_C \frac{F(z)\, e^{-zb}}{\sinh zb}\,dz = -\pi i [f(b) - f(-b)],$$

whence we get the well known result that the value of the series at the ends of the range is

$$\tfrac{1}{2}[f(b) + f(-b)].$$

With regard to the series (29), Fourier showed (*Théorie de la Chaleur*, p. 348 *et seq.*) how to expand an *odd* function in terms of sines. The coefficients a were therefore absent in his expansion. Picard in his *Cours d'Analyse* (pp. 179–183, first edition) has generalized Fourier's result, so as to include the even terms. But he has proceeded in what appears to be a rather arbitrary manner, with the result that he has introduced into his expansion a constant term which is unnecessary.

If we treat this example by the method of the present paper, putting

$$\psi(z) = \cosh zr - \frac{\lambda \sinh zr}{zr},$$

we find, by a process very similar to that used before,

$$F(z) = \frac{e^{rz}}{2}\left(1-\frac{\lambda}{rz}\right)\int_r^a e^{-zu} f(u)\,du + \frac{e^{-rz}}{2}\left(1+\frac{\lambda}{rz}\right)\int_{-r}^a e^{-zu} f(u)\,du$$
$$-\frac{\lambda}{2rz}\int_{-r}^{+r} f(u)\,du + \psi(z)\,G,$$

where G is some factor which we do not require to determine.

Whence, after some reductions,

$$f(x) = \sum_{s=1}^{s=\infty}\left\{\cos n_s x \frac{\int_{-r}^{r}\{\cos n_s u - \cos n_s r\} f(u)\,du}{r-(\sin 2n_s r)/2n_s}\right.$$
$$\left. + \sin n_s x \frac{\int_{-r}^{r} \sin n_s u\, f(u)\,du}{r-(\sin 2n_s r)/2n_s}\right\}, \tag{33}$$

which gives the expansion required.

The sine terms in this expansion agree with those given by Fourier (*loc. cit.*) for the expansion of an odd function.

Picard's result differs from (36) in that the coefficient of $\cos n_s x$ is

$$\frac{1}{\{r-(\sin 2n_s r)/2n_s\}}\int_{-r}^{r} \cos n_s u\, f(u)\,du,$$

instead of the coefficient given in (33), and there is an absolute term introduced.

That such an absolute term is not really required is obvious from the present work, since $z = 0$ is not a zero of $\psi(z)$. In fact (33) allows us to expand a constant in a series $\Sigma A_s \cos n_s x$, and when we replace the absolute term in Picard's result by its expansion in a series of cosines, the new expansion is found to agree with (33).

That (33) is the natural expansion may also be seen from the fact that

$$\int_{-r}^{r} (\cos n_s u - \cos n_s r) \cos n_t u\, du = 0,$$

if s and t are different—a result which is easily verified directly and which would allow us to obtain the expansion by a method analogous to that of normal functions.

Also, if we investigate as before the validity of this expansion by considering $\int_C F(z)\frac{\phi(zx)}{\psi(z)}\,dz$, we find that

$$\underset{|z|=\infty}{\mathrm{L}}\, zF(z) = \tfrac{1}{2}[f(r)+f(r)] - \frac{\lambda}{2r}\int_{-r}^{r} f(u)\,du.$$

Then it may be shown, as in Art. 5, that, if $|x|>r$, $|\phi(xz)/\psi(z)| = \infty$ over one half of the contour of integration, so that the expansion is then not legitimate, but that, if $|x|<r$, $|\phi(xz)/\psi(z)| = 0$ when $|z| = \infty$, except within an angle 2ϵ enclosing the imaginary axis; and the parts of the path of integration within this angle can easily be shown to contribute nothing ultimately to the integral, so that, if $|x|<r$, the expansion is valid.

When $x = r$, $\underset{z=\infty}{\mathrm{L}}\, \frac{\phi(xz)}{\psi(z)} = 2$ when the real part of z is positive, and $\underset{z=\infty}{\mathrm{L}}\, \frac{\phi(xz)}{\psi(z)} = 0$ when the real part of z is negative.

Hence, if $x = r$,

$$f(r) = \text{series} + \tfrac{1}{2}[f(r)+f(-r)] - \frac{\lambda}{2r}\int_{-r}^{r} f(u)\,du,$$

or

$$\text{series} = \tfrac{1}{2}[f(r)-f(-r)] + \frac{\lambda}{2r}\int_{-r}^{r} f(u)\,du. \qquad (34)$$

Similarly, if $x = -r$,

$$\text{series} = \tfrac{1}{2}[f(-r)-f(r)] + \frac{\lambda}{2r}\int_{-r}^{r} f(u)\,du. \qquad (35)$$

These two end values for the series are not given by Picard, and I have been unable to find them anywhere else.

It follows from (34) and (35) that when $f(x)$ is an odd function the expansion in sines holds right up to the limit $x = \pm r$. But, if $f(x)$ be an even function, there is a discontinuity at the ends of the range. This is precisely the reverse of what happens with the ordinary Fourier's series.

7. *Schlömilch's Expansion.*

Here we require to expand $f(x)$ in a series of Bessel functions of zero order in the form

$$f(x) = A_0 + A_1 J_0(x) + A_2 J_0(2x) + \dots.$$

Clearly, if this expansion is to hold for negative as well as for positive values of x, $f(x)$ must be an even function.

It will be more convenient to consider a more general expansion, where $f(x)$ is not restricted to be even, namely,

$$f(x) = A_0+A_1J_0(x)+A_2J_0(2x)+\ldots+B_1L_0(x)+B_2L_0(2x)+\ldots,$$

where
$$L_0(x) = \frac{2}{\pi}\left\{\frac{x}{1^2}-\frac{x^3}{1^2.3^2}+\frac{x^5}{1^2.3^2.5^2}-\ldots\right\}. \tag{36}$$

This function occurs in the theory of the vibrations of a circular plate, and its properties have been discussed by Lord Rayleigh (*Theory of Sound*, Vol. II., § 302). Rayleigh denotes this function by $K(x)$.

We write

$$Q(z) = 1+\frac{2}{\pi}\frac{z}{1^2}+\frac{z^2}{2^2}+\frac{2}{\pi}\frac{z^3}{1^2.3^2}+\frac{z^4}{2^2.4^2}+\frac{2}{\pi}\frac{z^5}{1^2.3^2.5^2}+\ldots, \tag{37}$$

and take
$$\phi(zx) = Q(zx), \qquad \psi(z) = \sinh \pi z.$$

We then find easily

$$\frac{1}{\phi^s(0)} = \int_0^1 \frac{st^{s-1}}{(1-t^2)^{\frac{1}{2}}}dt = \int_0^1 (1-t^2)^{-\frac{1}{2}}\frac{d}{dt}(t^s)\,dt \tag{38}$$

whether s be odd or even, with the exception of $1/\phi(0) = 1$.

Substituting into the expression (15) for $F(z)$, we find

$$F(z) = \psi_0(z)\frac{f(0)}{z}$$
$$+\left[\int_0^1 (1-t^2)^{-\frac{1}{2}}\frac{d}{dt}\left\{\frac{\psi_0(z)}{z}+\frac{tD\psi_1(z)}{z^2}+\ldots+\frac{t^rD^r\psi_r(z)}{z^{r+1}}+\ldots\right\}f(u)\,dt\right]_{u=0},$$

and, treating the series in curled brackets as was done in Art. 4, we find

$$F(z) = \psi_0(z)\frac{f(0)}{z}+\left[\int_0^1 (1-t^2)^{-\frac{1}{2}}\frac{d}{dt}\left\{\frac{\psi(tD)}{z-tD}\right\}f(u)\,dt\right]_{u=0}.$$

Hence, remembering that $[(tD)^rf(u)]_{u=0} = [D^rf(ut)]_{u=0}$,

$$F(z) = \psi_0(z)\frac{f(0)}{z}+\left[\frac{\psi(D)}{z-D}\int_0^1 (1-t^2)^{-\frac{1}{2}}uf'(ut)\,dt\right]_{u=0}. \tag{39}$$

(39) is the general expression for $F(z)$ whatever the form of $\psi(z)$. It is therefore applicable to all expansions of the type

$$\Sigma\{A_sJ_0(n_sx)+B_sL_0(n_sx)\}.$$

Taking now $\psi(z) = \sinh \pi z$, we find that

$$\frac{\sinh \pi D}{z-D}\chi(u) = -\tfrac{1}{2}\int_{-\pi}^{\pi} e^{z(\pi-u)}\chi(u)\,du+\psi(z)\,G.$$

Applying this result to (39),

$$F(z) = -\tfrac{1}{2}\int_{-\pi}^{\pi} du \int_0^1 dt(1-t^2)^{-\frac{1}{2}}\, e^{z(\pi-u)} u f'(ut) + \psi(z)\, G, \tag{40}$$

since $\psi_0(z) = 0$ in the present case.

There will be an absolute term, since $\psi(z)$ has a simple zero at the origin.

This absolute term is given by the first term in (22), namely,

$$\phi(0) \sum_{s=0}^{s=\infty} b_0 a_{s+1} \frac{f^s(0)}{\phi^s(0)} = \phi(0)\, b_0 \left[a_1 f(0) + \left(\frac{\psi(D)}{D}\int_0^1 \frac{u f'(ut)}{(1-t^2)^{\frac{1}{2}}}\, dt\right)_{u=0}\right], \tag{41}$$

which, when we put $\phi(0) = 1$, $b_0 = 1/\pi$, $a_1 = \pi$, $\psi(D) = \sinh \pi D$, becomes

$$f(0) + \frac{1}{2\pi}\int_{-\pi}^{\pi} du \int_0^1 \frac{u f'(ut)}{(1-t^2)^{\frac{1}{2}}}\, dt. \tag{42}$$

Thus, from (22), (40), (42),

$$f(x) = f(0) + \frac{1}{2\pi}\int_{-\pi}^{\pi} du \int_0^1 \frac{u f'(ut)}{(1-t^2)^{\frac{1}{2}}}\, dt$$
$$+ \sum_{\kappa} \frac{Q(\kappa x)}{2\pi \cosh \pi\kappa} \int_{-\pi}^{\pi} du \int_0^1 \frac{dt\, e^{\kappa(\pi-u)} u f'(ut)}{(1-t^2)^{\frac{1}{2}}},$$

$z = \kappa$ being any zero of $\sinh \pi z$ other than $z = 0$. Whence, grouping terms in pairs,

$$f(x) = f(0) + \frac{1}{2\pi}\int_{-\pi}^{\pi} du \int_0^1 \frac{dt\, u f'(ut)}{(1-t^2)^{\frac{1}{2}}} + J_0(nx)\, \frac{1}{\pi}\int_{-\pi}^{\pi} du \int_0^1 \frac{dt \cos nu\, u f'(ut)}{(1-t^2)^{\frac{1}{2}}}$$
$$+ L_0(nx)\, \frac{1}{\pi}\int_{-\pi}^{\pi} du \int_0^1 \frac{dt \sin nu\, u f'(ut)}{(1-t^2)^{\frac{1}{2}}}. \tag{43}$$

The even terms give Schlömilch's well known expansion. The odd terms complete this expansion, the function $L_0(x)$ having here to $J_0(x)$ the same relation that the sine has to the cosine.

In order to investigate the validity of this expansion, it is necessary to know the order of magnitude of $Q(z)$ when $|z|$ is large.

We have (see Rayleigh, *loc. cit.*)

$$Q(z) = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} e^{z \sin x}\, dx = \frac{2}{\pi}\int_0^1 \frac{e^{zt}}{(1-t^2)^{\frac{1}{2}}}\, dt.$$

Writing $(1-t) = u$,

$$Q(z) = \frac{2}{\pi}\, e^z \int_0^1 \frac{e^{-zu}\, du}{\sqrt{\{u(2-u)\}}}$$
$$= \frac{2}{\pi}\, e^z \int_0^1 \frac{e^{-zu}}{\sqrt{(2u)}}\left[1 + \frac{u}{4} + \left(\frac{u}{2}\right)^2 \frac{1.3}{2.4} + \ldots\right] du.$$

It may then be shown that, when z is large and its *real part is positive*, the most important term in $Q(z)$ is the first. This first term becomes, on writing $zu = v$,

$$\frac{2}{\pi}\frac{e^z}{\sqrt{(2z)}}\int_0^z \frac{e^{-v}\,dv}{\sqrt{v}},$$

the integrations with regard to v being taken along the straight line joining the origin to the point z.

By considering a contour $ABCD$ in the v-plane, AD being the arc of a very small circle, centre the origin, CB a concentric arc through the point $v = z$, CD a portion of the line joining $v = 0$ to $v = z$, and AB a portion of the real axis in the v-plane, we can readily show that the first term in $Q(z)$ is approximately equal to

$$\frac{2}{\pi}\frac{e^z}{\sqrt{(2z)}}\int_0^{|z|} \frac{e^{-v}\,dv}{\sqrt{v}}$$

when $|z|$ is large, the path of integration being now real.

Therefore the most important term in $Q(z)$ is

$$\frac{2}{\pi}\frac{e^z}{\sqrt{(2z)}}\int_0^{\infty} \frac{e^{-v}\,dv}{\sqrt{v}} = e^z\sqrt{\left(\frac{2}{\pi z}\right)}.$$

Take now the real part of z negative. Write $z = -\zeta$; then the real part of ζ is positive.

$$Q(z) = \frac{2}{\pi}\int_0^1 (1-t^2)^{-\frac{1}{2}} e^{-\zeta t}\,dt.$$

Now expand $(1-t^2)^{-\frac{1}{2}}$ by the binomial theorem

$$\begin{aligned} Q(z) &= \frac{2}{\pi}\int_0^1 e^{-\zeta t}\left(1+\frac{1}{2}t^2+\frac{1.3}{2.4}t^4+\ldots\right)dt \\ &= \frac{2}{\pi}\int_0^{\zeta} e^{-v}\left(\frac{1}{\zeta}+\frac{1}{2}\frac{v^2}{\zeta^3}+\frac{1.3}{2.4}\frac{v^4}{\zeta^5}+\ldots\right)dv. \end{aligned}$$

By reasoning similar to that employed above, the most important term in $Q(x)$ is

$$\frac{2}{\pi\zeta}\int_0^{\infty} e^{-v}\,dv = \frac{2}{\pi\zeta} = -\frac{2}{\pi z}.$$

The case where z is a pure imaginary has been worked out by Lord Rayleigh. Taking the value given for $L_0(z)$ [his $K(z)$] in ascending powers of $1/z$ in his *Theory of Sound*, § 302, we have

$$Q(i\zeta) = \frac{2i}{\pi\zeta}+e^{i(\zeta-\frac{1}{4}\pi)}\sqrt{\left(\frac{2}{\pi z}\right)} = -\frac{2}{\pi z}+e^z\sqrt{\left(\frac{2}{\pi z}\right)},$$

the same convention being adopted with regard to $\sqrt{z}$ as on p. 405. Similarly when $z = -i\zeta$.

Thus $$Q(z) = -\frac{2}{\pi z} + e^{z}\sqrt{\left(\frac{2}{\pi z}\right)} \tag{44}$$

will give approximately the order of $Q(z)$ for all large values of z.

It may then be readily shown

(*a*) That

$$\underset{z=\infty}{\mathrm{L}}\, zF(z) = \left[\sinh \pi D \int_0^1 (1-t^2)^{-\frac{1}{2}} u f'(ut)\, dt\right]_{u=0}$$

$$= \int_0^1 \tfrac{1}{2}(1-t^2)^{-\frac{1}{2}}\{\pi f'(\pi t) + \pi f'(-\pi t)\}\, dt,$$

which is finite.

(*b*) That, if $x > 0$,

$\dfrac{Q(zx)}{\sinh \pi z}$ tends to 0 if the real part of z is negative and approximates to $2\sqrt{2}(\pi z x)^{-\frac{1}{2}} e^{z(x-\pi)}$ if the real part of z be positive, that is, it tends to 0 or ∞ according as $x \not> \pi$ or $x > \pi$.

Similarly, if $x < 0$,

$\dfrac{Q(zx)}{\sinh \pi z}$ tends to 0 everywhere if $x \not< -\pi$, but tends to ∞ when the real part of z is negative and $x < -\pi$.

(*c*) That the parts of the contour integral in the neighbourhood of the imaginary axis are evanescent in the limit.

It follows that Schlömilch's expansion is valid if $-\pi \leqslant x \leqslant \pi$. No exception is to be made for the extremities of the range.

8. *Other Expansions in Bessel Functions of Zero Order.*

The method can also be applied to obtain an expansion of the type

$$f(x) = \Sigma\{A_s J_0(n_s x) + B_s L_0(n_s x)\}, \tag{45}$$

n_s being any root of the transcendental equation

$$J_0(na) = 0. \tag{46}$$

Such expansions occur frequently in mathematical physics in problems relating to vibrations where the boundaries are circular.

We have here $$\phi(zx) = Q(zx), \quad \psi(z) = J_0(iaz).$$

Thus $$\psi_0(z) = 1$$

and $$F(z) = \frac{f(0)}{z} + \left[\frac{J_0(iaD)}{z-D}\int_0^1 (1-t^2)^{-\frac{1}{2}} u f'(ut)\, dt\right]_{u=0}.$$

Proceeding as in Art. 4, this leads to

$$F(z) = \frac{f(0)}{z} - \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \int_0^{a\sin\theta} e^{z(a\sin\theta-u)}\, du \int_0^1 (1-t^2)^{-\frac{1}{2}} u f'(ut)\, dt + \psi(z)\, G$$

or, writing $t = \sin\phi$,

$$F(z) = \frac{f(0)}{z} - \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \int_0^{a\sin\theta} du \int_0^{\frac{1}{2}\pi} d\phi\, u f'(u\sin\phi)\, e^{z(a\sin\theta-u)} + \psi(z)\, G,$$

whence

$$f(x) = \sum_{s=1}^{s=\infty} \frac{J_0(n_s x)}{aJ_0'(n_s a)}$$
$$\times\left[-\frac{2f(0)}{n_s} - \frac{1}{\pi}\int_{-\pi}^{\pi} d\theta \int_0^{a\sin\theta} du \int_0^{\frac{1}{2}\pi} d\phi\, u f'(u\sin\phi)\sin n_s(a\sin\theta - u)\right]$$
$$+ \sum_{s=1}^{s=\infty} \frac{L_0(n_s x)}{aJ_0'(n_s a)}\left[-\frac{1}{\pi}\int_{-\pi}^{\pi} d\theta \int_0^{a\sin\theta} du \int_0^{\frac{1}{2}\pi} d\phi\, u f'(u\sin\phi)\cos n_s(a\sin\theta - u)\right]. \tag{47}$$

The form (47) is very different from the one usually employed, which gives

$$f(x) = \sum_{s=1}^{s=\infty} \frac{2J_0(n_s x)}{a^2\{J_0'(n_s a)\}^2}\int_0^a J_0(n_s x) f(x)\, x\, dx \tag{48}$$

when $f(x)$ is an even function.

The forms (47) and (48) are not easily comparable directly, but, if we go back to the form (15) for $F(z)$, it is found that, if $i\kappa = n_s$, so that κ is any zero of $\psi(z)$,

$$\psi_{2r+1}(\kappa) = \psi_{2r}(\kappa) = -2i(\tfrac{1}{2}\kappa)^{2r+1}\frac{a^{-1}}{(r!)^2}\int_0^a \frac{x^{2r+1} J_0(n_s x)}{J_0'(n_s a)}\, dx,$$

whence

$$F(\kappa) = -ia^{-1}\int_0^a \frac{xJ_0(n_s x)}{J_0'(n_s a)}\, dx\left[\Sigma \frac{x^{2r} D^{2r}}{(2r)!} f(u)\right]_{u=0}$$
$$-\frac{i\pi}{2\kappa}a^{-1}\int_0^a \frac{J_0(n_s x)}{J_0'(n_s a)}\left[\Sigma\frac{x^{2r+1}D^{2r+1}}{(2r+1)!}\left(\frac{1.3\ldots(2r-1)(2r+1)}{2.4\ldots(2r)}\right)^2 f(u)\right]_{u=0} dx$$
$$= -ia^{-1}\int_0^a \frac{xJ_0(n_s x)}{J_0'(n_s a)}\,\frac{f(x)+f(-x)}{2}\, dx$$
$$-\frac{2i}{\pi\kappa}a^{-1}\int_0^a \frac{J_0(n_s x)}{J_0'(n_s a)}\, dx\int_0^1 dt\int_0^1 dv$$
$$\times\tfrac{1}{2}(1-t^2)^{-\frac{1}{2}}(1-v^2)^{-\frac{1}{2}}\frac{d^2}{dt\,dv}\{f(xtv) - f(-xtv)\},$$

using the identity

$$\left(\frac{1.3\ldots(2r+1)}{2.4\ldots(2r)}\right)^2 = \frac{4}{\pi^2}\int_0^1 dt\int_0^1 dv\,(1-t^2)^{-\frac{1}{2}}(1-v^2)^{-\frac{1}{2}}\frac{d^2}{dt\,dv}(tv)^{2r+1}.$$

This leads to the form

$$f(x) = \sum_{s=1}^{s=\infty} \frac{J_0(n_s x)}{a^2\{J_0'(n_s a)\}^2}\int_0^a J_0(n_s x)\{f(x)+f(-x)\}\,x\,dx$$

$$+\sum_{s=1}^{s=\infty} \frac{L_0(n_s x)}{\pi n_s a^2\{J_0'(n_s a)\}^2}\int_0^a J_0(n_s x)\,dx\int_0^1 dt\int_0^1 dv$$

$$\times(1-t^2)^{-\frac{1}{2}}(1-v^2)^{-\frac{1}{2}}\frac{d^2}{dt\,dv}\{f(xtv)-f(-xtv)\}. \quad (49)$$

The even terms have the coefficients found by the usual methods. The odd terms have their coefficients in a new form.

The comparison of the coefficients in (47) and (49) will be found to yield several interesting theorems connecting definite integrals involving the function J_0, which it would be difficult to establish otherwise.

Returning to the expansion in the form (47) which presents itself more naturally in this connection, we find that

$$\underset{z=\infty}{\mathrm{L}}\, zF(z) = f(0)+\left[J_0(iaD)\int_0^{\frac{1}{2}\pi} uf'(u\sin\phi)\,d\phi\right]_{u=0}$$

$$= f(0)+\frac{1}{2\pi}\left[\int_{-\pi}^{\pi} e^{aD\sin\theta}\,d\theta\int_0^{\frac{1}{2}\pi} uf'(u\sin\phi)\,d\phi\right]_{u=0}$$

$$= f(0)+\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\int_0^{\frac{1}{2}\pi} a\sin\theta\, f'(a\sin\theta\sin\phi)\,d\phi.$$

By considering this integral as taken over the surface of a sphere of radius a, θ being the colatitude and ϕ the longitude, and y being $a\sin\theta\sin\phi$, we find

$$\int_0^{\pi} d\theta\int_0^{\frac{1}{2}\pi} a\sin\theta\, f'(a\sin\theta\sin\phi)\,d\phi = \int f'(y)\,a^{-1}dS,$$

taken over the area of the lune bounded by $\phi = 0$, $\phi = \frac{1}{2}\pi$,

$$= \pi\int_0^a f'(y)\,dy = \pi[f(a)-f(0)].$$

In like manner

$$\int_{-\pi}^{0} d\theta\int_0^{\frac{1}{2}\pi} a\sin\theta\, f'(a\sin\theta\sin\phi)\,d\phi = \pi[f(-a)-f(0)].$$

Thus
$$\underset{z=\infty}{\mathrm{L}}\, zF(z) = \tfrac{1}{2}[f(a)+f(-a)].$$

Also, when $|z|$ is large,

$$\frac{Q(zx)}{J_0(iaz)} = \frac{e^{zx}\left(\dfrac{2}{\pi zx}\right)^{\frac{1}{2}} - \dfrac{2}{\pi zx}}{(2\pi az)^{-\frac{1}{2}}(e^{az}+e^{-az+\frac{1}{2}\pi i})}$$

if the imaginary part of z is positive; and

$$\frac{Q(xz)}{J_0(iaz)} = \frac{e^{xz}\left(\frac{2}{\pi zx}\right)^{\frac{1}{2}} - \frac{2}{\pi zx}}{(-2\pi az)^{-\frac{1}{2}}(e^{-az}+e^{az+\frac{1}{2}\pi i})}$$

if the imaginary part of z is negative.

Reasoning strictly analogous to that used in previous cases shows that, if $|x| < a$, $|Q(zx)/J_0(iaz)|$ tends to zero when z tends to infinity in any direction save that of the imaginary axis.

If $x = a$, $Q(zx)/J_0(iaz)$ tends to 2 if the real part of z be positive, and to 0 if the real part of z be negative.

If $x = -a$, $Q(xz)/J_0(iaz)$ tends to 0 if the real part of z be positive, and to 2 if the real part of z be negative.

If $|x| > a$, $|Q(zx)/J_0(iaz)|$ tends to ∞ over one half of the contour C, and to 0 over the other half.

Finally, it may be proved that the neighbourhood of the imaginary axis contributes nothing in the limit to the contour integral.

Thus, if $|x| < a$, $\quad \mathrm{L}\, \frac{1}{2\pi i}\int_C F(z)\frac{\phi(zx)}{\psi(z)}\,dz = 0,$

and the series converges to $f(x)$.

If $|x| > a$, $\mathrm{L}\, \frac{1}{2\pi i}\int_C F(z)\frac{\phi(zx)}{\psi(z)}\,dz$ need not be finite.

If $x = \pm a$, $\mathrm{L}\, \frac{1}{2\pi i}\int_C \frac{F(z)\phi(zx)}{\psi(z)}\,dz = \frac{1}{2}[f(a)+f(-a)]$.

$$\left.\begin{array}{l}\text{Therefore value of series when } x = a \text{ is } \frac{1}{2}[f(a)-f(-a)] \\ \phantom{\text{Therefore }}\text{value of series when } x = -a \text{ is } \frac{1}{2}[f(-a)-f(a)]\end{array}\right\}. \tag{50}$$

This result shows that, whereas the even part of the series is, in general, discontinuous for the ends of the range of validity, being zero for $x = \pm a$ [which is, indeed, immediately obvious from the equation $J_0(n_s a) = 0$], the odd part remains continuous up to the ends of the range inclusive.

9. *Possibility of Extension of the above Results to Functions other than Polynomials.*

It is well known that every function $f(x)$ which can be represented by a Fourier's series between a and b can also be represented throughout the same range as the limit of polynomials.

Thus, let $f(x) = \underset{n=\infty}{\mathrm{L}} P_n(x)$, where $P_n(x)$ is a polynomial.

$P_n(x)$, by the preceding work, can be expanded in a series of suitable functions

$$P_n(x) = \sum_{r=1}^{r=\infty} A_{r,n}\phi(\kappa_r, x).$$

Thus,
$$f(x) = \underset{n=\infty}{\mathrm{L}} \sum_{r=1}^{r=\infty} A_{r,n}\phi(\kappa_r, x).$$

It seems difficult to prove generally that in the cases where the expansion of $P_n(x)$ is possible the limiting sign can be taken through the sign of summation.

If we *assume* that this can be done, then

$$f(x) = \sum_{r=1}^{r=\infty} A_r\phi(\kappa_r, x)$$

where
$$A_r = \underset{n=\infty}{\mathrm{L}} A_{r,n}.$$

In one fairly simple case, where $f(x)$ is expansible in an infinite power series, so that $P_n(x) =$ sum of the first n terms of the Taylor series for $f(x)$, we can prove that, under certain restrictions, the present method allows us to calculate the coefficients A_r—in other words, that $A_{r,n}$ tends to a limit when n increases.

We now proceed to prove this.

Consider the expression (15) for $F(z)$ and write it

$$\begin{aligned} F(z) = \psi(z)\left[\frac{f(0)}{\phi(0)}\frac{1}{z}+\frac{f'(0)}{\phi'(0)}\frac{1}{z^2}+\ldots+\frac{f^n(0)}{\phi^n(0)}\frac{1}{z^{n+1}}\right]& \\ -(a_1\ +a_2z\ +a_3z^2\ +\ldots)&\frac{f(0)}{\phi(0)} \\ -(a_2\ +a_3z\ +a_4z^2\ +\ldots)&\frac{f'(0)}{\phi'(0)} \\ -\ \ldots\qquad\ldots\qquad\ldots\qquad&\ldots \\ -(a_{n+1}+a_{n+2}z+a_{n+3}z^2+\ldots)&\frac{f^n(0)}{\phi^n(0)}. \end{aligned} \tag{51}$$

Consider first the part of $F(z)$ in square brackets. $\phi(z)$ is an integral function; hence by a well known result

$$|\phi^n(0)| < \frac{Mn!}{\zeta^n},$$

ζ being any positive constant, however large.

Also, if the power series for $f(x)$ have a finite radius of convergence ρ, then it is clear that we *cannot* have for *all* values of n, however large,

$$|\ f^n(0)| < \frac{\mu n!}{\rho'^n},$$

ρ' being any quantity greater than ρ.

Therefore terms must exist in the sum in square brackets which are numerically greater than

$$\frac{\mu}{M}\,\frac{\zeta^n}{|z|^{n+1}\rho'^n},$$

and this must occur for values of n as large as we please.

Thus, if we make n infinite, terms exceeding any given magnitude will appear in the series in square brackets, which is therefore divergent. If $f(x)$ were an integral function whose coefficients decreased at a sufficiently rapid rate, this part of $F(z)$ might be convergent; but this will be a comparatively rare case.

The divergence of this part of the expression for $F(z)$ is, however, immaterial, since in calculating the coefficients we put $z=\kappa$ in $F(z)$, where $\psi(\kappa)=0$. The part in question therefore disappears.

To deal with the other part we notice that in all the examples considered $\psi(z)$ has been an integral function.

We shall suppose that $\psi(z)$ is such a function, and further that from a certain value of r, $|a_r|<q^r/r!$, q being some positive quantity, a condition that will always be satisfied by integral functions of order zero. (See Poincaré, "Mémoire sur les fonctions entières," *Bulletin de la Société Mathématique de France*, 1883.)

Then, from this value of r,

$$\begin{aligned}|a_{r+1}+a_{r+2}z+\dots| &\leqslant |a_{r+1}|+|a_{r+2}|\,|z|+\dots\\ &\leqslant \frac{q^{r+1}}{(r+1)!}\left\{1+\frac{q\,|z|}{r+2}+\frac{(q\,|z|)^2}{(r+2)(r+3)}+\dots\right\}\\ &< \frac{q^{r+1}}{(r+1)!}\left(1-\frac{q\,|z|}{r+2}\right)^{-1}<\frac{q^{r+1}}{(r+1)!}\,\frac{\lambda}{\lambda-1},\end{aligned}$$

r being taken so large that $r+2>\lambda q\,|z|$, where λ is any fixed number greater than 1. Hence, if the series

$$\Sigma\frac{q^{r+1}}{(r+1)!}\,\frac{f^r(0)}{\phi^r(0)} \tag{52}$$

be absolutely convergent, the second part of $F(z)$ is also absolutely convergent. We may therefore increase n without limit, and use this series to calculate the limiting values of the coefficients. So far I have not been able to complete the demonstration and to show that these are the actual coefficients in the expansion of $f(x)$ itself.

10. *Application of the Theorem to Cases where $\phi(z, x)$ is not a Function of zx only.*

Return now to the more general case, where $\phi(z, x)$ is not a function of the product zx only. In this case the expansion in functions $\phi(\kappa_r, x)$ is known by (10), when the expansion (11) of $f(x)$ in terms of the functions $f_0(x) \dots f_n(x), \dots$, which are the coefficients in the expansion of $\phi(z, x)$ in powers of z, is known.

Now, if $f(x)$ be a polynomial, the expansion of $f(x)$ in functions $f_n(x)$ is easily obtained in the following case, namely, when $f_0(x), f_1(x), \dots, f_n(x)$ are themselves polynomials of increasing degrees 0, 1, 2, ..., n.

In this case, if $f(x)$ be a polynomial of degree n, all the quantities $p_{n+1}, p_{n+2}, \dots$ in the expansion (11) may be taken zero. Let

$$\left.\begin{aligned} f_0(x) &= q_{00} \\ f_1(x) &= q_{10}+q_{11}x \\ f_2(x) &= q_{20}+q_{21}x+q_{22}x^2 \\ \dots & \quad \dots \quad \dots \quad \dots \quad \dots \\ f_n(x) &= q_{n0}+q_{n1}x+q_{n2}x^2+\dots+q_{nn}x^2 \end{aligned}\right\}. \tag{53}$$

Let $$f(x) = a_0+a_1x+\dots+a_nx^n. \tag{54}$$

Then, to determine $p_0, p_1, \dots, p_n$, we have the $(n+1)$ equations

$$\left.\begin{aligned} p_0q_{00}+p_1q_{10}+p_2q_{20}+\dots+p_nq_{n0} &= a_0 \\ p_1q_{11}+p_2q_{21}+\dots+p_nq_{n1} &= a_1 \\ p_2q_{22}+\dots+p_nq_{n2} &= a_2 \\ \dots \quad \dots \quad \dots & \\ p_nq_{nn} &= a_n \end{aligned}\right\}, \tag{55}$$

of which the solution is

$$p_n = \frac{a_n}{q_{nn}},$$

$$p_{n-1} = \frac{a_{n-1}}{q_{n-1,\,n-1}} - \frac{a_n q_{n,\,n-1}}{q_{nn}q_{n-1,\,n-1}},$$

$$p_{n-2} = \frac{a_{n-2}}{q_{n-2,\,n-2}} - \frac{a_{n-1}q_{n-1,\,n-2}}{q_{n-1,\,n-1}\,q_{n-2,\,n-2}} + \frac{a_n \begin{vmatrix} q_{n-1,\,n-2} & q_{n,\,n-2} \\ q_{n-1,\,n-1} & q_{n,\,n-1} \end{vmatrix}}{q_{n,\,n}\,q_{n-1,\,n-1}\,q_{n-2,\,n-2}}$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

... ...

$$p_{n-r} = \frac{a_{n-r}}{q_{n-r,\,n-r}}$$

$$-\frac{a_{n-r+1}\,q_{n-r+1,\,n-r}}{q_{n-r+1,\,n-r+1}\,q_{n-r,\,n-r}}$$

$$+\frac{a_{n-r+2}\begin{vmatrix} q_{n-r+1,\,n-r} & q_{n-r+2,\,n-r} \\ q_{n-r+1,\,n-r+1} & q_{n-r+2,\,n-r+1}\end{vmatrix}}{q_{n-r+2,\,n-r+2}\,q_{n-r+1,\,n-r+1}\,q_{n-r,\,n-r}}$$

$$+\frac{(-1)^s a_{n-r+s}\begin{vmatrix} q_{n-r+1,\,n-r} & q_{n-r+2,\,n-r} & \cdots & q_{n-r+s-1,\,n-r} & q_{n-r+s,\,n-r} \\ q_{n-r+1,\,n-r+1} & q_{n-r+2,\,n-r+1} & \cdots & q_{n-r+s-1,\,n-r+1} & q_{n-r+s,\,n-r+1} \\ 0 & q_{n-r+2,\,n-r+2} & \cdots & q_{n-r+s-1,\,n-r+2} & q_{n-r+s,\,n-r+2} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & q_{n-r+s-1,\,n-r+s-1} & q_{n-r+s,\,n-r+s-1}\end{vmatrix}}{q_{n-r+s,\,n-r+s}\cdots q_{n-r,\,n-r}}$$

$$+\cdots$$

$$+\frac{(-1)^r a_n\begin{vmatrix} q_{n-r+1,\,n-r} & q_{n-r+2,\,n-r} & \cdots & q_{n-1,\,n-r} & q_{n,\,n-r} \\ q_{n-r+1,\,n-r+1} & q_{n-r+2,\,n-r+1} & \cdots & q_{n-1,\,n-r+1} & q_{n,\,n-r+1} \\ 0 & q_{n-r+2,\,n-r+2} & \cdots & q_{n-1,\,n-r+2} & q_{n,\,n-r+2} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & q_{n-1,\,n-1} & q_{n,\,n-1}\end{vmatrix}}{q_{n,\,n}\cdots q_{n-r,\,n-r}}; \tag{56}$$

so that the coefficients p are obtained in finite form. $F(z)$ is then known by (10) and the expansion (6) follows.

11. *Application to Functions occurring in the Theory of Elasticity.*

The equations for the mean stresses P, Q, S in the plane of an elastic plate are

$$\frac{dP}{dx}+\frac{dS}{dy}=0, \qquad \frac{dS}{dx}+\frac{dQ}{dy}=0,$$

and these are satisfied by

$$P=-\frac{d^2E}{dy^2}, \qquad Q=-\frac{d^2E}{dx^2}, \qquad S=\frac{d^2E}{dx\,dy},$$

where $\nabla^4E=0$. (See a paper by the author on "An Approximate Solution for the Bending of a Beam of Rectangular Cross-Section," *Phil. Trans.*, A, Vol. 201, pp. 63–155; and also a paper by John Dougall, M.A., "An Analytical Theory of the Equilibrium of an Isotropic Elastic Plate," *Trans. Roy. Soc. Edin.*, Vol. XLI., Pt. 1, No. 8, 1904. See also for an analysis of the above Professor Love's *Theory of Elasticity*, second edition, chapters v. and ix.)

If we take $\quad E = (C \sinh \kappa x + Dx \cosh \kappa x) \cos \kappa y,$

then
$$P = \kappa^2 [C \sinh \kappa x + Dx \cosh \kappa x] \cos \kappa y,$$
$$Q = -\kappa [(C\kappa + 2D) \sinh \kappa x + D\kappa x \cosh \kappa x] \cos \kappa y,$$
$$S = -\kappa [(C\kappa + D) \cosh \kappa x + D\kappa x \sinh \kappa x] \sin \kappa y.$$

If $P = 0$, $S = 0$, when $x = \pm b$, then
$$C \sinh \kappa b + Db \cosh \kappa b = 0,$$
$$(C\kappa + D) \cosh \kappa b + D\kappa b \sinh \kappa b = 0.$$

Hence
$$\sinh 2\kappa b - 2\kappa b = 0. \tag{57}$$

We have
$$Q = \text{const.} [\kappa x \cosh \kappa x \sinh \kappa b - \kappa b \cosh \kappa b \sinh \kappa x + 2 \sinh \kappa b \sinh \kappa x] \cos \kappa y.$$

It is easy to verify that when κ is a root of (57)
$$\int_{-b}^{b} Qx \, dx = 0. \tag{58}$$

Now for various purposes it is desirable to be able to expand a given function of x in terms of functions $\phi(\kappa_r, x)$ where
$$\phi(z, x) = zx \cosh zx \sinh zb - zb \cosh zb \sinh zx + 2 \sinh zb \sinh zx, \tag{59}$$
where $z = \kappa_r$ is any root of
$$\psi(z) = \sinh 2zb - 2zb = 0. \tag{60}$$

For example, if Q be given over $y = 0$, between $x = -b$ and $x = +b$, such an expansion will give us the coefficients of the typical terms which build up the complete solution.

We notice, however, that, if $f(x)$ be the function to be expanded, then, owing to (58),
$$\int_{-b}^{b} f(x)\, x \, dx = 0\,; \tag{61}$$
and therefore $f(x)$ is not entirely arbitrary.

The expansion of $\phi(z, x)$ in powers of z is as follows :—
$$\phi(z, x) = \sum_{r=1}^{r=\infty} \frac{z^{2r}}{(2r)!} [(b+x)^{2r} - (b-x)^{2r} - r(b^2 - x^2) \{(b+x)^{2r-2} - (b-x)^{2r-2}\}]. \tag{62}$$

Thus
$$f_{2r+1}(x) = 0,$$
$$f_0(x) = 0,$$
$$f_{2r}(x) = \frac{1}{(2r)!} [(b+x)^{2r} - (b-x)^{2r} - r(b^2 - x^2) \{(b+x)^{2r-2} - (b-x)^{2r-2}\}]$$
$$= 2 \sum_{s=1}^{s=r} \frac{x^{2s-1} b^{2r-2s+1} (2s-r)}{(2s-1)!\,(2r-2s+1)!}. \tag{63}$$

It follows that only an *odd* function will be suitable for $f(x)$.

If $$f(x) = a_1 x + a_3 x^3 + \ldots + a_{2r-1} x^{2r-1},$$

then, since from (76) it is seen that the highest power of x in $f_{2r}(x)$ is x^{2r-1}, we have

$$f(x) = p_2 f_2(x) + p_4 f_4(x) + \ldots + p_{2r} f_{2r}(x), \tag{64}$$

where $p_2, p_4, \ldots, p_{2r}$ can be found in the manner indicated in the preceding section.

The equations for $p_2, p_4, \ldots, p_{2r}$ may be here quoted: they are

$$\left.\begin{aligned}
\frac{(2-1)\,p_2 b^2}{1!} + \frac{(2-2)\,p_4 b^4}{3!} + \frac{(2-3)\,p_6 b^6}{5!} + \ldots + \frac{(2-r)\,p_{2r} b^{2r}}{(2r-1)!} &= \frac{1!\,a_1 b}{2} \\
\frac{(4-2)\,p_4 b^4}{1!} + \frac{(4-3)\,p_6 b^6}{3!} + \ldots + \frac{(4-r)\,p_{2r} b^{2r}}{(2r-3)!} &= \frac{3!\,a_3 b^3}{2} \\
\frac{(6-3)\,p_6 b^6}{1!} + \ldots + \frac{(6-r)\,p_{2r} b^{2r}}{(2r-5)!} &= \frac{5!\,a_5 b^5}{2} \\
\ldots \qquad \ldots \qquad \ldots \qquad & \ldots \\
\frac{(2r-r)\,p_{2r} b^{2r}}{1!} &= \frac{(2r-1)!\,a_{2r-1} b^{2r-1}}{2}
\end{aligned}\right\}. \tag{65}$$

12. *Zeroes of* $\sinh 2zb - 2zb = 0$.

We have now to consider the distribution of the zeroes of $\psi(z)$.

In the first place the expansion of $\sinh 2zb - 2zb$ begins with a term in z^3, so that the origin is a triple zero.

To find the other zeroes write

$$2zb = \xi + i\eta.$$

Then $$\sinh(\xi + i\eta) = \xi + i\eta,$$

and, equating real and imaginary parts,

$$\sinh\xi \cos\eta = \xi, \tag{66}$$

$$\sin\eta \cosh\xi = \eta. \tag{67}$$

If we put $\eta = 0$, we have $\sinh\xi = \xi$,

which is impossible, unless $\xi = 0$; and, if we put $\xi = 0$, we have

$$\sin\eta = \eta,$$

which is also impossible, unless $\eta = 0$.

Thus no root lies on the axes, except the triple root $z = 0$.

Consider now the position of the roots of very large modulus.

Squaring (66) and (67) and adding, we find

$$\cosh^2 \xi = \cos^2 \eta + \xi^2 + \eta^2.$$

If, therefore, $|\xi + i\eta| = \sqrt{(\xi^2+\eta^2)}$ is large, ξ must be large.

Hence, by (66), $$\cos \eta = \frac{\xi}{\sinh \xi},$$

and is small; therefore $$\eta = n\pi + \tfrac{1}{2}\pi$$

approximately. Putting this value into (67), then, since $\cosh \xi$ must be positive, n must be *even*.

Thus $$\eta = 2r\pi + \tfrac{1}{2}\pi, \qquad \cosh \xi = 2r\pi + \tfrac{1}{2}\pi$$

very nearly, or $$\xi = \log_e(4r\pi + \pi)$$

approximately. The roots fall into groups of four, symmetrically placed with regard to the axes, the four members of each group being given by $\pm \log_e(4r\pi+\pi) \pm i(2r\pi + \tfrac{1}{2}\pi)$ approximately.

18. *Additional Terms due to the Triple Zero at $z = 0$.*

Referring to the expression (21), we see that the additional terms due to the triple root $z = 0$ are

$$f_0(x) \sum_{s=0}^{s=\infty} (b_0 a_{s+3} + b_1 a_{s+2} + b_2 a_{s+1}) p_s + f_1(x) \sum_{s=0}^{s=\infty} (b_0 a_{s+2} + b_1 a_{s+1}) p_s$$
$$+ f_2(x) \sum_{s=0}^{s=\infty} (b_0 a_{s+1}) p_s,$$

and, since $f_0(x) = 0$, $f_1(x) = 0$, and $f_2(x) = 2xb$, this reduces to

$$2xb \sum_{s=0}^{s=\infty} (b_0 a_{s+1}) p_s.$$

Now $$\psi(z) = \frac{(2zb)^3}{3!} + \frac{(2zb)^5}{5!} + \ldots + \frac{(2zb)^{2r+1}}{(2r+1)!} + \ldots.$$

Thus $$a_0 = a_1 = a_2 = 0, \qquad a_{2r} = 0,$$

$$a_{2r+1} = \frac{(2b)^{2r+1}}{(2r+1)!}, \qquad b_0 = \frac{1}{a_3} = \frac{3!}{(2b)^3}.$$

Thus the additional term is

$$\frac{3x}{2b^2} \sum_{s=1}^{s=r} \frac{(2b)^{2s+1}}{(2s+1)!} p_{2s},$$

since, if $s > 2r$, $p_s = 0$. The quantities p_{2r} are here given by (65).

But this term may be evaluated as follows:—Rewriting the first expression (63) for $f_{2r}(x)$,

$$f_{2r}(x) = \frac{1}{(2r)!}\left[(r+1)\left\{(b+x)^{2r}-(b-x)^{2r}\right\}-2br\left\{(b+x)^{2r-1}-(b-x)^{2r-1}\right\}\right],$$

whence

$$\begin{aligned} xf_{2r}(x) = \frac{1}{(2r)!}\Big[&(r+1)\left\{(b+x)^{2r+1}+(b-x)^{2r+1}\right\} \\ &-b(3r+1)\left\{(b+x)^{2r}+(b-x)^{2r}\right\} \\ &+2rb^2\left\{(b+x)^{2r-1}+(b-x)^{2r-1}\right\}\Big] \end{aligned}$$

and

$$\begin{aligned} \int_{-b}^{b} xf_{2r}(x)\,dx &= \frac{1}{(2r)!}\left[\frac{r+1}{2r+2}\,2(2b)^{2r+2}-\frac{b(3r+1)}{2r+1}\,2(2b)^{2r+1}+\frac{2rb^2}{2r}\,2(2b)^{2r}\right] \\ &= \frac{2^{2r+1}b^{2r+2}}{(2r+1)!}. \end{aligned}$$

Thus
$$\begin{aligned} \sum_{s=1}^{s=r}\frac{(2b)^{2s+1}p_{2s}}{(2s+1)!} &= \frac{1}{b}\int_{-b}^{b} x\sum_{s=1}^{s=r} p_{2s}f_{2s}(x)\,dx \\ &= \frac{1}{b}\int_{-b}^{b} xf(x)\,dx \text{ by (64).} \end{aligned}$$

The additional term in the expansion is therefore

$$\frac{3x}{2b^3}\int_{-b}^{b} xf(x)\,dx. \tag{68}$$

Therefore

$$\begin{aligned} f(x) &= \frac{3x}{2b^3}\int_{-b}^{b} xf(x)\,dx-\Sigma\,\frac{F(\kappa_r)}{\psi'(\kappa_r)}\,\phi(\kappa_r, x) \\ &= \frac{3x}{2b^3}\int_{-b}^{b} xf(x)\,dx-\Sigma\,\frac{F(\kappa_r)\left[\kappa_r x\cosh\kappa_r x\sinh\kappa_r b-\kappa_r b\cosh\kappa_r b\sinh\kappa_r x+2\sinh\kappa_r b\sinh\kappa_r x\right]}{2b\left\{\cosh 2\kappa_r b-1\right\}} \end{aligned}$$

where κ_r is any root of (60), and

$$\begin{aligned} F(z) = \sum_{s=1}^{s=r}\frac{p_{2s}\psi_{2s}(z)}{z^{2s+1}} = {} & \frac{1}{z^2}\left\{p_4\frac{(2b)^3}{3!}+p_6\frac{(2b)^5}{5!}+\ldots+p_{2r}\frac{(2b)^{2r-1}}{(2r-1)!}\right\} \\ &+\frac{1}{z^4}\left\{p_6\frac{(2b)^3}{3!}+p_8\frac{(2b)^5}{5!}+\ldots+p_{2r}\frac{(2b)^{2r-3}}{(2r-3)!}\right\} \\ &+\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots \\ &+\frac{1}{z^{2r-2}}\left\{p_{2r}\frac{(2b)^3}{3!}\right\}. \end{aligned} \tag{69}$$

Consider the roots in groups of four. Connecting the two in each group which have opposite signs, then, since $F(z)$ contains only even powers of z, we may write

$$f(x) = \frac{3x}{2b^3}\int_{-b}^{b} xf(x)\,dx - \Sigma \frac{F(\kappa_r)\left[\kappa_r x\cosh\kappa_r x\sinh\kappa_r b - \kappa_r b\cosh\kappa_r b\sinh\kappa_r x + 2\sinh\kappa_r b\sinh\kappa_r x\right]}{b\left\{\cosh 2\kappa_r b - 1\right\}} \tag{70}$$

where the Σ now extends only to those roots of (60) of which the real part is positive.

As an example, if we wish to expand x^3 in such a series, we have from (65), putting $a_1 = 0$, $a_3 = 1$, and all the other a's zero,

$$p_4 = \frac{3}{2b}.$$

Hence
$$F(z) = \frac{2b^2}{z^2}$$

and
$$\int_{-b}^{b} xf(x)\,dx = \frac{2b^5}{5}.$$

Therefore

$$x^3 = \tfrac{3}{5}xb^2 - 2b\,\Sigma\frac{1}{\kappa_r^2}\frac{\left[\kappa_r x\cosh\kappa_r x\sinh\kappa_r b - \kappa_r b\cosh\kappa_r b\sinh\kappa_r x + 2\sinh\kappa_r b\sinh\kappa_r x\right]}{(\cosh 2\kappa_r b - 1)}.$$

We notice that the terms introduced by the zeroes at the origin ensure that the function which is represented by the sum of the series in (70) always satisfies the condition (61).

14. *Limits and Validity of this Expansion.*

Looking at the expression (69) for $F(z)$, we see that the most important terms when $|z|$ is large involve $1/z^2$, the terms in $1/z$ being absent from the expansion.

$$\underset{z=\infty}{\mathrm{L}}\, z^2F(z) = p_4\frac{(2b)^3}{3!} + p_6\frac{(2b)^5}{5!} + \ldots + p_{2r}\frac{(2b)^{2r-1}}{(2r-1)!}.$$

Now

$$f'_{2s}(x) = \frac{1}{(2s-1)!}\left[(s+1)\left\{(b+x)^{2s-1} + (b-x)^{2s-1}\right\} - (2s-1)\,b\left\{(b+x)^{2s-2} + (b-x)^{2s-2}\right\}\right].$$

Hence, when $s > 1$,
$$f'_{2s}(b) = \tfrac{3}{2}\frac{(2b)^{2s-1}}{(2s-1)!}.$$

Thus $p_4\dfrac{(2b)^3}{3!}+p_6\dfrac{(2b)^5}{5!}+\ldots+p_{2r}\dfrac{(2b)^{2r-1}}{(2r-1)!} = \tfrac{2}{3}\left[p_4 f_4'(b)+\ldots+p_{2r}f_{2r}'(b)\right]$

$$= \tfrac{2}{3}\left[f'(b)-p_2 f_2'(b)\right].$$

But $$f_2(x) = 2bx.$$

Therefore $$f_2'(b) = 2b.$$

Hence $$\underset{z=\infty}{\mathrm{L}}\, z^2F(z) = \tfrac{2}{3}f'(b)-\tfrac{4}{3}bp_2. \tag{71}$$

It follows that the remainder after n terms in (70)—the terms involving conjugate imaginaries being counted as one term—is given by

$$\frac{1}{2\pi i}\int_C \{z^2F(z)\}\left\{\frac{x\cosh zx\sinh zb-b\cosh zb\sinh zx+(2/z)\sinh zb\sinh zx}{\sinh 2zb-2zb}\right\}\frac{dz}{z}, \tag{72}$$

the contour C being a circle passing through the points $\pm i(4r+3)\,\pi/4b$.

Referring to the results of Art. 12, we see that when r is large

$$\frac{\xi}{\eta} = \frac{\log_e(4r+1)\,\pi}{\tfrac{1}{2}(4r+1)\,\pi},$$

and this tends to zero with r. The roots, after a certain value of r, are contained within an angle 2ϵ enclosing the imaginary axis, where 2ϵ may be taken as small as we please.

Again, if r be large enough, all the roots for which $\eta \leqslant (\pi/4b)(4r+1)$ lie inside the circle C.

For, if (ξ', η) be *on* the circle and

$$\eta = (\pi/4b)(4r+1), \qquad \xi'^2 = (\pi/2b)(2R-\pi/2b),$$

R being the radius of the circle, that is $R = (\pi/4b)(4r+3)$. Thus

$$\xi' = \pi/2b\,\sqrt{(4r+2)}$$

and $$\xi/\xi' = \log\{4r+1)\,\pi\}/\pi\sqrt{(4r+2)} = 0$$

when $r=\infty$.

Therefore, if r be large enough, (ξ, η) lies inside the circle C, and the roots for which $\eta < (\pi/4b)(4r+1)$ can easily be shown to lie inside the circle C.

We will consider first those parts of the contour integral which lie inside the angle 2ϵ on the arc which is bisected by the positive half of the imaginary axis; the work for the arc which is bisected by the negative half of the imaginary axis is precisely similar.

Let $z = p+iq$, as before, be any point on the circle C. Then

$$\left|\frac{\phi(z, x)}{z\psi(z)}\right| = \left|\frac{(x\cosh zx \sinh zb - b\cosh zb \sinh zx + (2/z)\sinh zb \sinh zx)}{\sinh 2zb - 2zb}\right|$$

$$\leqslant \frac{|x|\,|\cosh zx|\,|\sinh zb| + b\,|\cosh zb|\,|\sinh zx| + (2/R)\,|\sinh zb|\,|\sinh zx|}{|\sinh 2zb - 2zb|}.$$

Now $$|\cosh(px+iqx)| < \tfrac{1}{2}|e^{px+iqx}| + \tfrac{1}{2}|e^{-px-iqx}|$$
$$< \tfrac{1}{2}(e^{px}+e^{-px}) < e^{p|x|}.$$

Similarly $|\sinh zb| < e^{pb}$, $|\cosh zb| < e^{pb}$, $|\sinh zx| < e^{p|x|}$.

Thus $$\left|\frac{\phi(z, x)}{z\psi(z)}\right| < \frac{\{|x|+b+2/R\}\, e^{p(b+|x|)}}{|\sinh 2zb - 2zb|}.$$

Take p so large that $\sinh 2pb - 2Rb > \tfrac{1}{2}\lambda\, e^{2pb}$, (73)

λ being some positive quantity < 1. Then

$$|\sinh 2zb - 2zb| > |\sinh 2zb| - |2zb|,$$

and $$|\sinh 2zb| > \tfrac{1}{2}|e^{2zb}| - \tfrac{1}{2}|e^{-2zb}| > \tfrac{1}{2}e^{2pb} - \tfrac{1}{2}e^{-2pb} > \sinh 2pb.$$

Thus $$|\sinh 2zb - 2zb| > \sinh 2pb - 2Rb > \tfrac{1}{2}\lambda\, e^{2pb},$$

and for values of p which satisfy the inequality (73)

$$\left|\frac{\phi(z, x)}{z\psi(z)}\right| < \frac{2}{\lambda}\left\{|x| + b + \frac{2}{R}\right\} e^{p[|x|-b]};$$

and is therefore finite if $-b \leqslant x \leqslant +b$, however large R may be.

The first value of p which satisfies the inequality (73) is given by

$$\tfrac{1}{2}(1-\lambda)\, e^{2pb} = \tfrac{1}{2}e^{-2pb} + 2Rb.$$

This leads to a large value of p when R is large, and thus, to a first approximation, this limiting value of p is given by

$$\tfrac{1}{2}(1-\lambda)\, e^{2pb} = 2Rb$$

or $$2pb = \log\{4bR/(1-\lambda)\}.$$

Thus $$\frac{p}{\xi'} = \frac{\log\{4bR/(1-\lambda)\}}{\pi\sqrt{(4r+2)}} = \frac{\log\left\{\dfrac{\pi(4r+3)}{1-\lambda}\right\}}{\pi\sqrt{(4r+2)}}.$$

This ratio becomes very small as r (and therefore R) increases. Hence the parts of the contour for which p does not satisfy the inequality (73) are on a small arc $2\epsilon'$ bisected by the imaginary axis and ultimately very small compared with the arc 2ϵ.

Consider now the values of $\left|\dfrac{\phi(z, x)}{z\psi(z)}\right|$ over this arc $2\epsilon'$. We have

$$R-q<p^2/R, \qquad q>R-p^2/R.$$

Hence, *a fortiori*,

$$q>(4r+3)\frac{\pi}{4b}-\frac{1}{4b^2R}\left[\log\left(\frac{4Rb}{1-\lambda}\right)\right]^2,$$

by increasing R the second term can be made numerically less than any assigned quantity $\theta/2b$, and we have

$$\tfrac{1}{2}(4r+3)\pi>2qb>\tfrac{1}{2}(4r+3)\pi-\theta$$

all over the arc $2\epsilon'$.

Now

$$|\sinh 2zb-2zb| = |\sinh 2pb\cos 2qb-2pb+i(\cosh 2pb\sin 2qb-2qb)|$$
$$\geqslant |\cosh 2pb\sin 2qb-2qb|.$$

Now $\sin 2qb$ lies between -1 and $-\cos\theta$. Therefore

$$\cosh 2pb\sin 2qb-2qb$$

lies between $-\cosh 2pb-2qb$ and $-\cosh 2pb\cos\theta-2qb$.

Thus $|\cosh 2pb\sin 2qb-2qb|>\cosh 2pb\cos\theta+2qb>\tfrac{1}{2}e^{2pb}\cos\theta$.

Accordingly, inside the arc $2\epsilon'$

$$\left|\frac{\phi(z, x)}{z\psi(z)}\right|<2\left\{|x|+b+\frac{2}{R}\right\}e^{p[|x|-b]}\sec\theta,$$

and this is finite for $-b\leqslant x\leqslant b$, since in this case $e^{p[|x|-b]}\leqslant 1$.

It follows that the parts of the integral due to the whole arc 2ϵ are less than

$$2\epsilon' 2\left\{|x|+b+\frac{2}{R}\right\}\sec\theta+(2\epsilon-2\epsilon')\frac{2}{\lambda}\left\{|x|+b+\frac{2}{R}\right\},$$

and. since we can take $\cos\theta>\lambda$, this is less than

$$\frac{4\epsilon}{\lambda}\left\{|x|+b+\frac{2}{R}\right\},$$

which tends to zero with ϵ. These parts then contribute nothing in the limit to the integral.

Now consider the parts outside the angle 2ϵ. It is easy to show that, if $|x|<b$, then over those parts

$$\underset{R=\infty}{\mathrm{L}}\left|\frac{\phi(z, x)}{z\psi(z)}\right| \doteq 0,$$

and they ultimately contribute nothing to the contour integral; the latter then vanishes when $R = \infty$, and the expansion holds.

If $x = \pm b$,

$$\left| \frac{\phi(z, x)}{z\psi(z)} \right| = \left| \frac{2/z \sinh^2 zb}{\sinh 2zb - 2zb} \right| = \frac{1}{R}$$

when R is large. In this case also the contour integral vanishes when $R = \infty$. The expansion holds therefore for the ends of the range of validity.

If $|x| > b$,
$$\left| \frac{\phi(z, x)}{z\psi(z)} \right| = \infty$$

when $R = \infty$, and the expansion is not valid.

PARTIAL DIFFERENTIAL EQUATIONS: SOME CRITICISMS AND SOME SUGGESTIONS

By A. R. FORSYTH.

Presidential Address.

[Delivered at the Annual General Meeting, November 8th, 1906.]

THE rules of our Society are discreetly silent concerning Presidential Addresses: but there are two unformulated customs which maintain absolute sway over these performances. One of the customs is marked by a lack of logic which could easily be remedied, if remedy were desired. Such an address is never delivered by the President: yet there may be wisdom in our practice, which does not allow the authority of high office to confer, upon the expression of individual opinions, an importance that is beyond their merits. By the other custom, an indulgent freedom is conceded to the immediate ex-President: he is unrestricted in the selection of a subject; and our records shew that the freedom has been amply used in the past. Following the examples set by my predecessors, I too have made a choice of a subject that is of some interest to me. It is undoubtedly technical in character; but no apology is offered on that score, because technical studies are the bond of the Society. I invite your attention to the consideration of some aspects of the theory of partial differential equations, with especial reference to the present state of the practical methods adopted for the integration of such equations. My remarks are an estimate of work that is old, rather than a summary of recent advances or an enunciation of new results: they belong to the domain of history as much as to current thought, and they hardly trench upon the domain of prophecy. Amid the occasional fevered rushes after new developments, there is sometimes an advantage in a careful review of old ground. Such a review can indicate gaps to be filled; it can recall to mind the forgotten reasons why progress in various directions has not gone beyond some particular stage; and, if nothing else is done, there is a reminder of old problems which, not proved to be insoluble, yet remain unsolved. But, while indicating some probable advantages to be derived from a careful review of an old subject, I ought to warn you at the very threshold that this review will be distinctly limited, not least

by the conditions under which it is presented. For instance, I shall say nothing about the applications of partial differential equations to geometry or to physics, nothing of modern developments in the region of boundary problems, nothing as to the association of characteristics with the theory of the equations, nothing of the formal results obtained by the introduction of groups into the theory. As already indicated, my chief topic is related to the practical integration of the equations, dealing mainly with the significance and the limitations of the methods adopted.

I.

As integration continues to be an inverse process in the present state of analysis, it is interesting to select and isolate all the actual inverse operations which are performed during the construction of an integral of a partial differential equation. Restricting attention for the moment to equations of the first order, we have to recognise that, in every method and notwithstanding every disguise, these operations consist ultimately and entirely of varieties of a single kind of operation—the integration of an ordinary equation (or of ordinary equations) in a single independent variable, of the first order, and homogeneous and linear in the differential elements of all the variables. When once an integral (or the set of integrals) of the subsidiary ordinary equations has been obtained, much still remains to be done in completing the construction of the integral of the partial equation; but, with the exception of a possible quadrature (and, when this is required, it also is obtained through a succession of ordinary quadratures), all the remaining operations in any one stage are direct and not inverse. Thus we integrate ordinary equations in order to build up an integral of a partial equation: but we never actually integrate the partial equation; we always integrate something else. The fact is that we can invert a differential operation of the first order, when there is only a single independent variable: we cannot invert a full differential operation of the first order, when there are several independent variables; and so we have to proceed by stages such as those which have just been indicated.

But what is the consequence of this indirect mode of obtaining an integral of a partial equation? When an integral of an ordinary equation has been obtained, the simplest inspection of its form will give much information as to the possibilities of its comprehensiveness: but, when the materials for the integral of a partial equation have been combined so as to yield its expression, one of the earliest enquiries is as to the range of the integral. Often, it is true that an integral of a partial equation can

be obtained, sufficiently good for the solution of some problem in physics: when it has been obtained, there is no concern or question about other integrals. But the mental repose of the happy physicist, when his mathematical ambition has been satisfied, is denied to the inquisitive mathematician. The latter wishes to know how far an integral, already obtained, is comprehensive, either in its own form, or in respect of other integrals for the construction of which it can be used. Not merely has he had to build the integral by inverse operations—that is a necessity of the case: he has had also to employ indirect processes no one of which can give any hint or clue about the comprehensiveness of the integral when it has been built. So he initiates a new investigation the results of which are to be the answer to his new questions: what are the tests and what are the signs by which it is possible to recognise whether an integral is sufficiently comprehensive to include every integral of the equation?

Such questions may spring merely from a subjective desire for completeness in his work: they can be stirred by a generalisation from his knowledge of the character of the integrals of ordinary equations, and, even if the generalisation has not been justified by an adequate tale of reasons, still, the questions are stirred and they demand an answer: the imperfections of the indirect analysis must somehow be covered. In our inquisitive mathematician's new investigation, he notes all the elements of generality (if the phrase may be used) that have occurred in integrals which he has hitherto constructed: they consist of arbitrary constants and of arbitrary functional forms, and these are to be specified or specialised, so that his integral may (if possible) comprehend every other integral which belongs to the equation or equations. By calculations, some of which have little intrinsic connection with the matter in hand, and by arguments, some of which are not always valid, he succeeds in obtaining various classes of integrals. Thus, in the simplest case, when there are only two independent variables, he obtains one integral which he calls *general*, another which he calls *complete*, and sometimes another which he calls *singular*. Having assigned tests for the recognition of the classes, he usually is satisfied that at last he possesses all the integrals of the equation: to endow himself with a contented mind, he proceeds to prove the completeness of his mathematical possessions.

II.

At this stage (and still dealing only with equations of the first order), two remarks have to be made from the practical side: one of them notes

a difficulty, the other suggests a doubt. As regards the first remark, let the first step in the subsidiary process (or the initial step in any stage of the subsidiary process in the Jacobian method) be considered: in theory, it presents a clear and definite issue, free from difficulty when certain conditions are fulfilled: in practice, when those conditions are fulfilled quite absolutely, there remains the difficulty of the actual integration of a set of ordinary equations. But it is not a difficulty, which appertains specially to the theory under consideration. It has arisen earlier, in the practical integration of ordinary equations, even when their theoretical solution can be regarded as known: it would be removed with the removal of the earlier obstacle: it must be recognised solely as a limitation to the practical working of a system of operations, which have arisen in earlier discussions: and it is not specially caused by any quality of the present problem. It is solved for many equations of quite simple form: it remains unsolved for equations of general form. As regards the second remark, suggesting a doubt, let me review for a moment the general argument by which the various classes of integrals are established. In the course of that discussion, all the equations considered occur in quite general forms; absolutely no account is taken of the peculiarities or the limitations of equations of special forms; it is implicitly assumed that an equation of general form has no limitations: and, if any specialities do occur, they are usually ignored as trivial. But, if it should happen that the peculiarities of an individual equation are not trivial, either in relation to itself or in relation to the integral which has been obtained, the quality may be fatal to the validity of the argument which aims at establishing the completeness of the classified set of integrals: the investigator may find that integrals exist which are not included in his complete set. Thus, on the one hand, there is difficulty in practically effecting the earliest necessary processes in the construction of an integral, unless (speaking broadly) the equation is of quite simple form; and, on the other hand, when the individual equation is of quite simple form, its intrinsic qualities may be such as to invalidate the argument that leads to the usual classification of integrals.

The difficulty, as regards the preliminary process of integrating the subsidiary ordinary equations when the partial equation is of general form, is so familiar to all who have to work the process that no illustration of the remark is needed: it is too common an experience to find equations which cannot be integrated. But the doubt is more of a novelty, and some illustrations may shew its force: two will suffice.

It is a customarily accepted result that the five equations

$$f(x, y, z, p, q) = 0,$$

$$\frac{\partial f}{\partial p} = 0, \quad \frac{\partial f}{\partial q} = 0, \quad \frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z} = 0, \quad \frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z} = 0,$$

when they co-exist, determine an integral of the equation

$$f(x, y, z, p, q) = 0,$$

which is free from all arbitrary elements and is usually called the singular integral: in attaining this result, it is assumed (without proof, and even without tests) that the five consistent equations do actually determine z in terms of x and y alone; in other words, it is implicitly assumed that the five consistent equations are equivalent to three independent equations which define z, p, q, in terms of x and y. If the assumption is justified in fact for any given equation, then certainly the value of z is an integral of the equation, and the values of p and q are the derivatives of z. But no regard is paid to the possibility that, while the equations may be consistent, they do not determine z, p, q, in terms of x and y; still less is there any hint of their significance when this possibility is fact. For example, in the case of the equation

$$f = (px+qy-z)^2 + \frac{z^2}{x^2+y^2-1} - p^2 - q^2 = 0,$$

the five equations in question are consistent with one another; they are satisfied completely by $z = 0$, $p = 0$, $q = 0$, and $z = 0$ is to be regarded as a singular integral. But the five equations also are satisfied completely in virtue of the two relations

$$p = \frac{xz}{x^2+y^2-1}, \qquad q = \frac{yz}{x^2+y^2-1},$$

a contingency not contemplated in the customary argument. These two relations lead to a special integral

$$z^2 = c\,(x^2+y^2-1),$$

involving an arbitrary constant c; it has been obtained in exercising the rule that leads to singular integrals, yet it does not conform to the description of a singular integral. A complete integral is

$$z(x^2+y^2-1)^{-\frac{1}{2}} = a + b\tan^{-1}\frac{y}{x} + b\tan^{-1}\{(x^2+y^2-1)\}^{\frac{1}{2}},$$

where a and b are arbitrary constants; a general integral is

$$\frac{z^2}{x^2+y^2-1} = \phi\left[\tan^{-1}\frac{y}{x} + \tan^{-1}\{(x^2+y^2-1)^{\frac{1}{2}}\}\right],$$

where ϕ is an arbitrary function. The preceding special integral can summarily be placed as a particular instance of the complete integral, and also as a particular instance of the general integral; but the argument, which would justify either classification, requires reconstruction and completion.

As my other illustration, let me take a theorem which is affected by a different kind of exception. In connection with the equation

$$Pp+Qq = R,$$

it is customary to say that, if $u =$ constant and $v =$ constant be any two integrals (the simpler the better) of the ordinary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R},$$

and if any integral of the partial equation be given by $\psi = 0$ (which, for the sake of simplicity, can be regarded as irresoluble), then a function f can always be found such that

$$\psi = f(u, v).$$

Clearly, if the theorem is universally true, the general integral given by an equation

$$F(u, v) = 0,$$

where F is an arbitrary function, is completely comprehensive; for then the function F can be specialised so that any integral can be given. The theorem is an old one, due to Lagrange; the ancient proof was allowed, during many years, and by many writers and readers alike, to pass muster; but it is not valid, because it rests upon a tacit assumption which cannot be justified. A more careful discussion, on the lines of an investigation which (so far as I know) was first indicated by Goursat, shews that the theorem is not universally true. Some cases can be constructed in which the theorem is only partially true, that is to say, an integral given by $\psi = 0$ will be given by $f(u, v) = 0$, with an appropriate choice of the function f; but this is not part of Lagrange's theorem. Other cases can be constructed which definitely falsify the theorem, and some of the reasons of the falsification can be traced.

In particular, for the equation

$$xp+yq = z,$$

we can take

$$u = \frac{z}{x}, \qquad v = \frac{z}{y},$$

and

$$\psi = z - \frac{x^2}{y} = 0$$

furnishes an integral; but

$$\psi = z\left(1-\frac{v}{u^2}\right),$$

so that ψ is not of the form $f(u, v)$: the theorem is what I have described as only partially true. Again, for the equation

$$(z-xy)^2 p+yq = z,$$

we can take $$u = \frac{z}{y}, \qquad v = \tfrac{1}{2}y^2+\frac{y}{z-xy},$$

and $$\psi = z-xy = 0$$

furnishes an integral; also, for the equation

$$\{1+(z-x-y)^{\frac{1}{2}}\}\, p+q = 2,$$

we can take $$u = 2y-z, \qquad v = y+2\,(z-x-y)^{\frac{1}{2}},$$

and $$\psi = z-x-y = 0$$

furnishes an integral; in neither of these instances can ψ be expressed in terms of u and v alone, and in neither of them is the theorem even partially true. More generally, there are indications that, whenever values of x, y, z which satisfy $\psi = 0$ constitute a non-regular place for either of the quantities u and v, the theorem is not valid; but such contingencies do not constitute the aggregate of the exceptions. Meanwhile, the customary classification of the integrals of $Pp+Qq = R$, which has no singular integral in the usual sense of the term, makes no provision for the placing of such integrals as are given by $\psi = 0$ in the instances adduced.

It may be urged (and, if urged, this old extenuating plea would be acknowledged) that the instances are of very special forms. They have been artificially made in order to furnish the simplest illustrations in a challenge of usually accepted results: and they can be created in any number. In mathematics, as in other sciences, particular (and even isolated) cases can be the first to reveal new regions of knowledge, attainable initially only by proceeding to wide issues under the guidance of quite simple results: to quote the words of a great living Viennese scholar, "All beginnings are obscure, whether owing to their minuteness or their apparent insignificance: where they do not escape perception, they are liable to elude observation." In the present discussion, the particular equations have been adduced to shew that exceptions can occur in even the simplest cases, and that these exceptions pass without recognition in the orthodox classification: they are sufficient to raise more than doubts in my mind concerning the accuracy and the completeness of that

classification for any general equation

$$f(x, y, z, p, q) = 0.$$

It appears to me that there is a very definite need for a re-examination and a revision of the accepted classification of integrals of equations even of the first order: in the usual establishment of the familiar results, too much attention is paid to unspecified form, and too little attention is paid to organic character, alike of the equations and of the integrals. Also, it appears to me possible that, at least for some classes of equations, these special integrals may emerge as degenerate forms of some semi-general kinds of integrals; but it is even more important that methods should be devised for the discovery of these elusive special integrals.

III.

If this criticism of the present state of the classification of integrals is valid in the case of equations of the first order, it holds with much greater force for equations of the second order and of higher orders. For such equations, there is an almost fortuitous aggregate of integrals; an integral, when it is found, is labelled by a descriptive name: but there is no systematic discussion of the relations of integrals to one another of a kind that can lead to organic classification, the so-called classes usually being mere collections of specimens. One reason for this deficiency is to be found in the fact that the methods of integration of equations of the second order are exceedingly limited in the range of their application, as will be seen later when their properties are briefly indicated. Even of the integral, which has proved most extensively useful in application to physical problems when they lead to particular equations—I mean the integral called general—there are two distinct definitions, almost rivals to one another, and both of them subject to grave limitations, so far as regards comprehensiveness. According to Ampère, who propounded one of the definitions, an integral is general when the only relations, which are free from the arbitrary elements in the integral and to which the integral leads among the variables and the derivatives of the dependent variable, are those expressed by the differential equation itself and by equations deduced from the equation by differentiation. At first sight, the demand made seems sufficiently extensive to justify the appropriation of the title: but detailed examination of the demand in actual working leads to hesitation and to doubt. All the requirements are satisfied, for the equation $s = yq$, by the integral

$$z = \int_a^y e^{xu} \phi(u)\, du,$$

where ϕ is an arbitrary function; on Ampère's definition, the integral would be entitled to claim the name general. But simple quadratures lead to an integral

$$z' = \psi(x) + \int_a^y e^{xu}\phi(u)\,du,$$

where ϕ and ψ are arbitrary functions; it is clear that the value of z is not sufficiently general to include the value of z'. So much then for Ampère's definition of a general integral: it undoubtedly represented a large view, as was so often the characteristic of Ampère's genius at work; and at its time it led to important contributions to knowledge. The particular equation just mentioned will shew that it can be subject to grave limitations.

According to Darboux, who propounded the other definition, an integral is general when the arbitrary elements which it contains can be determined so as to give the integral, set out in Cauchy's existence-theorem involving functional values assigned to z and to one of its derivatives in specified circumstances. So far as a comparison of integrals graduating under the rival definitions is concerned, it is easy to see that an integral, which is general under the Darboux definition, is general also under the Ampère definition; and equations can be constructed—an example has already been given—which possess integrals that are general in the Ampère sense and are not general in the Darboux sense. The significance of the later class is therefore more extensive than that of the earlier class: yet its comprehensiveness is limited. The utmost that is achieved by the proper assignment of the arbitrary elements in the expression is the possession of the integral which is the concern of Cauchy's theorem; and the latter is not merely a regular function of the variables, but it has its very source in an ostentatious ignoration of deviations from regularity, alike in the assigned initial conditions and in the form of the differential equation. Any deviation from regularity of any of the functions involved reduces the Cauchy existence-theorem to silence; and therefore there is some exaggeration in calling the Darboux-Cauchy integral general, when it is made to depend entirely upon another integral which, however important in itself, is throughout characterised by special limitations. A return to this matter, in quite a different relation, will be made almost immediately; meanwhile, we note that an integral may be called general under either definition, and yet not be comprehensive.

As there is a vague incompleteness about the only integral which is by way of being based upon a broad definition, there is an incompleteness more than vague about other kinds of integrals which are found to be

possessed, some by one equation, some by another. Thus one kind of integral, which is possessed by equations of the second order in two independent variables, involves five arbitrary independent constants: this is the greatest number of constants that could be eliminated from an unspecialised integral equation so as to lead to a single partial equation of the second order: and so the integral is endowed with the name *complete*, surviving from the days of Lagrange, when differential equations were deduced from integral relations in finite forms. But Ampère gave reasons for his opinion that these complete integrals are particular examples of the class which he himself called *general*. Again, some equations of the second order possess integrals, which can be called singular integrals, and which are at least as special to the order as are the singular integrals for equations of the first order. Moreover, these integrals appear almost casually, in response to one or other of the groping ordeals of procedure: there has been little success in any attempt, either to connect them, or to derive them from one another. The method of variation of parameters, applied by Lagrange to equations of the first order, and there effective in obtaining a provisional classification of integrals (though the classification cannot be regarded as complete), proves not applicable to equations of the second order: when applied to a complete integral, it demands the solution of a problem more difficult than the integration of the original equation. It has been applied by Imschenetsky to a more limited integral, and there, by the integration of an equation of the second order which is not necessarily of a form amenable to known methods, it leads from the limited integral to an integral of the type called general; but it furnishes no information as to other classes of integrals.

Such integrals as have been mentioned often are called primitives: they are relations, either explicit or implicit, between the dependent variable and the independent variables. But there are other relations, in virtue of which an equation of the second order can be satisfied: they involve the derivatives of the dependent variable, and their expression contains an arbitrary function or a couple of arbitrary constants: they are called *intermediate integrals*. Most equations, which possess such integrals, are very special in form, and all of them can be recognised by definite tests: they happen, however, to be amenable to processes which lead to the construction of an integral: and they occur very freely in geometrical investigations. Yet, even for them, while the titles *general intermediate integral* and *complete intermediate integral* are used, there is little discussion of possible organic relation between the two kinds of integral: and there is no attempt to make a systematic classification or to

estimate their comprehensiveness. That the general intermediate integral is not completely comprehensive, can be seen easily from a couple of examples. The equation

$$x(r+s)-y(s+t) = p+q-xy$$

has a general intermediate integral

$$f(u, v) = 0,$$

where $$u = x(p+q-xy), \qquad v = y(p+q-xy),$$

and it has a special intermediate integral

$$w = p+q-xy = 0;$$

no form can be assigned to f which will make

$$w = f(u, v),$$

though the integral $w = 0$ can be given by special forms $u = 0$, $v = 0$. Again, the equation

$$x^2r-y^2t = (px+qy-z)^2$$

has a general intermediate integral

$$f(ye^{(-1)/(px+qy-z)}, xy) = 0,$$

and it has a special intermediate integral

$$\phi = px+qy-z = 0;$$

no form can be assigned to f, which will make $\phi = f$, or which will change $f = 0$ into $\phi = 0$. Thus there is a limit to the generality of the general intermediate integral: and manifestly there is good reason, as well as ample range, for a critical investigation of the classification, and of the organic relations with one another, of integrals and primitives of equations of the second order.

IV.

But more can be said as regards the classification of integrals of equations, even only of the second order. The earliest investigations, as the subject was being developed, shewed that integrals could be obtained in the form of explicit equations, which might be single and then would constitute the integral by itself, or which might consist of several equations and then superfluous quantities would have to be eliminated. But the applications in mathematical physics soon led to solutions of a new type, in the form of definite integrals: that these solutions were particular, being made so by the conditions laid down in the problem, is irrelevant to

the present issue: they were sufficient to shew that integrals of the equations exist, the expressions of which involve quadratures. It is true that the quadratures can sometimes be effected in finite terms, and then their form is unessential; we know, from the theory of ordinary linear equations and asymptotic expansions, that quadratures can sometimes be effected in terms of infinite series, and the same holds of the integrals of partial equations: examples will be given immediately. Thus there certainly are two kinds of equations, discriminated by the expressions of their integrals: one kind has integrals expressible by means of explicit equations in finite form and free from quadratures, and they were called the *first class* by Ampère who did not, however, further develop the classification: the other kind possesses integrals in the expression of which quadratures (often called *partial quadratures*) occur. The difference between the two classes of equations is more substantial than the formal difference between the absence and the presence of partial quadratures: it affects the relation between the dependent variable and its derivatives. In the case of equations which belong to Ampère's first class, the successive derivatives of the dependent variable introduce new derivatives of the arbitrary functions which occur in the integral, so that the number of arbitrary elements in the aggregate of derived equations increases with increase of the number of derivations effected. On the other hand, when a dependent variable is expressed by means of quadratures which cannot be evaluated in finite terms, derivation with respect to the independent variable can make substantial changes in the subject or subjects of quadrature: the arbitrary elements in derivatives can be different from the arbitrary elements in the dependent variables.

While the two types of integral mentioned are the types of most frequent occurrence in investigations that have been actually carried out, they are not completely comprehensive of all possible types. For example, the arbitrary elements in an integral can conceivably be defined in connection with another differential equation which is simpler than the original equation, through the occurrence of fewer independent variables or from the fact of lower order; and it is equally conceivable that precise assignment of such modes of occurrence of the arbitrary elements in an integral might lead to the determination of new classes of integrable equations.

Of the two classes of equations discriminated by Ampère—and here let me say that his memoirs, ancient as their date 1815 now seems, amply repay the fullest perusal in spite of a superficial difficulty of notation—the first class has received much more attention from mathematicians than the other. One obvious reason for this preference is that the statement of the problem is more definite: another is that the problem itself

is easier; and a third is that there are few hints or suggestions, which can prove an initial guide to progress in the development of the theory of those equations having integrals that involve partial quadratures. But I seem to see signs of an awakening, though they do not yet point to any reduction in the difficulty of the theory. In 1895 Borel published a paper in which he shewed, in the case of a large class of linear equations of any order, how to construct (by means of partial quadratures) a regular integral of the equation which is not otherwise finite in form. When his method is applied, for example, to the equation $r = q$ (which is a well known equation under a thin disguise), the Cauchy integral, which is to be such that $z = f(y)$ and $p = g(y)$ when $x = 0$, can be expressed in a form

$$z = \frac{1}{\pi}\int_0^{2\pi} e^{x\cos a + y\sin a} \cos(a + x\sin a + y\sin 2a)\, G(a)\,da,$$

where $G(a)$ is an arbitrary function constructed in a quite definite manner from the coefficients in $f(y)$ and in $g(y)$; also,

$$\frac{1}{\pi}\int_0^{2\pi} e^{y\cos 2a} \cos(a + y\sin 2a)\, G(a)\,da = f(y),$$

$$\frac{1}{\pi}\int_0^{2\pi} e^{y\cos 2a} \cos(2a + y\sin 2a)\, G(a)\,da = g(y),$$

results to which I shall return almost immediately. Again, we have Dr. Whittaker's solution of Laplace's equation $\nabla^2 V = 0$ in a form

$$V = \int_0^{2\pi} f(z + ix\cos u + iy\sin u,\, u)\,du,$$

where, so far as the discussion in his memoir is concerned, f is an arbitrary function of its two arguments subject to the restrictions that it is a regular function of its first argument and that it is capable of expansion in a Fourier series of multiples of u; and, as in the last instance, this is the regular Cauchy integral defined by the initial conditions

$$V = h(x, y), \qquad \frac{\partial V}{\partial z} = k(x, y),$$

when $z = 0$, where h and k are regular functions, the actual evaluation of the coefficients in the expansion of f in terms of the coefficients in the expansions of h and k being comparatively simple.

Results such as these, when restated in a different manner, lead me to a suggestion which has been a dream of mine for a little time. Thus the specified integral of the equation $r = q$ can be regarded as known, when once the function $G(a)$ can be regarded as determinate; and $G(a)$

can be regarded as determined by the two equations involving $f(y)$ and $g(y)$ respectively. Now, when $G(a)$ is the unknown quantity, these two equations belong to the class called integral equations. This subject, though taking its rise in a paper of Abel's dated 1823, has only undergone any real development in recent years, as may be seen from the bibliography in Mr. Bateman's recently published paper on the subject in our own *Proceedings;* and an application to ordinary linear equations has already been made by Dini. Perhaps my dream is fanciful; but I am not without hope that the developments of the subject of integral equations may throw some light upon the comparatively neglected class of equations (even if only of linear equations) whose integrals involve partial quadratures.

V.

Now let me return to the practical integration of equations, so as to consider the processes from another point of view: as before, it is convenient to begin with equations of the first order. There are various methods for the purpose, and each of the methods has undergone some modification or amplification subsequent to its original creation: so that, in a history of the subject, many names would need to be mentioned. Broadly speaking, there are six methods which stand out as more important than the rest: the six are Lagrange's method (limited to equations linear in the derivatives), Charpit's method (limited to equations in two independent variables), Cauchy's method of characteristics, the Jacobi-Hamiltonian method (based upon Hamilton's researches in theoretical dynamics, and sometimes called Jacobi's first method), Jacobi's general method (sometimes called Jacobi's second method), and Lie's methods (by contact transformations, and by groups of functions). I do not stop to discuss the comparative theoretical merits of these methods: to become effective in practice, all of them, at some stage or other in the construction of an integral of a partial equation, demand integration (complete or incomplete) of a subsidiary system of ordinary equations. When this subsidiary system has been formed, intrinsic difficulties in obtaining an integral or integrals of the system are ignored; in theory, the problem of the partial equation is regarded as solved when the fitting use of the integral or integrals of the system has been devised. As regards the comparative practical merits of the methods, when there is a question of constructing an integral of a propounded equation, my own preference is for Charpit's method and for Jacobi's second method. It is one of the little ironies of mathematical history that, though Charpit's method is so well known and is extensively used, no book or memoir

was ever published under Charpit's name: he did present a memoir to the Academy of Sciences at Paris in 1784, but he died soon afterwards and his memoir was never printed; and the results which it contained were rescued apparently from oblivion by Lacroix, who may have been at the meeting in 1784 and who later expounded Charpit's method in a great treatise on the differential and integral calculus. As already mentioned, the method applies only to equations in two independent variables: Lacroix indicates that Charpit tried to extend it to equations in more than two independent variables. This extension was reserved for Jacobi, and is contained in his second method: it was not published until 1862, eleven years after his death. This second method of Jacobi allows the number of independent variables to be n, any whatever: it contains Charpit's method when $n = 2$, and, unlike so many unilluminating generalisations, which contain for a general value nothing that is not contained for a special value, Jacobi's extension solves one of the difficulties which may have faced Charpit in his unsuccessful attempts—the use to be made of second integrals and of further integrals of the subsidiary system.

Still, as practical methods, both require one or more of the integrals of the subsidiary system: consequently, if such integrals cannot be obtained, even with all the aids furnished by modifications that have been made in detail, the methods do not lead to an integral of the partial equation. They fail; and then, it may be added, the other methods also fail in practice. In such a case, what is the state of our knowledge as regards an integral of the partial equation? It is very limited; but I think that a demand for increase of that knowledge presents a practicable problem. For the moment, we have to fall back on Cauchy's existence-theorem, which asserts the existence of an integral in the form of a regular function of the variables under functional conditions completely limited by regularity; moreover, it gives an expression for this integral in the form of a converging series. But Cauchy's theorem makes no provision for the occurrence of irrationalities, or singularities, or any other deviations from regularity, either in the equation itself or in the postulated conditions: and here there is an ample field for investigations which shall determine the functional effect upon an integral, and the conditions under which it can exist, if the regularity everywhere demanded in Cauchy's conditions is not possessed in some quarter or another. We all know the substantial extension of knowledge of the integrals of ordinary equations in the vicinity of singularities initiated by the work of Briot and Bouquet, and the later (and even more considerable) extension of knowledge of the integrals of ordinary linear equations in the

vicinity of singularities initiated by the work of Fuchs. The names of Klein, Painlevé, Picard, and Poincaré at once recall new tracts of ground that have been opened up by departure from the high road of regular integrals. In the region of partial differential equations, even of the first order only, I feel as confident as one can be before the event that there are at least as great possibilities which are yet undeveloped. You cannot expect me to be able to indicate the main lines of development; for I do not possess the vision of a seer. But a beginning could be made as regards equations that are uniform in expression by a consideration of the effect of singularities upon the integrals: it would be rather analogous to the work of Briot and Bouquet, and Charpit's method might supply the principal terms at least, and might indicate some properties. Something has been done by Darboux in this direction, in reference to characteristics. I should expect that a more general discussion would lead to results that are not within the present range of knowledge.

VI.

When we come to review the methods that are applied to the practical integration of equations of the second order, we find that they are extraordinarily less effective than for equations of the first order; but this need not cause surprise, because analysis has not found a means of performing an irresoluble inverse operation of the second order. When the operation is resoluble into two inverse operations of the first order, the complete integration can sometimes be achieved: simple and familiar instances are $s=0$ and $r+t=0$; but, of course, resoluble operations are special, not general, in character. In consequence, there is no method which can be effectively applied to all equations; there are particular methods which are useful for particular classes of equations, and these constitute the utmost range of present attainment. Among the methods that have long been known, there are three which are of prominent mark: they are associated with the names of Laplace, Monge, and Ampère respectively.

Laplace's method deals solely with equations that are linear in the dependent variable and its derivatives up to the second order inclusive. After securing a canonical form $s+ap+bq+cz=0$, it modifies the equation by one or other of two transformations: if either of the transformations, effected a finite number of times, should produce an equation characterised by the vanishing of a certain invariantive combination of the coefficients, the original equation can be integrated, and the integral is finite in form. But, if neither of the transformations should ever lead to such an equation, the method fails. The class of linear equations, for

which the method is useful, is thus limited: their general expression has been obtained.

Monge's method depends essentially upon the existence of a general intermediate integral of the form $\phi(u, v) = 0$, where ϕ is arbitrary, and where u and v are specific functions of x, y, z, p, q; and so it can only be applied to equations of the form $U(rt-s^2)+Rr+2Ss+Tt = V$, where the coefficients U, R, S, T, V do not involve the second derivatives. But it does not apply even to all of these equations: certain relations, undiscovered and undiscoverable by the method, have to be satisfied by the coefficients in order that the supposed intermediate integral may exist. The actual process of the method requires the integration of a subsidiary system of simultaneous equations, homogeneous and linear in the differential elements: in their formation, the existence of the intermediate integral is used. The quantities u and v arise through equations $u = a$, $v = b$, which are integrals of the subsidiary system; when they have been obtained, the intermediate integral can at once be formed, being a partial equation of the first order: its primitive is a primitive of the original equation. The construction of the integrals $u = a$ and $v = b$ of the subsidiary system is unassisted by any hint or rule in the method: everything depends upon the skill of the worker in framing integrable combinations. Thus the method is gravely limited, both in its range and its practicability.

Ampère's method does not postulate the existence of any general intermediate integral, though it works more easily when such an integral exists. It applies to equations having integrals in finite terms free from partial quadratures; and, beginning with this assumption as to the character of the integral, it proceeds to the construction of a subsidiary system of ordinary equations, effected by a change of independent variables. It is not limited to equations of the form suited to Monge's method; but, when it is applied to such equations, the subsidiary system obtained is formally the same as Monge's system, though its significance is now quite altered. Again, integrable combinations of the equations in the subsidiary system are required, though the method is no more helpful in hints as to their construction than in Monge's; but the combinations need not be so many as in Monge's method, and the use made of them is entirely different, for they become functions of the new variable quantity deliberately omitted in the construction of the subsidiary system. Further integrable combinations of selected equations in the subsidiary system are required; but, again, there are no compelling rules for their construction, and the actual success of the method as applied to any given equation depends largely on the manipulative

skill of the investigator. Subject, however, to the limitation imposed on the character of the integrals to be obtained, Ampère's method is general: it can be applied to equations of order higher than the second; and it can be applied, in theory at least, when the number of independent variables is greater than two, though the manipulative difficulties are vastly greater for a greater number of independent variables.

Laplace's method was first given in a memoir dated 1773; Monge's method (applied only to the equation $Rr+Ss+Tt = V$) was first given in a memoir dated 1784; and Ampère's two memoirs were published in 1815 and 1819. For many years after the last of these dates, real advances in the practical integration of partial equations of the second order were slight, though there were occasional changes of form. Thus Boole's method, as expounded in a memoir dated 1862, is essentially a rearrangement of Monge's method: it makes the same assumption as to the existence of a general intermediate integral, and so is limited to the same equations of the second order; but by constructing the subsidiary equations in the form of partial equations of the first order, instead of the form of ordinary equations in differential elements, he was able to obtain the relations among the coefficients which must be satisfied if the assumed intermediate integral exists. Further, after the publication of Jacobi's researches, the manipulation of partial equations, which are homogeneous and linear of the first degree, has become comparatively easy: and this is the form of the equations in Boole's subsidiary system.

The next (and, to my mind, the latest) considerable advance in the practical integration of partial equations was effected by Darboux, in a memoir published in 1870: the memoir relates to the integration of equations of the second order, but the extension to equations of higher order requires nothing but an easy extension of simple analysis. Moreover, the memoir is important, not solely in itself, but also because it has given rise to many additional investigations and to not a few independent establishments of results, attained from a different point of view, yet mainly covered by his work and by consequences that are its immediate mathematical sequel. The methods of Monge and of Boole had been dependent upon the existence of an intermediate integral of a partial equation of the second order; and Ampère's method, based in its practice upon the same subsidiary equations as were obtained by Monge for the equations of a limited type, was rendered more easy in practice by the existence of any intermediate integral, even if more limited in significance and expression than the general intermediate integral postulated by Monge. The real advance made by Darboux lies in an idea which, in the

light of the past, seems obvious when once it is formulated, but, like every other idea that is at once obvious and useful, seems to require individual discovery: it can be described merely as the extension (and, in practice, a repeated extension) of the notion that is the basis of the Charpit method and the Jacobi method for the integration of partial equations of the first order. In both of these methods, the practical aim is the construction of an equation (or of equations) of the same order as, and compatible with, the original equation of the first order. In methods devised before Darboux's memoir, the dominant practice was either the construction of a compatible equation of order lower than that of the original equation, or the ignoration of any compatible equation. What Darboux did was to develop the notion that an equation of the second order, though it could not have an intermediate integral, might still admit a compatible equation of its own order: thus, to choose an example solely for purposes of illustration, which shall not imply that it is best treated by Darboux's method, the equation

$$r-t-2\frac{p}{x}=0$$

has no intermediate integral: it admits the two compatible equations

$$r+2s+t=xf(x+y), \qquad r-2s+t=xg(x-y),$$

where f and g are arbitrary functions, and the three equations are analytically and algebraically independent of one another.

Yet such an illustration does not explain or suggest the whole extent of the new process. Darboux's method is progressive: that is to say, if the analysis, which (after Jacobi's results about partial equations of the first order) can be made exhaustive, should shew that an equation of the second order does not admit two equations compatible with itself, a similar process will lead to the construction of an equation (or of equations) of the third order which may be compatible with the original equation. If that should fail, then similarly for a possible equation or equations of the fourth order; and so on, for successive orders, though manifestly there is a finite limit of such operations in practice. Not merely so; but, if at any stage (say the first stage) it should appear that the method will give only a single equation compatible with the original equation, instead of two equations, still there are processes for the construction of a primitive. But, if the method proves effective by the discovery of equations of any order compatible with the original equation, the subsequent actual construction of the primitive is merely a matter of simple quadratures. For

instance, the equation

$$r-t-4\frac{p}{x}=0$$

leads to immediate derivatives

$$\alpha-\gamma=4\left(\frac{r}{x}-\frac{p}{x^2}\right), \qquad \beta-\delta=4\frac{s}{x};$$

it admits of compatible equations

$$\alpha+3\beta+3\gamma+\delta=x^2\phi(x+y), \qquad \alpha-3\beta+3\gamma-\delta=x^2\psi(x-y),$$

which are independent of one another and of the derived equations: the four equations determine simultaneous and possible values of α, β, γ, δ: the rest of the analysis is merely quadrature.

At this stage, and having regard to the spirit and the content of Darboux's method, a few questions arise. The first (not in order of importance, but merely the first as they occur to me) is: what is the most general form of equation of the second order in two independent variables that is amenable to Darboux's method in a finite number of operations? I dare not even play with a question as to a form of the method, if it possesses a form, when the number of operations is conceived as being capable of unlimited increase. Let me only say that the first question, as asked, has been discussed for equations

$$f(r, s, t)=0$$

by De Boer, and for equations

$$s=f(x, y, z, p, q)$$

by Goursat. Much remains to be done for equations of the second order, as regards the discovery of classes that are integrable by Darboux's method: meanwhile, Darboux declares that the method is effective for all equations of the second order which belong to Ampère's first class, that is, which possess integrals in finite form free from partial quadratures: and, in this aspect, I shall suggest (in a few moments) the same enquiry in somewhat different terms.

The range of the method is not peculiarly limited to equations of the second order in two independent variables: it applies to equations of order higher than the second: it applies equally to equations which involve more than two independent variables. In face of the reluctance of analysis, at the present time, to venture far into such regions, it may seem premature to suggest an enquiry into the classes of such equations for which the method is effective: but, unless some entirely new and more comprehensive process applicable to equations of all orders in any number of variables is devised and illuminates the whole field of analysis connected with partial

equations, my question (now premature perhaps) may come to be more insistent upon an answer than it is to-day.

Another question arises when the general idea in Darboux's method is regarded retrospectively, rather than prospectively, from the point of view of the order of equations that are compatible with an original equation of the second order. Monge (and, following him, Boole) considered only the possibility of a general intermediate integral, that is, a compatible equation of the first order, which involves an arbitrary function and (on elimination of the arbitrary function) leads to the original equation. But a so-called complete intermediate integral, that is, a compatible equation of the first order which involves two arbitrary constants, can also lead to an equation of the second order on the elimination of the two arbitrary constants (it being understood that we still are limited to a couple of independent variables). The general idea of Darboux's method, translated into the appropriate analysis, leads to the discovery of such complete intermediate integrals when they exist: and, indeed, this translation provides the most general method known to me for the construction of intermediate integrals of any type. Thus the equation

$$(sq-tp)^2 = (rt-s^2)(sp-rq),$$

treated by Darboux's method, is found to possess a complete intermediate integral

$$z+ap+a^2q = b,$$

where a and b are arbitrary constants; similarly, for the equation

$$z(rt-s^2)^2-(tp^2-2spq+rq^2)(rt-s^2)+(tp-sq)(sp-rq) = 0,$$

there is a complete intermediate integral

$$z = ap+bq+ab;$$

and these integrals can be definitely formed by synthetic construction. It might be possible to construct all the classes of equations which have complete intermediate integrals of the form

$$\theta(x, y, z, p, q, a, b) = 0:$$

and, in connection with equations which possess intermediate integrals of any type, it is right to mention the investigations of Goursat.

VII.

Such, then, in outline, are the principal methods used for the integration of equations of the second order. As will be seen, they have not the same power of compelling a result as have the most effective

methods applied to equations of the first order: some of the limitations of the methods, applied to equations of the second order, are intrinsic to the order of the equation and, in addition, they are subject to all the limitations that affect the methods applied to equations of the first order. Accordingly, an equation of the second order may be propounded for integration and yet not be amenable to any of the customary methods: it would be a vain demand (though not a novel one) to ask for the invention of a new method, because the whole theory of equations of higher order has long waited for further methods. As was the alternative to the practical failure of the methods used for equations of the first order, so here also the only available plan is to have recourse to Cauchy's existence-theorem, and to use the integral supplied by that theorem. The integral is not very convenient, because it is in the form of a converging series which usually cannot be expressed in a more commodious form: it is a regular function and it is subject to limiting conditions, which make z and one of its derivatives regular functions of one variable for some assigned value of the other variable, but no deviation from regularity is admitted or even contemplated: still, there is an integral, and it is all that can be obtained with the analytical means at our disposal.

Having thus come to the end of our practical resources, we can at least indicate one or two obvious demands, even though they cannot be met or though ultimately some less direct demands should be met more easily and more usefully. We can hope for a new method: but, during the interval of waiting for the realisation of that deferred hope, there are other tasks which can be undertaken and which, if accomplished, will lead to extensions of knowledge. One of these is the determination of classes of equations of the second order which are amenable to Ampère's method: later on, I shall propound the same question in another connection and in a different form. Similarly, as already stated, it would be important to possess the complete determination of such equations as are amenable to Darboux's method. Again, as for partial equations of the first order, so for equations of the second order, it is permissible to conceive a wide extension of existence-theorems that shall take some account of the more commonly recognised deviations from regularity. It is true that occasional grumbling is heard, and more than occasional grudge is felt, about these existence-theorems which, like the abstract resolutions adopted by visionary legislators, go far beyond the needs or the possibilities of practice. But they have their use: they indicate whether an integral of a particular kind does or does not exist, as well as the conditions of its existence; and, if the practical

investigator is inclined to reproach them for the unpractical clumsiness of the integrals they produce, he might at least remember that, clumsy as the results are, they often are the only results which can be obtained. So, while the practical man is left pondering in his ungrateful and dissatisfied mood, let me suggest an extension of the existence-theorem which shall take account of some of the commoner irregularities. The object is difficult: nor is the sense of difficulty diminished by a remembrance of the corresponding stage in the theory of ordinary equations, when we pass beyond the first order. In that theory, a parametric value of the independent variable cannot be an essential singularity of the integral of any equation of the first order $y' = f(y, x)$, where f is rational in y and is uniform in x: a parametric value of the independent variable is an essential singularity of the integral of an equation of the second order so simple as $y'' = y'^2$. It may be expected, or even feared, that the complete investigation will be complicated by the existence of latent irregularities which do not appear by inspection. If so, the complication is only another penalty for having entered into the possession of much knowledge already obtained: the easy problems, which are worth solving, have been solved before our time; and we are apt to stigmatise as difficult all those which yet remain unsolved. As regards the particular question suggested, even an incomplete investigation might make a substantial addition to our knowledge of the properties of integrals of partial equations.

VIII.

In the brief description that was given of the methods effective in practice for the integration of equations of the first order, a deliberate choice was expressed for the method of Charpit and (specially as including that method when the number of independent variables is greater than two) for the second method of Jacobi. If, in unprovoked wilfulness, a similar choice is to be made among the practical methods for integrating equations of the second order or of higher orders, my vote will be cast for Darboux's method, provided its retrospective added application to the construction of intermediate integrals (when they exist) be included as part of the method. I cannot discuss the comparative extent of the two classes of equations which are amenable respectively to Ampère's method and to Darboux's method; but the latter is more systematic than the former, less dependent upon the happy intuitions of purely analytical skill: and, in the absence of some new and more comprehensive process, the later method is to be preferred to the earlier in that it offers work on recognised lines to the mason when there is no sculptor, and work

on recognised lines to the builder when there is neither architect nor engineer.

Any account of the theory of partial equations of the first order must recognise the importance of Lie's contributions to the subject and would indicate how, alike by the theory of contact transformations and of groups of functions, new light has been thrown upon the equations and a definite method has been provided for the integration of a single equation and of a system of equations in involution. It is natural to consider whether the idea of transformation in general can be usefully extended to equations of order higher than the first. Particular instances are known: thus Laplace's method of integrating a homogeneous linear equation of the second order is a special example. Accordingly, some general questions almost propound themselves. What are the transformations among the variables, whether dependent or independent, which keep the order of an equation unchanged? What are the limitations upon the form of an equation which must be satisfied in order that it may be changed into another equation of similar form and the same order? When two equations can be transformed into one another, what are the relations between their integrals? Some answers to these, and to similar questions, are to be found in the investigations of Bäcklund, who approached the matter as one concerned, first with the transformation of surfaces in ordinary space, and next with simultaneous equations of the first order. Thus far, the results obtained relate to equations of the second order in two independent variables, which are of the Monge-Ampère type and are not of even the most general form belonging to that type. Additions to Bäcklund's results have been made by Goursat and by Clairin; but the subject, important as it is, has not received extensive consideration and the literature is distinctly limited. Much remains to be done before we can declare that we have obtained the most general theory of transformation of equations of order higher than the first; and, as I am convinced, it offers a field for fruitful, if difficult, research.

Partly as an independent subject of enquiry, and partly because they arise in one mode of application of Ampère's method to equations of the second order, I would suggest that simultaneous equations of the first order, in number equal to the number of dependent variables, offer problems, perhaps not less difficult than those which I have just mentioned, yet distinct from them alike in range and probable modes of attack. The operation of integrating such equations is an operation of class greater than unity in general, that is to say, it cannot generally be performed solely with the help of quadratures: but classes of such equations as can thus be integrated have been already discussed and

obtained by Hamburger, König, Königsberger, and others. I do not feel confident that the last word worth speaking has yet been spoken about such equations integrable by operations of the first class. Again, an attempt to extend Jacobi's second method to a pair of equations

$$f(z_1, z_2, p_1, q_1, p_2, q_2, x, y) = 0, \qquad g(z_1, z_2, p_1, q_1, p_2, q_2, x, y) = 0,$$

in two dependent variables and two independent variables (to take only the simplest case), leads to a couple of subsidiary lineo-linear equations in two new dependent variables, and an increased number of independent variables; it would be of interest to know for what classes of equations $f = 0$, $g = 0$, these lineo-linear equations can be integrated, because their integral leads to the primitive of the original equations.

There is a temptation to wander into other regions of investigation, so far as this account is inspired by temptations and repressed by limitations. One other method, synthetic in idea and companionable (if such a word is permitted) in form, was initiated by Riemann: it is based upon the use of adjoint equations which, useful and important in the theory of ordinary differential equations, were used by him to obtain the integral of a partial linear equation of the second order, when the integral is subject to general conditions, by means of an integral of an equation of the same character subject to more (sometimes, to very) special conditions. The process can hardly be called one of general application. The idea, and developments of the idea, have been applied by many investigators, among whom Picard and Volterra deserve particular mention, to equations of special form, as arising in discussions of problems in mathematical physics, and also to equations of somewhat specialised type, so as to obtain existence-theorems. But I am coming perilously near the topic of boundary problems, such as occur in connection with partial equations: and that topic, by a self-imposed self-denial, is forbidden to me. The subject has had its fluctuations of vitality: strong in the past, promising now, it may form the subject of an enterprising address to this Society on a day that is to come.

IX.

Any general review of the position of the theory of partial equations could be deemed guilty of a grave omission if it did not refer to what is often described briefly as "integration in finite terms." Such a result was the aim of an inverting practice among classical analysts, like Euler and Lagrange, who started from integral equations possessing generality in some form or other, and constructed the differential equations where the general elements no longer appeared; they sought to invert the process

by passing from the differential equations back to corresponding integral equations with an attenuated success explained only in later discussions, ranging over ideas quite foreign to the original creators of the theory. Such a result is often the aim of questions, submitted to skilful youth in elaborate examinations: they tend to give a wholly false perspective, at once to the theory and to the practice of the integration of partial equations. But the general notion has, nevertheless, its value in the development of the subject: not for the historical aspect, which in earlier days was the main course of growth: not for the modern investigator who is as eager as any classical analyst to obtain an integral in finite terms, if such an integral can be obtained: but because, in the absence of methods generally effective for equations of order higher than the first, the prescribed character for the integral leads to definitely prescribed classes of equations. For the purposes of this range of investigation, it is unnecessary to take any account of equations of the first order: such deficiencies as exist in the process of their integration are not intrinsic, being specially connected with the subsidiary processes which, when effective, can lead to an integral. Nor is it necessary to do more than mention those equations of the second order, which possess a general intermediate integral and are therefore of the conditioned Monge-Ampère type; in particular, equations of the Monge-Ampère type which possess two general intermediate integrals are known (by a result due to Lie and Darboux) to be capable of change into $s=0$ by means of contact transformations only. It is not necessary to mention those equations which possess a complete intermediate integral, but have not hitherto been included in any prescribed form: in all these cases, the integral can be definitely obtained by regular processes, the difficulties in which are not intrinsic. But mention must be made of Liouville's equation $s=e^z$, having a primitive

$$e^z = \frac{2X'Y'}{(X+Y)^2},$$

where X is an arbitrary function of x and Y is an arbitrary function of y: it is very special in form; yet the slightly different equation $s+c=e^z$, where c is a constant distinct from zero, does not possess an integral similarly expressible. Mention must also be made of the equation

$$r+ap+bq+cz=0,$$

where a, b, c are functions of x and y only; the integral, save in a trivial case, is not expressible in similar explicit finite terms. Also, there is

Laplace's linear equation

$$s+ap+bq+cz = 0,$$

the primitive of which can be expressed, wholly or partly, in finite terms according as two conditions, or only one condition, in a regulated succession of forms can be verified by the coefficients a, b, c, again functions of the independent variables only. These diverse results naturally raise a question as to the most general class of equations of the second order in two independent variables, primitives of which can be expressed explicitly, in finite terms, without partial quadratures, and by a single equation. The first professed answer to the question was given in a memoir by Moutard, presented to the Academy of Sciences at Paris in 1870; but the memoir, in its entirety, was never published, for it perished during the fires of the Commune, and only a portion of it was rewritten some years later. (May I add a fact, which here is irrelevant, but which all mathematicians must regret? Those same fires consumed the manuscript of what would have been the third and the concluding volume of Bertrand's treatise on the calculus, and it would have been devoted to differential equations: the gravity of the loss is beyond words.) The first complete proof of Moutard's enunciated theorem is contained in a note by Cosserat, near the end of the fourth volume of Darboux's *Théorie générale des Surfaces*, published in 1896: but it will be interesting to the members of our Society to know that many, if not all, of the results were previously obtained in a memoir by Prof. Lloyd Tanner, which appears in the volume of our *Proceedings* for the year 1876. The general result of the investigation is as follows: the equations, which cannot be transformed to Liouville's equation or to Laplace's linear equation, are reducible to the form

$$s = \frac{\partial}{\partial x}(ae^{z}) - \frac{\partial}{\partial y}(be^{-z}),$$

where a and b are functions of x and y satisfying certain conditions, the expression of which is unnecessary for our present purpose: and the integrals are expressible in a form

$$z = f(x, y, X, X_1, \ldots, X_m, Y, Y_1, \ldots, Y_n),$$

where X and Y are arbitrary functions of x and of y respectively; $X_1, \ldots, Y_1, \ldots$, are their respective derivatives; m and n are finite integers; and f is a definite function, affecting (and affected by) the form of a and b.

Now this result is general: it completes, in one direction, the work of the early classical analysts: when the transformations are known—and they are indicated in the course of the proof—we are able to settle

whether a given equation of the second order, which arises for general integration and is not associated with some hint-giving conditions, shall have a general integral endowed with the prized quality of concise expression. And the result is useful, as marking off one class of equations with defined properties. Nor is it useful only as an achievement: for it suggests three definite questions, the answers to which will mean progress of a peaceful kind though, even if completely answered, they will effect no revolution in the subject. One of these questions relates to equations of the second order: the other two relate to equations of the third order.

The first of the questions requires the extension of Moutard's theorem. In that theorem, the integral of the equation is to be given by a single relation between the variables: but we know that integral relations are often found as expressions for x, y, z, in terms of two independent parameters: for example, this is the mode in which the most general integral of the equation of minimal surfaces has been expressed by Monge, Ampère, Legendre, and Weierstrass. But the problem, which corresponds to Moutard's for equations of the second order having integrals expressible by a single relation, has not been solved for those equations having integrals expressible by three relations of the form

$$x = f(u, v, U, U_1, \ldots, U_m, V, V_1, \ldots, V_n),$$

$$y = g(u, v, U, U_1, \ldots, U_m, V, V_1, \ldots, V_n),$$

$$z = h(u, v, U, U_1, \ldots, U_m, V, V_1, \ldots, V_n),$$

where U and V are arbitrary functions of u and of v respectively; $U_1, \ldots, U_m, V_1, \ldots, V_n$ are their derivatives; m, n are finite integers; and f, g, h are definite functions. When this problem has been solved, it will incidentally give all those equations of the second order which, integrable in finite terms by Ampère's method, have integrals that are free from partial quadratures.

The first of the two suggested questions about equations of the third order is the application of Laplace's method to linear equations of that order. It would require the discussion of those transformations which leave the equation

$$B\beta + C\gamma + Rr + Ss + Tt + Pp + Qq + Zz = 0$$

(where the coefficients are functions of x and y alone) unaltered in character, the determination of those combinations of the coefficients which are invariant under the transformations, the significance of these invariants, and generally the construction of such equations of this type as have their integrals, some or all, of finite rank. The other suggestion

about equations of the third order is the immediate analogue of Moutard's theorem for equations of the second order: what are the various types of equations of the third order in two independent variables which have their integrals given by a single equation between x, y, z, three arbitrary functions of the variables and the derivatives of these functions up to any finite order? Pending the discovery of new processes of integration, the solution of these two problems will give definite and useful information about equations of the third order; it is the kind of information which, some time or other, must be obtained before we shall be justified in regarding our knowledge of equations of the third order as comparable with the knowledge of equations of the second order. And there need be no hesitation in grappling with equations of the third order: we can no longer be warned off the field by an Astronomer Royal for the reason that was adduced years ago: mathematical physicists themselves are not now always able to pursue their investigations by the use of equations of order not higher than the second.

Yet, having made a suggestion as to the consideration of equations of the third order, and being ready to extend it even to equations of the fourth order, I am not prepared to recommend the expenditure of intellectual energy (at least, in the present state of mathematical knowledge) upon discussions of equations, of unspecified finite order and involving any unspecified finite number of variables. Crucial difficulties can arise in the passage from an equation of the first order to an equation of the second order: the process, which solves them or surmounts them, is sometimes equally effective for the removal of the corresponding difficulty in passing to an equation of any order: unless something specially characteristic of the order is obtained in the process, the generalisation of the method cannot be regarded as standing in the first line of contribution to knowledge. Similarly, crucial difficulties can arise in the passage (say) from two independent variables to three, even from three to a greater number—such an instance is provided by Jacobi's second method as compared with Charpit's method, the new results being a real extension of knowledge and providing a real increase of practical power: and crucial difficulties can arise in the progressive passage between an even number, and an odd number, of independent variables—such an instance occurs in the reduction of a Pfaffian equation to a canonical form. Unless some essential difficulty is solved, or some essential novelty is obtained, mere formal increase in the order of the equation solved, or in the number of independent variables involved, or even in both of these respects, is an insufficient ground for the concession of the homage that is paid to progress in knowledge, as distinguished from mere accumulation of formulæ.

As this address does not pretend to be a comprehensive account of the present state of knowledge of the theory of partial equations,* I make no claim of having included all the important aspects of that theory, and I offer no apology for having omitted many of them. Those, who wish for such an account, must consult the skeletons to be found in a work like the German *Encyclopædia* and must study the fuller treatment given, in cosmopolitan fashion, by writers belonging to not a few nations and races. If a critic should remark that my selection of topics has been strangely guided by an individual caprice, I shall not challenge the accuracy of his remark: my only retort would be that a similar remark could be made concerning most addresses of the kind, delivered in similar circumstances: and there might be an added declaration, which he would believe, that the individual caprice has been the result of many years of more or less steady labour at the subject. The aim of my address has been to shew that the theory of partial differential equations, so far from being exhausted or effete, is full of possible developments almost from its very foundations. These developments admit of the exercise alike of constructive power and of critical faculty; and I commend their consideration to my fellow-workers in this Society.

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* References to the writings of the mathematicians mentioned will be found in Part IV. of my *Theory of Differential Equations*.

The matter, indicated on pp. 436, 437 (*ante*), is partly discussed in a Note which will be included in the *British Association Report* (York Meeting, 1906).

ON THE INVERSION OF A DEFINITE INTEGRAL*

By H. BATEMAN.

[Received October 31st, 1906.—Read November 8th, 1906.]

CONTENTS.

Introduction.

The present memoir is devoted to the discussion of a problem which was considered at some length in a paper which appeared in these *Proceedings* some months ago.

The problem in question is to determine a function $\phi(t)$ so that, for a given range of values of s, we may have

$$f(s) = \int_L \kappa(s, t)\, \phi(t)\, dt$$

where it is supposed that the path of integration and the functions $f(s)$ and $\kappa(s, t)$ are known.

Equations of this type occur in potential problems in which the value of the potential function is given at points on a curve or surface. On this account alone they are worthy of close attention; but there is another object which a systematic theory of these equations would accomplish—it would group together the innumerable isolated results in the subject of definite integrals, thus giving us a means of classifying them, besides indicating the fundamental principles upon which the formulæ depend.

* This paper is an elaboration of one which was presented to the Society on May 9th, and afterwards withdrawn, as the subsequent researches of the author required that it should be remodelled.

With this object in view, I have attempted in § 2 to classify the integral equations themselves according to what may be called the fundamental formulæ on which the solution depends. A practical method of finding the inversion formula depending on the use of a differential equation is then suggested, and is employed to obtain the solutions of a number of particular equations. I am unable, however, to give a rigorous investigation of the theory of this method.

A second method, which also depends upon the use of linear differential equations, is indicated in § 8; it seems to be full of possibilities, and throws some light upon the theory of a certain type of partial differential equation.

§ 10 consists of an extension of the general method which was given by the author,* and an attempt is made to obtain an existence theorem.

1. *The Integral Equation considered as the Limit of a System of Linear Equations.*

The integral equation

$$f(s) = \int_a^b \kappa(s, t)\, \phi(t)\, dt \tag{1}$$

is in many respects analogous to the system of linear equations

$$f_s = \sum_{t=1}^{n} \kappa_{s,t}\, \phi_t \quad (s = 1, 2, \ldots, m), \tag{2}$$

and can, in fact, be obtained from it by a limiting process in which m and n are finally made infinite. The important point is that in this process of passing to the limit many of the properties of the system of linear equations are preserved.

Now the properties of a system of linear equations depend upon the relation between the number of equations and the number of unknown quantities.

(1) If $m > n$, we shall be able to construct a number of relations of the form

$$\sum_{s=1}^{m} a_s \kappa_{st} = 0,$$

which are satisfied for all values of t, and then the equations (2) will be inconsistent unless the quantities f satisfy linear relations of the same type

$$\sum_{s=1}^{m} a_s f_s = 0.$$

* *Proc. London Math. Soc.*, Ser. 2, Vol. 4, Part 2 (1906).

In passing to the limit, a relation of this type may take several forms, such as

$$\int_c^d a(s) f(s)\, ds = 0,$$

$$\left[\Sigma A_r \frac{d^r f}{ds^r}\right]_{s=s_0} = 0,$$

$$\Sigma B_r f(s_r) = 0;$$

or it may be expressible as a linear combination of such forms.

In any case the corresponding property possessed by the integral equation is that in general a necessary condition that a function $f(s)$ may be expressed in the form (1) is that it should satisfy all the linear relations that are satisfied by $\kappa(s, t)$ for all values of t. This condition is only necessary so long as the function $\phi(t)$ is restricted to be such that the operation of forming the linear relation may be interchanged with that of integration in equation (1). We cannot say at present whether it is sufficient or not, because the conditions which are to be laid on $\phi(t)$ have not been determined, and it has been found that in certain cases a function represented by equation (1) satisfies linear conditions (such as continuity*) which the function $\kappa(s, t)$ does not.

(2) If $m = n$, there is in general a unique set of quantities ϕ_t and these are determined by a set of linear equations of the form

$$\phi_t = \sum_{s=1}^{m} \bar{\kappa}_{ts} f_s \quad (t = 1, \ldots, n), \tag{3}$$

provided no relation of the form

$$\Sigma a_s \kappa_{st} = 0$$

is satisfied for all values of t.

(3) If $m < n$, there are an infinite number of sets of solutions, but we may single out one set by imposing $n-m$ linear conditions on the quantities ϕ_t.

We conclude from this, that, in general, the solution of equation (1) will not be unique, but that it may be rendered unique by imposing a number of linear conditions upon the function $\phi(t)$. The whole question depends, of course, upon the range of values for which $f(s)$ is given:†

* The condition of continuity must be regarded as being equivalent to a number of linear conditions.

† In some cases $f(s)$ may only be given for an enumerable set of values of s; but, by properly choosing the conditions to be satisfied by ϕ, we can make the solution unique, as, for instance, in Stieltjes' problem of the moments. (*Annales de la Faculté des Sciences de Toulouse*, t. VIII., 1894.)

for one range of values the function $\phi(t)$ may be uniquely determinate, while for a smaller range this will, in general, not be the case, and it may be necessary to restrict $\phi(t)$ to be zero for a certain portion of the range of integration in order to render the solution unique under the new conditions.

Supposing, then, that conditions have been chosen which will make the solution of (1) unique, we shall expect, in analogy to (3), to obtain an expression for $\phi(t)$ of the form

$$\phi(t) = L_t f(t) \tag{4}$$

where L_t denotes a linear operator which may be built up of terms of the types

$$p(t)\frac{d^r f}{dt^r}, \quad f(t+a), \quad \int_c^d F(t, s) f(s)\, ds.$$

It frequently happens that the inversion formula takes the simple form

$$\phi(t) = \int_L \bar{\kappa}(t, s) f(s)\, ds; \tag{5}$$

but it is to be remarked that this integral is in general of a different character from that which occurs in equation (1).

This may be seen by considering a particular example in which the limits a and b are finite and the function $\kappa(s, t)$ remains finite and continuous within the range. By choosing a particular function $\phi(t)$ which experiences a sudden change of value at a point x within the range of integration, we obtain a continuous function $f(s)$. The integral in equation (5), on the other hand, must represent a discontinuous function, and this can only be the case if the integral is an improper one.

The exceptional character of the integral in equation (5) may either be due to the limits being infinite or to a discontinuity in the function $\bar{\kappa}(t, s)$.

When the solution of the equation

$$f(s) = \int_a^b \kappa(s, t)\, \phi(t)\, dt$$

is given by a formula of the type

$$\phi(t) = \int_L \bar{\kappa}(t, s) f(s)\, ds,$$

in which the path L does not depend upon the value of t, we can, in general, assert that the solution of the associated equation

$$\chi(t) = \int_L \kappa(s, t)\, \psi(s)\, ds$$

is, under suitable conditions, given by the formula

$$\psi(s) = \int_a^b \bar{\kappa}(t, s)\,\chi(t)\,dt.$$

For we have

$$\int_L f(s)\,\psi(s)\,ds = \int_L \int_a^b \kappa(s, t)\,\psi(s)\,\phi(t)\,ds\,dt$$
$$= \int_a^b \chi(t)\,\phi(t)\,dt = \int_a^b \int_L \bar{\kappa}(t, s)\,\chi(t) f(s)\,dt\,ds$$

and $f(s)$ is arbitrary; hence we must have

$$\psi(s) = \int_a^b \bar{\kappa}(t, s)\,\chi(t)\,dt.$$

2. *The Classification of Integral Equations of the First Kind.*

Integral equations of the first kind may be classified according to the principles on which their inversion formulæ depend.*

The method which we adopt to ascertain the nature of an equation is as follows:—

Let $\kappa(s, r)$ be substituted for $f(s)$ in the inversion formula with the view of obtaining, if possible, a function $\phi(t) = h(t, r)$ which will give a representation of $\kappa(s, r)$ in the form

$$\kappa(s, r) = \int_a^b \kappa(s, t)\,h(t, r)\,dt \tag{6}$$

for values of r lying between a and b.

Now, although the function $\kappa(s, r)$ satisfies all the linear relations that are satisfied by $\kappa(s, t)$, it can, in general, only be expressed in this form if the integral is an improper one. For we know that, in the case of a proper integral, the equation †

$$\psi(r) = \int_a^b \psi(t)\,h(t, r)\,dt$$

is only satisfied for a finite number of functions ψ if at all; whereas, by giving different values to s in equation (6), we shall obtain an infinite number of linearly independent functions ψ, unless it happens that the function $\kappa(s, t)$ can be expressed as a finite sum of the form

$$\kappa(s, t) = \sum_{n=1}^{m} \phi_n(s)\,\psi_n(t).$$

When $\kappa(s, r)$ is substituted for $f(s)$ in the inversion formula it frequently happens that the result takes the form of a divergent definite

* An integral equation is regarded here as consisting of the equation itself *plus* a number of conditions which will render the solution unique.

† Fredholm, *Acta Math.*, Vol. XXVII. (1903).

integral or series, in this case we associate a value with it by applying one of the known methods of summation such as Borel's exponential method.*

If this method is applied to all the equations for which the formulæ for inversion are known, it will be found that the function $h(t, r)$ takes one of a limited number of distinct forms; and so the equations may be classified accordingly.

In equations of type 1 the function $h(t, r)$ is zero, except in the vicinity of $t = r$. As an example of an equation of this type, we may take Hilbert's equation

$$f(s) = \int_0^1 \kappa(s, t)\, \phi(t)\, dt$$

where $\kappa(s, t) = s(1-t),\ s \leqslant t;\ = t(1-s),\ s \geqslant t,$

the solution of which is given by

$$\phi(t) = -\frac{d^2}{dt^2} f(t).$$

Now
$$\frac{d}{dt}\{\kappa(t, r)\} = 1 - r \quad \text{for } t \leqslant r$$
$$= -r \quad \text{for } t \geqslant r,$$

and so is discontinuous at the point $t = r$; but for any other point we may differentiate again, obtaining

$$h(t, r) = -\frac{d^2}{dt^2}\{\kappa(t, r)\} = 0.\dagger$$

The equations of the first type form a very large class; other types depend on the following forms of formula (6):—

(2) $$\kappa(s, r) = \frac{1}{2\pi i}\int_C \frac{\kappa(s, t)}{t - r}\, dt,$$

(3a) $$\kappa(s, r) = \frac{1}{\pi}\int_{-\infty}^{\infty} \kappa(s, t)\, \frac{\sin(t-r)}{t-r}\, dt.$$

(4a) $$\kappa(s, r) = \int_0^{\infty} \kappa(s, t)\, h(t, r)\, dt$$

where $$h(t, r) = t\int_0^1 J_0(tx)\, J_0(rx)\, x\, dx,$$

the numbers being chosen as above because the types (1), (2), (3), and (4)

* This method is applied to definite integrals in a paper by G. H. Hardy, *Quarterly Journal*, Vol. xxxv., p. 22.

† [*Note added December 11th.*—When, however, the solution of the integral equation is not unique the function $h(t, r)$ may differ from one of the forms given below by a multiple of a function $\phi(t)$ which makes the definite integral zero.]

are connected in some way with the differential equations

$$\frac{d^2V}{dx^2} = 0, \quad \frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} = 0, \quad \frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} + V = 0,$$

and
$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} + \frac{d^2V}{dw^2} + V = 0$$

respectively.

It is exceedingly probable that the equations of other types may be made to depend eventually upon what may be called the fundamental formula of type 1, and so we proceed to consider this formula in detail.

3. *The Fundamental Formula for Integral Equations of the First Type.*

The equation which I have referred to as the fundamental formula for integral equations of the first type is strictly not a mathematical equation at all, and cannot be used in a rigorous demonstration until it has been rendered more precise by a determination of the class of functions to which it is applicable and of the necessary and sufficient conditions to be satisfied in order that operations such as differentiation and integration under the integral sign may be performed upon it.

These the present writer does not feel competent to give, but some justification of our employing it to attain the ends we have in view may be derived from the following considerations:—

Let $f(t)$ be a function which possesses a continuous derivative for all values of t within the range (a, b), and let $F(x, t)$ be a function which is defined as follows:—

$$\left.\begin{aligned} F(x, t) &= -1 \quad (t < x) \\ &= +1 \quad (t > x) \end{aligned}\right\}; \tag{7}$$

then
$$\int_a^b F(x, t)\, f'(t)\, dt = f(b) + f(a) - 2f(x).$$

Now let us suppose, for the moment, that we can integrate this equation by parts; then we shall have, by the ordinary rule,

$$\int_a^b F(x, t)\, f'(t)\, dt = f(b) + f(a) - \int_a^b \frac{\partial}{\partial t} F(x, t)\, f(t)\, dt. \tag{8}$$

Accordingly, if the improper integral

$$\int_a^b \frac{\partial}{\partial t} F(x, t)\, f(t)\, dt$$

be defined by this equation, we have

$$f(x) = \tfrac{1}{2} \int_a^b \frac{\partial}{\partial t} F(x, t)\, f(t)\, dt, \tag{9}$$

and this is the fundamental formula to which I have referred. It is clear that the limits a and b can be made infinite and the function $F(x, t)$ replaced by $F(x, t)+\psi(x)$ without altering the argument; but it is not clear whether this formula can be considered to hold when $f(x)$ does not possess a continuous derivative.

A geometrical interpretation of the formula may be obtained by writing $F(x, t) = \frac{2\theta}{\pi}$ where θ is the angle which the radius vector, from the point x to the point t, makes with the line through x perpendicular to the axis. If the point x is excluded from the range of integration by a small semi-circle of radius ϵ, the formula may be written

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\partial\theta}{\partial t} f(t)\,dt = \frac{1}{\pi}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} f[x+\epsilon e^{i(\theta-\frac{1}{2}\pi)}]\,d\theta,$$

and this is easily seen to be true if $f(x+h)$ can be expanded in a Taylor's series.

Now let us consider an integral equation of the first type

$$f(s) = \int_a^b \kappa(s, t)\,\phi(t)\,dt,$$

for which the inversion formula is

$$\phi(t) = \int_L \bar{\kappa}(t, s) f(s)\,ds.$$

Then, by hypothesis, the function

$$h(t, r) = \tfrac{1}{2}\frac{\partial}{\partial t} F(r, t)$$

is obtained when $\kappa(s, r)$ is substituted for $f(s)$, *i.e.*,

$$\tfrac{1}{2}\frac{\partial}{\partial t} F(r, t) = \int_L \bar{\kappa}(t, s)\,\kappa(s, r)\,ds\ ?, \tag{10}$$

the sign $=?$ being used to denote that the integral may be divergent, in which case $\tfrac{1}{2}\frac{\partial}{\partial t} F(r, t)$ is the value associated with the divergent integral.

Accordingly, if we wish to find the inversion formula, we must look for a relation of the form (10) : *i.e., if the path L is known, we must solve the integral equation*

$$\tfrac{1}{2}\frac{\partial}{\partial t} F(r, t) = \int_L \chi(s)\,\kappa(s, r)\,ds,$$

for $\chi(s)$.

I shall indicate later a method depending on the theory of linear

differential equations by which relations of the form (10) may be constructed. For the present I shall content myself by showing that the relation (9) may be considered to be connected with many of the well known representations of a function.

In the case of Fourier's double integral

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty \cos s(x-t)\, f(t)\, ds\, dt$$

the equation (10) has the form

$$\tfrac{1}{2}\frac{\partial}{\partial t} F(x, t) = \frac{1}{\pi}\int_0^\infty \cos s(x-t)\, ds\ ?,$$

the representation for $F(x, t)$ being the exact equation

$$F(x, t) = -\frac{2}{\pi}\int_0^\infty \frac{\sin s(x-t)}{s}\, ds$$

and the limits a and b being $\pm\infty$ respectively.

The representation of a function as a series of Legendre polynomials may be considered to be based on the exact equation *

$$F(x, t) = P_1(t)\, P_0(x) + \sum_1^\infty \left[P_{n+1}(t) - P_{n-1}(t)\right] P_n(x)\,;$$

the relation corresponding to (10) is

$$\tfrac{1}{2}\frac{\partial}{\partial t} F(x, t) = \sum_1^\infty \frac{2n+1}{2}\, P_n(x)\, P_n(t)\ ?\,;$$

and the limits are $a = +1,\ b = -1$.

4. *Study of a particular Equation.*

We shall now consider the integral equation

$$f(s) = \int_{-1}^{+1} \frac{\phi(t)\, dt}{(1-2ts+s^2)^\nu} \quad (\nu > 0). \tag{11}$$

If $|s| < 1$ and $f(s)$ satisfies certain conditions, it will be shown that the inversion formula is

$$\phi(s) = \frac{(1-s^2)^{\nu-\frac{1}{2}}}{\pi}\int_0^\pi \left[\nu f(x) + x f'(x)\right] \sin^{2\nu-1} a\, da \tag{12}$$

where $\qquad x = s + i\sqrt{1-s^2}\cos a.$

---

* The necessary and sufficient conditions that a function satisfying the conditions laid down in Dirichlet's proof of Fourier's theorem may be expanded in a series of Legendre polynomials are given by Darboux, "Approximation des Fonctions de grands Nombres," *Liouville's Journal* (3e série), t. IV., p. 393 (1878). The function $F(x, t)$ satisfies these conditions, and so may be expanded in the above form.

Substituting $f(x) = \dfrac{1}{(1-2xr+x^2)^\nu}$ according to the rule, we obtain

$$\phi(t) = h(t, r) = \frac{(1-t^2)^{\nu-\frac{1}{2}}}{\pi} \int_0^\pi \frac{\nu(1-x^2)}{(1-2xr+x^2)^{\nu+1}} \sin^{2\nu-1} a\, da$$

where $$x = t+i\sqrt{1-t^2}\cos a$$

or $$h(t, r) = \frac{\nu}{i\pi} \int_{t-i\sqrt{(1-t^2)}}^{t+i\sqrt{(1-t^2)}} \frac{(1-x^2)(1-2tx+x^2)^{\nu-1}}{(1-2tx+x^2)^{\nu+1}}\, dx$$

$$= \frac{1}{2i\pi} \int_{t-i\sqrt{(1-t^2)}}^{t+i\sqrt{(1-t^2)}} \frac{1}{r-t} \frac{d}{dx} \left\{ \left(\frac{1-2tx+x^2}{1-2rx+x^2}\right)^\nu \right\} dx$$

$$= 0, \quad \text{unless } r = t.$$

The integral equation is thus seen to be of the first type.

We can also show that $h(t, r)$ is the derivative of a discontinuous function of the type required, for

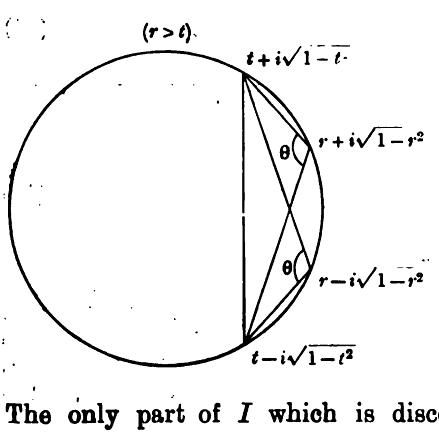
$$\int h(t, r)\, dr = \frac{1}{2i\pi} \int_{t-i\sqrt{(1-t^2)}}^{t+i\sqrt{(1-t^2)}} \frac{1-x^2}{x} \frac{(1-2tx+x^2)^{\nu-1}}{(1-2rx+x^2)^\nu}\, dx,$$

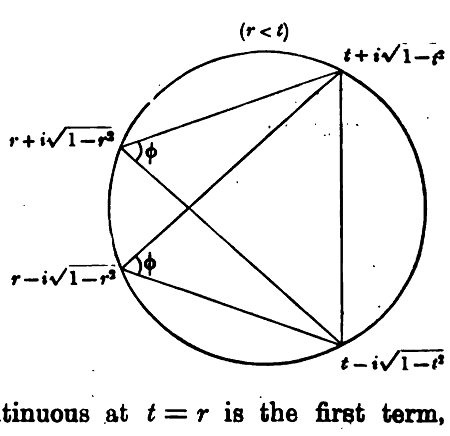
and this integral is discontinuous at $t = r$. To determine the change of value, we write it in the form

$$I = \frac{1}{2\pi i} \int \left\{ \frac{2(r-x)}{1-2rx+x^2} + \frac{1}{x} + \frac{A(t-r)}{(1-2rx+x^2)^2} \right.$$
$$\left. + \text{ higher powers of } (t-r) \right\} dx$$

$$= -\frac{1}{2\pi i} \Big[\log (x-r-i\sqrt{1-r^2})(x-r+i\sqrt{1-r^2}) \Big]_{t-i\sqrt{(1-t^2)}}^{t+i\sqrt{(1-t^2)}}.$$

Let the path of integration be the straight line joining the two points $t \pm i\sqrt{1-t^2}$, then we have the two figures





The only part of I which is discontinuous at $t = r$ is the first term,

and this is equal to $+\theta/\pi$ if $r > t$ and $-\phi/\pi$ if $r < t$. Hence, since $\theta+\phi = \pi$, I will suddenly increase by $+1$ as r increases through the value t; accordingly, if we write

$$F(t, r) = 2\int h(t, r)\,dr,$$

the formula (9) will give

$$\phi(t) = \int_{-1}^{+1} h(t, r)\,\phi(r)\,dr$$

$$= \frac{\nu}{i\pi}\int_{-1}^{+1}\int_{t-i\sqrt{(1-t^2)}}^{t+i\sqrt{(1-t^2)}} \frac{(1-x^2)\,\phi(r)}{(1-2rx+x^2)^{\nu+1}}\,(1-2tx+x^2)^{\nu-1}\,dx\,dr,$$

and, if we assume that the order of integration can be changed, we may write this:

$$\phi(t) = \frac{\nu}{i\pi}\int_{t-i\sqrt{(1-t^2)}}^{t+i\sqrt{(1-t^2)}} (1-2tx+x^2)^{\nu-1}\,\chi(x)\,dx, \tag{12$'$}$$

$$\chi(x) = \int_{-1}^{+1} \frac{(1-x^2)\,\phi(r)\,dr}{(1-2rx+x^2)^{\nu+1}} = f(x) + \frac{x}{\nu}\,f'(x)$$

where
$$f(x) = \int_{-1}^{+1} \frac{\phi(r)\,dr}{(1-2rx+x^2)^{\nu}}.$$

The formula (12)$'$ is easily seen to be equivalent to (12) when the substitution

$$x = t+i\sqrt{1-t^2}\cos a$$

is made; accordingly the inversion formula can actually be constructed by means of equation (9), and so the integral equation is proved to be of the first type.

We may find a *sufficient* set of conditions to be satisfied by $f(s)$ in order that it may be represented in the form (11) by using Dini's method of expansion, that is, by representing the function $\kappa(s, t)$ in the form

$$\kappa(s, t) = \Sigma a_n \psi_n(s)\,\theta_n(t).$$

In the present case we use the expansion

$$(1-2st+s^2)^{-\nu} = \sum_{n=0}^{\infty} s^n C_n^{\nu}(t) \quad (|s| < 1). \tag{13}$$

The properties of the polynomials $C_n^{\nu}(t)$ are fairly well known, but it will be convenient to furnish proofs of the different relations that will be required, as I do not know exactly where some of them are to be found.

LEMMA I.—The functions $C_n^\nu(t)$ satisfy the integral relations

$$\int_{-1}^{+1}(1-t^2)^{\nu-\frac{1}{2}}C_n^\nu(t)C_m^\nu(t)\,dt = 0 \qquad (m \neq n)$$

$$= \frac{\pi}{2^{2\nu-1}(\nu+n)}\,\frac{\Gamma(n+2\nu)}{\Gamma^2(\nu)\,n!} \qquad (m = n). \quad (14)$$

To prove these, we observe first of all that the function

$$V = (1-2ts-s^2)^{-\nu}$$

satisfies the partial differential equation

$$(t^2-1)\frac{\partial^2 V}{\partial t^2}+(2\nu+1)t\frac{\partial V}{\partial t} = s^{1-2\nu}\frac{\partial}{\partial s}\left\{s^{2\nu+1}\frac{\partial V}{\partial s}\right\}.$$

Replacing V by the series and equating coefficients of s^n, we find that $C_n^\nu(t)$ satisfies the differential equation

$$(t^2-1)\frac{d^2y}{dt^2}+(2\nu+1)t\frac{dy}{dt}-n(n+2\nu)y = 0.$$

Writing the equations satisfied by the two functions $C_n^\nu(t)$, $C_m^\nu(t)$ in the form

$$\frac{d}{dt}\left[(t^2-1)^{\nu+\frac{1}{2}}\frac{d}{dt}C_n^\nu(t)\right] = n(n+2\nu)(t^2-1)^{\nu-\frac{1}{2}}C_n^\nu(t),$$

$$\frac{d}{dt}\left[(t^2-1)^{\nu+\frac{1}{2}}\frac{d}{dt}C_m^\nu(t)\right] = m(m+2\nu)(t^2-1)^{\nu-\frac{1}{2}}C_m^\nu(t),$$

multiplying the first by $C_m^\nu(t)$, the second by $C_n^\nu(t)$, and subtracting, we obtain

$$\frac{d}{dt}\left[(t^2-1)^{\nu+\frac{1}{2}}\left\{C_m^\nu(t)\frac{d}{dt}C_n^\nu(t)-C_n^\nu(t)\frac{d}{dt}C_m^\nu(t)\right\}\right]$$

$$= (n-m)(n+m+2\nu)(t^2-1)^{\nu-\frac{1}{2}}C_n^\nu(t)\,.\,C_m^\nu(t).$$

Now $C_m^\nu(t)$ is a polynomial in t, and so remains finite when $t = \pm 1$; accordingly the quantity inside the square bracket vanishes for these values of t, and so, if $n \neq m$, we have

$$\int_{-1}^{+1}(1-t^2)^{\nu-\frac{1}{2}}C_n^\nu(t)\,C_m^\nu(t)\,dt = 0.$$

To find the value of the integral when $m = n$, we require the recurrence formula satisfied by $C_n^\nu(t)$, viz.,

$$(n+1)C_{n+1}-2(n+\nu)t\,C_n+(n+2\nu-1)C_{n-1} = 0. \quad (15)$$

To prove this, we differentiate equation (13) with regard to s, obtaining

$$2\nu(t-s)(1-2ts+s^2)^{-\nu-1} = \Sigma n s^{n-1}C_n^\nu(t),$$

and compare with the former expansion, whence we get the relation

$$(1-2st+s^2)\Sigma ns^{n-1}C_n^\nu(t) = 2\nu(t-s)\,\Sigma s^n C_n^\nu(t),$$

from which the recurrence formula is obtained by equating coefficients of s^n. Multiplying (15) by $(1-t^2)^{\nu-\frac{1}{2}}C_{n+1}^\nu(t)\,dt$ and integrating, we obtain the relation

$$(n+1)\int_{-1}^{+1}(1-t^2)^{\nu-\frac{1}{2}}\{C_{n+1}^\nu(t)\}^2dt = 2(n+\nu)\int_{-1}^{+1}t(1-t^2)^{\nu-\frac{1}{2}}C_n^\nu(t)\,C_{n+1}^\nu(t)\,dt.$$

Similarly, multiplying by $(1-t^2)^{\nu-\frac{1}{2}}C_{n-1}^\nu(t)\,dt$ and integrating, we obtain

$$(n+2\nu-1)\int_{-1}^{+1}(1-t^2)^{\nu-\frac{1}{2}}\{C_{n-1}^\nu(t)\}^2dt$$
$$= 2(n+\nu)\int_{-1}^{+1}t(1-t^2)^{\nu-\frac{1}{2}}C_n^\nu(t)\,C_{n-1}^\nu(t)\,dt.$$

Changing n into $n+1$ and comparing with the last equation, we find that

$$I_{n+1} = \int_{-1}^{+1}(1-t^2)^{\nu-\frac{1}{2}}\{C_{n+1}^\nu(t)\}^2dt = \frac{(n+\nu)(n+2\nu)}{(n+\nu+1)(n+1)}I_n.$$

Now $C_0^\nu(t) = 1$, and, if we put $x = \dfrac{1-t}{2}$ in the integral

$$I_0 = \int_{-1}^{+1}(1-t^2)^{\nu-\frac{1}{2}}dt,$$

it becomes $$I_0 = 2^{2\nu}\int_0^1 x^{\nu-\frac{1}{2}}(1-x)^{\nu-\frac{1}{2}}dx = 2^{2\nu}\frac{\Gamma^2(\nu+\frac{1}{2})}{\Gamma(2\nu+1)}.$$

Hence the relation $$I_{n+1} = \frac{(n+\nu)(n+2\nu)}{(n+\nu+1)(n+1)}I_n$$

will give $$I_n = 2^{2\nu}\frac{\nu}{n+\nu}\frac{1}{n!}\frac{\Gamma^2(\nu+\frac{1}{2})\,\Gamma(n+2\nu)}{\Gamma(2\nu+1)\,\Gamma(2\nu)}.$$

This expression may be simplified by using the relation*

$$\Gamma(\nu)\,\Gamma(\nu+\tfrac{1}{2}) = 2^{1-2\nu}\sqrt{\pi}\,.\,\Gamma(2\nu),$$

and we finally obtain $$I_n = \frac{\pi}{2^{2\nu-1}(n+\nu)}\frac{\Gamma(n+2\nu)}{\Gamma^2(\nu)\,n!}.$$

The solution of the integral equation for $f(s) = s^n$ may now be obtained, for, if we multiply the expansion (13) by $(1-t^2)^{\nu-\frac{1}{2}}C_n^\nu(t)\,dt$ and integrate between -1 and $+1$, we obtain

$$\int_{-1}^{+1}\frac{(1-t^2)^{\nu-\frac{1}{2}}C_n^\nu(t)\,dt}{(1-2st+s^2)^\nu} = \frac{\pi s^n}{2^{2\nu-1}(n+\nu)}\frac{\Gamma(n+2\nu)}{\Gamma^2(\nu)\,n!};$$

* Whittaker's *Analysis*, p. 180.

so that the corresponding function $\phi_n(t)$ is given by

$$\phi_n(t) = \frac{2^{2\nu-1}}{\pi}(n+\nu)\frac{\Gamma^2(\nu)\, n!}{\Gamma(n+2\nu)}(1-t^2)^{\nu-\frac{1}{2}}C_n^\nu(t).$$

By giving n different values, we see that the solution for

$$f(s) = \Sigma a_n s^n$$

should be

$$\phi(t) = \Sigma a_n \phi_n(t).$$

Now, in order to establish the inversion formula, we must prove first of all that

$$C_n^\nu(t) = \frac{\Gamma(n+2\nu)}{2^{2\nu-1}\Gamma^2(\nu)\, n!}\int_0^\pi \{t+i\sqrt{1-t^2}\cos a\}^n \sin^{2\nu-1} a\, da. \tag{16}$$

If we call the right-hand side F_n and substitute in the recurrence formula (15), we find that

$$(n+1)F_{n+1} - 2(n+\nu)\, t\, F_n + (n+2\nu-1)F_{n-1}$$

$$= \frac{\Gamma(n+2\nu)\sqrt{t^2-1}}{2^{2\nu-1}\Gamma^2(\nu)\, n!}\left[\int_0^\pi (t+i\sqrt{1-t^2}\cos a)^n (2\nu \sin^{2\nu-1} a \cos a\, da) \right.$$
$$\left. - \int_0^\pi n(t+i\sqrt{1-t^2}\cos a)^{n-1} i\sqrt{1-t^2}\sin^{2\nu+1} a\, da\right].$$

But, on integrating the first integral by parts, we find that the quantity under the square brackets is zero; hence F_n satisfies the same recurrence formula as $C_n^\nu(t)$. Also, when $n = 0$, we have

$$F_0 = \frac{\Gamma(2\nu)}{2^{2\nu-1}\Gamma^2(\nu)}\int_0^\pi \sin^{2\nu-1} a\, da = 1,$$

$$F_1 = \frac{\Gamma(2\nu+1)}{2^{2\nu-1}\Gamma^2(\nu)}\int_0^\pi (t+i\sqrt{1-t^2}\cos a)\sin^{2\nu-1} a\, da = 2\nu t,$$

and so the first two values of F_n coincide with those of $C_n^\nu(t)$; therefore F_n is equal to $C_n^\nu(t)$ for all positive integral values of n.

We are now in a position to prove the following theorem :—

THEOREM.—*If $f(s)$ is a function which can be expanded in a power series*

$$f(s) = \Sigma a_n s^n,$$

which converges within the unit circle in such a way that the series

$$\Sigma |(n+\nu) a_n|$$

is convergent, then $f(s)$ can be expressed by means of the definite integral

$$f(s) = \int_{-1}^{+1} \frac{\phi(t)\,dt}{(1-2st+s^2)^{\nu}} \quad (|s| < 1),$$

and the function ϕ is given by the formula

$$\phi(t) = \frac{(1-t^2)^{\nu-\frac{1}{2}}}{\pi} \int_0^{\pi} [\nu f(x) + x f'(x)] \sin^{2\nu-1} a\, da,$$

where $$x = t + i\sqrt{1-t^2}\cos a.$$

Since $$(1-2ts+s^2)^{-\nu} = (1-se^{i\phi})^{-\nu}(1-se^{-i\phi})^{-\nu},$$

we find, on expanding both factors and calculating the coefficient of s^n, that

$$C_n^{\nu}(t) = A_n \cos n\phi + A_{n-2}\cos(n-2)\phi + \dots$$

where the coefficients A_r are all positive; for all the coefficients in $(1-x)^{-\nu}$ are all positive.

The modulus of $C_n^{\nu}(t)$ will therefore certainly be less than its value when $\phi = 0$, *i.e.*, when $t = 1$, and then we have

$$C_n^{\nu}(1) = \frac{\Gamma(n+2\nu)}{\Gamma(2\nu)\, n!}.$$

Now we saw that for $f(s) = \sum_0^{\infty} a_n s^n$ we had the formal solution

$$\phi(t) = \sum_0^{\infty} \frac{2^{2\nu-1}}{\pi} (n+\nu) \frac{\Gamma^2(\nu)\, n!}{\Gamma(n+2\nu)} (1-t^2)^{\nu-\frac{1}{2}} a_n C_n^{\nu}(t),$$

and, if we take out the factor $(1-t^2)^{\nu-\frac{1}{2}}$, we shall have a series which is absolutely and uniformly convergent for $|C_n^{\nu}(t)| \leqslant \frac{\Gamma(n+2\nu)}{\Gamma(2\nu)\, n!}$, and by hypothesis the series $\Sigma|(n+\nu)a_n|$ is convergent.

To show that when this series is substituted for $\phi(t)$ in the integral it can be integrated term by term, we employ the rule given by G. H. Hardy,* viz.: If $\phi = \Sigma\phi(t)$ is uniformly convergent throughout $(a, A-\epsilon)$, however small be the positive quantity ϵ, $f(t)$ is continuous throughout (a, A), and $\int_a^A \bar{\phi}(t)$ is convergent where

$$\bar{\phi}(x) = \Sigma|\phi_n(t)|,$$

then $$\int_a^A \phi f(t) = \Sigma \int_a^A \phi_n f\, dt.$$

* *Mess. of Math.*, Vol. xxxv., No. 8, p. 126.

Taking $f = \dfrac{1}{(1-2st+s^2)^\nu}$, it is easily seen that, if $|s| < 1$, the condition is satisfied; hence, on integration, we shall obtain the series for f, and the first part of the theorem is proved.

Next, to prove that the function ϕ is actually given by the inversion formula, we again apply Hardy's rule; observing that

$$\nu f(x)+xf'(x) = \Sigma(n+\nu)a_n x^n,$$

$$|x| = \sqrt{\cos^2 a+t^2\sin^2 a},$$

we see that, if
$$\bar{\phi} = \Sigma\,|(n+\nu)a_n x^n \sin^{2\nu-1} a\,|,$$

then $\int_0^\pi \bar{\phi}\,da$ is convergent, and so the integration term by term may be effected.

The formula (12) is thus proved for any function which satisfies the conditions laid down; these conditions are sufficient, but not necessary. It is important to notice that they are satisfied in the case of a function $f(s)$ which can be expanded in a power series whose radius of convergence is greater than unity.

The solution of the integral equation for the case in which $|s| > 1$ may be deduced from the inversion formula by writing $s = \dfrac{1}{s'}$.

5. *Applications of the preceding Formula.*

If in equations (11) and (12) we write $\nu = 1$, the results may be expressed in a simpler form; for we have

$$\phi(t) = \frac{i}{\pi}\Big[xf(x)\Big]_0^\pi$$

$$= \frac{1}{i\pi}\Big[(t+i\sqrt{1-t^2})\,f(t+i\sqrt{1-t^2})-(t-i\sqrt{1-t^2})\,f(t-i\sqrt{1-t^2})\Big], \tag{17}$$

$$f(s) = \int_{-1}^{+1}\frac{\phi(t)\,dt}{1-2ts+s^2}. \tag{18}$$

Writing
$$s = \mu+i\sqrt{1-\mu^2},$$

we have
$$sf(s) = \psi(\mu) = \tfrac{1}{2}\int_{-1}^{+1}\frac{\phi(t)\,dt}{\mu-t}. \tag{19}$$

The function $\psi(\mu)$ is, in general, a many-valued function of μ, and there are two values of s corresponding to each value of μ. The inversion

formula may be written *

$$\phi(t) = \frac{1}{i\pi} \operatorname*{Lt}_{\epsilon=0} [\psi(\mu - i\epsilon) - \psi(\mu + i\epsilon)], \tag{20}$$

thus allowing for the multiformity of the function ψ.

In particular, if $\phi(t) = P_n(t)$, we have $\psi(\mu) = Q_n(\mu)$, and the formula

$$Q_n(\mu) = \tfrac{1}{2} P_n(\mu) \log\left(\frac{1+\mu}{1-\mu}\right) + R_n(\mu) \tag{21}$$

where $R_n(\mu)$ is single valued shows at once once how formula (20) will give the correct value of ϕ.

Again, if $f(s) = \dfrac{\sin ns}{s}$, we find that $\phi(t) = \dfrac{2}{\pi} \cos nt \sinh n\sqrt{1-t^2}$; and so we have the formula

$$\frac{\sin ns}{s} = \frac{2}{\pi} \int_{-1}^{+1} \frac{\sinh n\sqrt{1-t^2} \cos nt}{1-2ts+s^2}\, dt, \tag{22}$$

which gives for $s = 0$

$$\int_{-1}^{+1} \sinh (n\sqrt{1-t^2}) \cos nt\, dt = \frac{n\pi}{2}.$$

Next let us consider the equation associated with equation (18), viz.,

$$f(s) = \int_{-1}^{+1} \frac{\phi(t)\, dt}{1-2st+t^2}. \tag{23}$$

To obtain a solution of this, we determine a function $h(r, s)$ such that

$$\frac{1}{1-2tr+r^2} = \int_{-1}^{+1} \frac{h(r, s)\, ds}{1-2st+t^2} \quad (|r| < 1);$$

then we shall have

$$\int_{-1}^{+1} \frac{\phi(t)\, dt}{1-2tr+r^2} = \int_{-1}^{+1} \int_{-1}^{+1} \frac{h(r, s)\, \phi(t)\, ds\, dt}{1-2st+t^2}$$

$$= \int_{-1}^{+1} h(r, s)\, f(s)\, ds = \psi(r) \quad \text{(say)},$$

and so the function $\phi(t)$ will be determined by solving the equation †

$$\psi(r) = \int_{-1}^{+1} \frac{\phi(t)\, dt}{1-2tr+r^2}.$$

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* The sign to be chosen is obtained by considering a particular function. This formula is similar to the one given by Stieltjes for the integral $F(z) = \int_0^\infty \frac{f(u)\, du}{z+u}$. See Borel's *Leçons sur les Séries divergentes*, p. 69.

† The method adopted here is capable of more general application.

Applying the inversion formula (17), we find

$$h(r, s) = \frac{1}{i\pi}\left[\frac{s+i\sqrt{1-s^2}}{1+r^2-2r(s+i\sqrt{1-s^2})} - \frac{s-i\sqrt{1-s^2}}{1+r^2-2r(s-i\sqrt{1-s^2})}\right]$$

$$= \frac{2}{\pi}\,\frac{\sqrt{1-s^2}\,(1+r^2)}{(1-2rs+r^2)^2+4r^2(1-s^2)}.$$

Hence the inversion formula for the equation

$$f(s) = \int_{-1}^{+1}\frac{\phi(t)\,dt}{1-2st+t^2} \quad (|s|<1)$$

is

$$\phi(t) = \frac{1}{i\pi}[F(t+i\sqrt{1-t^2})-F(t-i\sqrt{1-t^2})]$$

where

$$F(r) = \frac{2}{\pi}\,r(1+r^2)\int_{-1}^{+1}\frac{f(s)\sqrt{1-s^2}\,ds}{1+6r^2+r^4-4rs(1+r^2)}.$$

If we substitute

$$f(s) = \frac{1}{1-2sz+z^2} \tag{24}$$

in this, we find

$$\phi(t) = 0 \quad (t \neq z);$$

accordingly, this integral equation is also of the first type.

When $\nu = \frac{1}{2}$, we may write equations (11) and (12) in the form

$$\chi(s) = f(s)+2sf'(s) = \int_{-1}^{+1}\frac{(1-s^2)\,\phi(t)\,dt}{(1-2ts+s^2)^{\frac{3}{2}}}, \tag{25}$$

$$\phi(t) = \frac{1}{2\pi}\int_0^{\pi}\chi(t+i\sqrt{1-t^2}\cos a)\,da, \tag{26}$$

which shows at once that they arise from a potential problem.

For we know that

$$V = \frac{1}{\pi}\int_0^{\pi}\chi(z+i\sqrt{x^2+y^2}\cos a)\,da \tag{27}$$

represents a potential function symmetrical about the axis, and, since (26) gives us the values of this function at points on the sphere $x^2+y^2+z^2=1$, the values at all other points, and in particular those on the axis, may be determined by Green's formula

$$V = \frac{1}{4\pi}\iint V\frac{\partial G}{\partial n}\,ds,$$

which is equivalent to (25), but formula (27) gives $V = \chi(z)$ on the axis.

6. *The Determination of the Inversion Formula by means of a Linear Differential Equation.*

We shall now consider the case in which the function $\kappa(s, t)$ in an integral equation of the first type satisfies a linear differential equation of the form

$$P_t(u)+\psi(s)\,Q_t(u) = 0, \tag{28}$$

where $P_t(u)$ and $Q_t(u)$ are used to denote the expressions

$$p_0(t)\frac{d^n u}{dt^n}+p_1(t)\frac{d^{n-1}u}{dt^{n-1}}+\dots$$

and

$$q_0(t)\frac{d^n u}{dt^n}+p_1(t)\frac{d^{n-1}u}{dt^{n-1}}+\dots$$

respectively.

The adjoint linear differential equation will clearly be of the same form, and may be written

$$\bar{P}_t(u)+\psi(s)\,\bar{Q}_t(u) = 0. \tag{29}$$

The quantity $\kappa(s, t)$, being a solution of the original equation, will be an integrating factor of the above expression.

Now let $u(r, t)$ be a solution of the equation

$$\bar{P}_t(u)+\psi(r)\,\bar{Q}_t(u) = 0\,;$$

then we have

$$[\psi(s)-\psi(r)]\,\kappa(s, t)\,\bar{Q}_t[u(r, t)] = \kappa(s, t)\,[\bar{P}_t(u)+\psi(s)\,\bar{Q}_t(u)] = \frac{dW}{dt}\,;$$

therefore*

$$\int_{t_0}^{t_1}\kappa(s, t)\,\bar{Q}_t[u(r, t)]\,\psi'(r)\,dt = \frac{\psi'(r)}{\psi(s)-\psi(r)}\,[W]_{t_0}^{t_1}. \tag{30}$$

Now, let the limits be chosen so that W takes the same value (usually zero) at both; then we shall have

$$\psi'(r)\int_{t_0}^{t_1}\kappa(s, t)\,\bar{Q}_t\,[u(r, t)]\,dt = 0 \quad (r \neq s). \tag{31}$$

If the limits cannot be thus chosen, we try to determine them so that the quantity W oscillates very rapidly in their vicinity; we can then write

$$[W]_{t_0}^{t_1} = 0\,?$$

and

$$\psi'(r)\int_{t_0}^{t_1}\kappa(s, t)\,\bar{Q}_t\,[u(r, t)]\,dt = 0\,? \quad (r \neq s). \tag{31$'$}$$

* The factor $\psi'(r)$ is inserted purposely, as it appears later in the inversion formula.

In either case we have a function of the type required for formula (9), provided that we can show that the function obtained by integrating with regard to r has a discontinuity at the point $r = s$. The above equation will then give us the inversion formula for one or both of the integral equations

$$\left.\begin{aligned} f(s) &= \int_{t_0}^{t_1} \kappa(s, t)\, \phi(t)\, dt \\ \phi(t) &= A \int_m^n \bar{Q}_t[u(r, t)]\, \psi'(r) f(r)\, dr \end{aligned}\right\}. \tag{32}$$

To illustrate the method we shall take the following examples :—

(1) If $\kappa(s, t) = J_m(st)$, the differential equation is

$$\frac{d}{dt}\left(t\frac{dv}{dt}\right) - \frac{m^2}{t}v + s^2 tv = 0,$$

which is self-adjoint. Hence, if $u = J_m(rt)$, we have

$$\frac{d}{dt}\left[t\left(v\frac{du}{dt} - u\frac{dv}{dt}\right)\right] = (s^2 - r^2)\, tuv.$$

Now the quantity within the square brackets is zero when $t = 0$ and oscillates very rapidly near $t = \infty$; for we have approximately

$$J_m(z) = \sqrt{\frac{2}{\pi z}} \cos\left\{z - (m + \tfrac{1}{2})\frac{\pi}{2}\right\};$$

hence we may write

$$h(s, r) = r\int_0^\infty J_m(st) J_m(rt)\, t\, dt = 0\,? \quad (r \neq s),$$

and we are led to consider the possibility of an equation of the form

$$f(s) = \int_0^\infty \int_0^\infty J_m(st) J_m(rt)\, tr f(r)\, dr\, dt. \tag{33}$$

The actual formula is well known and was given by Hankel* in 1875. We shall not stop to verify that $h(s, r)$ is the derivative of a discontinuous function.

(2) If $V = (1 - 2st + t^2)^{\nu-1} = \kappa(s, t)$, we have the differential equation

$$(1 - 2st + t^2)\frac{dV}{dt} - 2(\nu - 1)(t - s)V = 0,$$

which is adjoint to

$$(1 + t^2)\frac{du}{dt} + 2\nu tu - 2s\left[t\frac{du}{dt} + \nu u\right] = 0.$$

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* *Math. Ann.*, Bd. VIII., p. 482.

Here
$$u(r, t) = (1-2rt+t^2)^{-\nu},$$
$$Q_t[u(r, t)] = t\frac{du}{dt}+\nu u = \frac{(1-t^2)\nu}{(1-2rt+t^2)^{\nu+1}},$$
$$2(r-s)\,Q_t[u(r, t)]\,\kappa(s, t) = \frac{d}{dt}[(1-2st+t^2)^{\nu}u(r, t)],$$

and the quantity under the square brackets vanishes if
$$t = s \pm i\sqrt{1-s^2}.$$
Hence we have
$$\int_{s-i\sqrt{(1-s^2)}}^{s+i\sqrt{(1-s^2)}} (1-2st+t^2)^{\nu-1}\frac{(1-t^2)}{(1-2rt+t^2)^{\nu+1}}\,dt = 0 \quad (r \neq s), \tag{34}$$

which leads us to the equation considered in § 4.

(3) Consider the equation
$$f(s) = \int_0^\infty J_0(st)\,K_0(st)\,t\,\phi(t)\,dt, \tag{35}$$

where $J_0(z)$ and $K_0(z)$, the two solutions of Bessel's equation of order zero, are so defined that their approximate values for large positive values of z are
$$\sqrt{\frac{2}{\pi z}}\cos\left(\frac{\pi}{4}-z\right) \quad\text{and}\quad \sqrt{\frac{2}{\pi z}}\sin\left(\frac{\pi}{4}-z\right)$$
respectively.

The function $\kappa(s, t) = tJ_0(st)\,K_0(s, t)$ satisfies the self-adjoint differential equation
$$\frac{d^3v}{dt^3}+\left(4s^2+\frac{1}{t^2}\right)\frac{dv}{dt}-\frac{v}{t^3} = 0,$$

of which another solution is $tJ_0^2(st)$.

If, then, $u(r, t) = tJ_0^2(rt)$, we have
$$4(s^2-r^2)\,v\frac{du}{dt} = \frac{d}{dt}\left[v\frac{d^2u}{dt^2}-\frac{dv}{dt}\frac{du}{dt}+u\left\{\frac{d^2v}{dt^2}+\left(4s^2+\frac{1}{t^2}\right)v\right\}\right],$$

v being written for $\kappa(s, t)$.

The quantity inside the square brackets is zero when $t = 0$, and oscillates very rapidly when $t = \infty$; hence, since in this case
$$Q_t(u) = \frac{du}{dt},$$

we have
$$r\int_0^\infty J_0(st)\,K_0(st)\,\frac{d}{dt}\left\{tJ_0^2(rt)\right\}t\,dt = 0\,? \quad (r \neq s),$$

and we are led to consider an inversion formula of the form

$$\phi(t) = A\int_0^\infty \frac{d}{dt}\{tJ_0^2(rt)\}\, rf(r)\,dr.$$

We may verify by means of the equations

$$J_0(2x) = \int_0^{\frac{1}{2}\pi} \frac{d}{dx}\{xJ_0^2(x\sin\theta)\}\sin\theta\,d\theta,$$

$$\frac{\pi}{2}J_0(zt)\,K_0(zt) = \int_0^\infty J_0(2zt\cosh u)\,du,$$

and Hankel's inversion formula given in (1) that this inversion formula is correct if $A = 2\pi$: but the formula was discovered originally from the differential equation.

The formula that we have just obtained, viz.,

$$\left.\begin{aligned} f(s) &= \int_0^\infty J_0(st)\,K_0(st)\,t\phi(t)\,dt \\ \phi(t) &= 2\pi\int_0^\infty \frac{d}{dt}\{tJ_0^2(rt)\}\,rf(r)\,dr \end{aligned}\right\}, \tag{36}$$

is analogous to Hankel's formula

$$\left.\begin{aligned} \psi(s) &= \int_0^\infty J_m(st)\,t\,\phi(t)\,dt \\ \phi(t) &= \int_0^\infty J_m(rt)\,r\psi(r)\,dr \end{aligned}\right\}, \tag{37}$$

in one respect: either of the functions $f(s)$ or $\phi(t)$ may be taken to be zero for values of s or t greater than a given quantity a. The solution of the integral equation with finite limits is then given by a definite integral with an infinite limit.

The method which we have just explained does not apply to all functions, because, in general, it is not possible to construct a linear differential equation of the required form which is satisfied by $\kappa(s, t)$ for all values of t. It can, however, be extended a little by the introduction of mixed linear equations in which definite integrals and finite differences can also occur.

For this purpose we require the equation adjoint to a mixed linear equation, and this may be obtained by writing down the adjoint expressions of its various constituents, the expression adjoint to

$$I(\psi) \equiv \int_a^b \kappa(s, t)\,\psi(t)\,dt - \kappa\lambda(s)\,\psi(s)$$

being understood to be

$$J(\chi) \equiv \int_a^b \kappa(s, t)\, \chi(s)\, ds - \kappa\lambda(t)\, \chi(t);$$

for, if $\chi(s)$ is a function satisfying the equation $J(\chi) = 0$, we have

$$\int_a^b \chi(s)\, I(\psi)\, ds \equiv \int_a^b \chi(s)\, ds \int_a^b \kappa(s, t)\, \psi(t)\, dt - \int_a^b \lambda(s)\, \psi(s)\, \chi(s)\, ds \equiv 0.$$

The theory of mixed linear equations is rather difficult; so we shall not pursue these enquiries any further in the present paper; we may mention, however, that one of the chief difficulties we are faced with is that of determining whether our equations can be satisfied for a continuum of values of the arbitrary parameter or only for an enumerable set of values.

7. *Equations of Type 3a.*

The fundamental formula upon which the inversion formulæ of integral equations of this type depend is the following:—

$$f(r) = \frac{1}{\pi} \int_{-}^{\infty} \frac{\sin(r-t)}{r-t} f(t)\, dt. \tag{38}$$

It is at once evident that this formula is not satisfied by a perfectly arbitrary function: we must therefore find a convenient description of a class of functions to which the formula is applicable.

Now the function $\dfrac{\sin(r-t)}{r-t}$ can be written in the form

$$\tfrac{1}{2} \int_{-1}^{+1} e^{ir\mu - it}\, d\mu.$$

Accordingly, if the order of integration can be changed, we shall have

$$f(r) = \int_{-1}^{+1} e^{ir\mu}\, \psi(\mu)\, d\mu \tag{A}$$

where

$$\psi(\mu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-it\mu} f(t)\, dt. \tag{B}$$

We shall therefore assume that $f(r)$ is a function defined by means of a definite integral of the form (A), and is such that the integral (B) is uniformly convergent for the range $\mu = (-1, 1)$; so that the order of integration in our double integral can be changed.

Now, in order to use this formula to solve integral equations, we must express the function $\dfrac{\sin(r-t)}{r-t}$ as a definite integral of the required type. The method to be adopted is similar to that used in § 6, and is best illustrated by means of an example.

The function $J_0(x-t)$ satisfies the differential equation

$$(x-t)\frac{d^2u}{dx^2}+\frac{du}{dx}+(x-t)u=0;$$

also $v=J_0(x-s)$ is an integrating factor of the equation with s written instead of t; therefore we have

$$\frac{d}{dx}\left[(x-s)\left(v\frac{du}{dx}-u\frac{dv}{dx}\right)\right]=(t-s)v\left[\frac{d^2u}{dx^2}+u\right].$$

Now for large real values of x we have approximately

$$J_0(x-t)=\frac{2\cos\left(\dfrac{\pi}{4}-x+t\right)}{\sqrt{2\pi(x-t)}}$$

and
$$\frac{d}{dx}J_0(x-t)=\frac{2\sin\left(\dfrac{\pi}{4}-x+t\right)}{\sqrt{2\pi(x-t)}};$$

hence for $x=\infty$ the quantity under the square brackets becomes equal to

$$\frac{2}{\pi}\sin(t-s),$$

and for $x=s$ it is zero.

Further, we have

$$u+\frac{d^2u}{dx^2}=-\frac{1}{x-t}\frac{d}{dx}J_0(x-t)=\frac{J_1(x-t)}{x-t};$$

therefore the equation gives

$$\int_s^\infty J_0(x-s)\frac{J_1(x-t)}{x-t}\,dx=\frac{2}{\pi}\frac{\sin(s-t)}{s-t}. \tag{39}$$

Combining this with the fundamental formula, we obtain

$$\int_{-\infty}^\infty\int_s^\infty J_0(x-s)\frac{J_1(x-t)}{x-t}f(t)\,dx\,dt=2f(s), \tag{40}$$

an equation which can, in general, be written in the form

$$\left.\begin{aligned}\phi(x)&=\int_{-\infty}^\infty\frac{J_1(x-t)}{x-t}f(t)\,dt,\\ f(s)&=\tfrac{1}{2}\int_s^\infty J_0(s-x)\,\phi(x)\,dx\end{aligned}\right\}. \tag{41}$$

This relation can be verified directly if we assume for $\phi(x)$ an expression of the form

$$\phi(x) = \int_{-1}^{+1} e^{ix\mu}\,\psi(\mu)\,d\mu,$$

in which $\psi(\mu)$ is finite and continuous along the path of integration.

The method which we have just explained is only applicable to functions $\kappa(s, t)$ which oscillate near $s = \infty$ and which admit an asymptotic representation involving a circular function, so that the term $\sin(t-r)$ can appear. A similar method can be used for equations of type 2; for it is easy to see that, if $[W]$ is a constant instead of zero, we shall obtain a definite integral of the required form equal to $\dfrac{1}{t-r}$.

8. *The Problem of solving a Linear Differential Equation by means of a Definite Integral of a given Type.*

We shall now consider the case in which the function $f(s)$ is not explicitly given, but is to be derived from a given linear differential or integral equation

$$L_s(f) = 0. \tag{42}$$

We shall suppose that $f(s)$ satisfies a set of linear conditions which are sufficient to distinguish it from other solutions of equation (42), and that these conditions are included in or are the same as those to be satisfied by $f(s)$ in order that it may be represented in the form

$$f(s) = \int_a^b \kappa(s, t)\,\phi(t)\,dt. \tag{43}$$

The success of the method to be adopted depends upon the possibility of finding a relation of the form

$$L_s\{\kappa(s, t)\} = M_t\{h(s, t)\} \tag{44}$$

where M_t is an operator of the form

$$p_0(t)\frac{d^n}{dt^n} + p_1(t)\frac{d^{n-1}}{dt^{n-1}} + \dots.$$

For, if $\phi(t)$ is an integrating factor of the expression $M_t(v)$, we shall have

$$L_s(f) = \int_a^b L_s\{\kappa(s, t)\}\,\phi(t)\,dt = \int_a^b M_t\{h(s, t)\}\,\phi(t)\,dt = \int_a^b dR, \tag{45}$$

provided the interchange of operations in the first line is permissible.

There will, in general, be more than one integrating factor of $M_t(v)$; we choose one so that R takes the same value (usually zero) at a and b or at two points α and β within the range (a, b). The function ϕ is thus

a solution of the linear differential equation adjoint to $M_t(v) = 0$, and the above requirement may suffice to distinguish it from other solutions of this equation. If the points α and β (which must be independent of s) are different from a and b, we reduce the range of integration to (α, β) by defining the function $\phi(t)$ to be zero outside this range and equal to the given integrating factor inside.

We have seen that the solution of the integral equation is often given by a formula of the same type

$$\phi(t) = \int_c^d \bar{\kappa}(s, t) f(s) ds.$$

Now, according to the previous work, $\phi(t)$ should satisfy the equation adjoint to $M_t(v) = 0$, which we may write

$$\overline{M}_t(v) = 0.$$

Consequently, if the same kind of analysis applies for this integral equation as for the previous one, we shall expect to have an identical relation of the form

$$\overline{M}_t\{\bar{\kappa}(t, s)\} \equiv \overline{L}_s\{\bar{h}(t, s)\}. \tag{46}$$

We shall now simplify matters by assuming that the functions $\kappa(s, t)$ and $h(s, t)$ are the same and that the corresponding functions $\bar{\kappa}$ and $\bar{h}$ are also the same; in this way we may lose a certain amount of generality, but the analysis becomes more manageable.

The functions $\kappa(s, t)$ and $\bar{\kappa}(s, t)$ then satisfy the partial differential equations

$$\left.\begin{aligned} L_s(u) &= M_t(u) \\ \overline{L}_s(v) &= \overline{M}_t(v) \end{aligned}\right\} \tag{47}$$

and

respectively.

Now this is an important fact, because, if we know the solutions of the integral equation

$$f(s) = \int_a^b \kappa(s, t)\, \phi(t)\, dt$$

corresponding to a few particular forms of $f(s)$, *we may be able to determine a number of partial differential equations of the form*

$$L_s(u) = M_t(u)$$

which are satisfied by $\kappa(s, t)$. *If, then, the integral equation is amenable to this treatment, the corresponding partial differential equations*

$$\overline{L}_s(v) = \overline{M}_t(v)$$

will all be satisfied by the function $\bar{\kappa}(s, t)$; *and so the problem is reduced to that of finding the common solution of a number of partial differential equations.*

Thus, for example,* if the quantities a_{nm} are quite arbitrary and

$$L_s(u) \equiv \sum_m \sum_n a_{nm} s^n \frac{d^m u}{ds^m},$$

$$M_t(u) \equiv \sum_m \sum_n a_{nm} t^m \frac{d^n u}{dt^n},$$

the partial differential equation

$$L_s(u) = M_t(u)$$

is always satisfied by $u = \kappa(s, t) = e^{st}$.

The adjoint expressions are

$$\bar{L}_s(v) = \Sigma\Sigma(-1)^m a_{nm} \frac{d^m}{ds^m}(s^n v),$$

$$\bar{M}_t(v) = \Sigma\Sigma(-1)^n a_{nm} \frac{d^n}{dt^n}(t^m v),$$

and, since $(-1)^m \frac{d^m}{ds^m}(s^n e^{-st}) \equiv (-1)^n \frac{d^n}{dt^n}(t^m e^{-st})$,

the corresponding partial differential equation

$$\bar{L}_s(v) = \bar{M}_t(v)$$

is always satisfied by $v = \bar{\kappa}(s, t) = e^{-st}$.

This example corresponds to Pincherle's well known formula

$$f(s) = \frac{1}{2\pi i}\int_c e^{st}\phi(t)\,dt,$$

$$\phi(t) = \int_0^\infty e^{-st} f(s)\,ds.$$

Another fact which is worth noticing is the reciprocal nature of the pair of equations

$$L_s(u) = M_t(u),$$

$$\bar{L}_s(v) = \bar{M}_t(v).$$

If $\psi(t)$ is a solution of the equation $M_t(w) = 0$, we shall expect a function $\chi(s)$ for which

$$\psi(t) = \int_c^d \kappa(s, t)\,\chi(s)\,ds,$$

* This example is deduced from some work of Petzval's (*Integration der linearen Differential-gleichungen*, pp. 472–473).

to be given by $\bar{L}_s(\chi) = 0$, and the second equation suggests that it should be given by a definite integral of the type

$$\chi(s) = \int_a^b \bar{\kappa}(s, t)\, \psi(t)\, dt.$$

This corresponds exactly to the result which is predicted in § 1.

9. *The Partial Differential Equation* $L_s(u) = M_t(u)$.

We have seen that a system of partial differential equations of the form

$$L_s(u) = M_t(u) \tag{48}$$

may be connected with the two integral equations

$$\left.\begin{aligned} f(s) &= \int_a^b \kappa(s, t)\, \phi(t)\, dt \\ \psi(t) &= \int_c^d \kappa(s, t)\, \chi(s)\, ds \end{aligned}\right\}. \tag{49}$$

Now we assumed in § 1 that the function $\kappa(s, t)$ satisfied a number of linear conditions in s independently of t: this assumption was made to make the work perfectly general. Accordingly, when we consider both equations, we must admit functions $\kappa(s, t)$ which satisfy a number of linear conditions in t independently of s. The system of partial differential equations will then possess a solution which satisfies a number of linear conditions in both s and t.

Let

$$L_s[\kappa(s, t)] \equiv F(s, t) = M_t[\kappa(s, t)];$$

then $\kappa(s, t)$ is the solution of the ordinary linear differential equations

$$L_s(u) = F(s, t),$$
$$M_t(u) = F(s, t)$$

which satisfies certain linear conditions in both s and t.

Now, if these differential equations possess Green's functions* $G(s, x)$

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* The characteristic property of the Green's function for a linear differential equation $L_s(w) = 0$ and a set of linear conditions is that the solution of

$$L_s(u) + f(s) = 0$$

which satisfies the given linear conditions can be expressed in the form of a definite integral

$$u = \int_c^d G(s, x)\, f(x)\, dx.$$

If the differential equation is of the n-th degree, the function $G(s, x)$, which is called the Green's function, will be a continuous function of s, x satisfying the given linear conditions for all values of x, but its $(n-1)$-th derivative will experience a sudden change of value at the point $x = s$. The linear conditions usually take the form of relations between the value of the function and

and $H(x, t)$, corresponding to the given linear conditions, we shall have

$$u = \kappa(s, t) = -\int_c^d G(s, x)\, F(x, t)\, dx$$

and
$$u = \kappa(s, t) = -\int_a^b H(x, t)\, F(s, x)\, dx;$$

whence
$$\int_c^d G(s, x)\, F(x, t)\, dx = \int_a^b H(x, t)\, F(s, x)\, dx. \tag{50}$$

We have shown elsewhere* that an integral relation of this type implies that the numbers λ for which the equations

$$\left.\begin{aligned} \phi(s) - \lambda \int_c^d G(s, x)\, \phi(x)\, dx &= 0 \\ \text{and} \qquad \chi(t) - \lambda \int_c^b H(x, t)\, \chi(x)\, dx &= 0 \end{aligned}\right\} \tag{51}$$

can possess solutions different from zero are, in general, the same.

In the demonstration it is necessary to assume that the function $F(s, x)$ is such that no functions $a(s)$ and $b(x)$ exist for which

$$\int_c^d a(s)\, F(s, x)\, ds = 0,$$

for all values of x,

and
$$\int_a^b F(s, x)\, b(x)\, dx = 0,$$

for all values of s.

Suppose, then, that λ is a quantity such that the homogeneous equation

$$\phi(s) - \lambda \int_c^d G(s, x)\, \phi(x)\, dx = 0$$

possesses a solution $\phi(s)$ which is not identically zero.

The equation†
$$\theta(x) - \lambda \int_c^d G(s, x)\, \theta(s)\, ds = 0$$

its first $(n-1)$ derivatives at the points a and b, but Hilbert has shown that, when these points are singularities of the differential equation, conditions of remaining finite or becoming infinite in a specified way may be introduced. It is probable that linear conditions expressed by definite integrals can be added to these to complete the generality of the theory.

The one-dimensional Green's function is, in many respects, analogous to the function used by Green in electrostatics. It was discovered by Burkhardt, and its properties have been developed by the following writers :—Burkhardt, *Bull. Soc. Math.*, Bd. XXII. (1894) ; Böcher, *Amer. Bull.* (1901), p. 297 : Dunkel, *Amer. Bull.* (1902), p. 288 ; Mason, Diss. Gött. (1903), *Trans. Amer. Math. Soc.*, Vol. v., No. 2, pp. 220-225 ; Hilbert, *Gött. Nachr.* (1904), Heft 3.

* *Trans. Camb. Phil. Soc.*, Vol. XX., No. 10, p. 234.

† Fredholm, *Acta Math.*, p. 27 (1903).

will also possess a solution $\theta(x)$ different from zero for the same value of λ; accordingly, if

$$\chi(t) = \int_c^d F(x, t)\,\theta(x)\,dx,$$

$\chi(t)$ is not identically zero, and we shall have

$$\chi(t) = \lambda \int_c^d \int_c^d G(s, x)\,F(x, t)\,\theta(s)\,ds\,dx$$

$$= \lambda \int_c^d ds \int_a^b H(x, t)\,F(s, x)\,\theta(s)\,dx$$

$$= \lambda \int_a^b H(x, t)\,\chi(x)\,dx.$$

Conversely, if λ is a quantity for which this equation holds, a function ψ will also exist for which

$$\psi(x) = \lambda \int_a^b H(x, t)\,\psi(t)\,dt$$

and, if

$$\phi(s) = \int_a^b F(s, x)\,\psi(x)\,dx,$$

we have

$$\phi(s) = \lambda \int_a^b \int_a^b F(s, x)\,H(x, t)\,\psi(t)\,dt$$

$$= \lambda \int_a^b \int_c^d G(s, x)\,F(x, t)\,\psi(t)\,dt$$

$$= \lambda \int_c^d G(s, x)\,\phi(x)\,dx.$$

Now, in the present case, a function $\phi(s)$ which satisfies the homogeneous integral equation

$$\phi(s) = \lambda \int_c^d G(s, x)\,\phi(x)\,dx$$

will satisfy the differential equation

$$L_s\phi + \lambda\phi = 0,$$

and will also satisfy the linear conditions associated with the function G. Hence the values of λ for which a solution of the homogeneous integral equation exists are the values of λ for which the above differential equation can possess a solution satisfying the given linear conditions.

Similarly, the values of λ for which a solution of

$$\chi(t) = \lambda \int_a^b H(x, t)\,\chi(x)\,dx$$

exists are the values of λ for which a solution of

$$M_t\chi(t)+\lambda\chi(t) = 0$$

can satisfy the linear conditions associated with the function H.

Hence, since the values of λ for the two integral equations are the same, the values of λ for the two differential equations are also the same.

We conclude from this that the partial differential equation will, in general, only possess a solution satisfying the linear conditions identically both in s and t, when the equations

$$L_s(u)+\lambda u = 0,$$

$$M_t(v)+\lambda v = 0$$

can possess solutions of the required type for the same values of λ.

We can also show that, in general, any function which satisfies the relation

$$\int_c^d G(s, x)\, f(x, t)\, dx = \int_a^b f(s, x)\, H(x, t)\, dx$$

must be a solution of the partial differential equation

$$L_s(w) = M_t(w).$$

Let $$g(s, t) = \int_c^d G(s, x)\, f(x, t)\, dx = \int_a^b f(s, x)\, H(x, t)\, dx;$$

then $g(s, t)$ will, in general, satisfy the linear conditions associated with both G and H; and so we shall have

$$f(s, t) = -L_s(g),$$

$$f(s, t) = -M_t(g),$$

whence $$L_s(f) = -L_s M_t(g) = M_t(f).$$

The partial integral equation

$$\int_c^d G(s, x)\, f(x, t)\, dx = \int_a^b f(s, x)\, H(x, t)\, dx \tag{52}$$

is thus satisfied by a certain group of solutions of the partial differential equation, but we cannot say that it is satisfied by every solution of the partial differential equation. If the equation be written in the symbolical form

$$G_s\{f(s, t)\} = H_t\{f(s, t)\}$$

where G and H are linear distributive operators, it is possible to regard the operation $G_s - H_t$ as a factor of the operation $L_s - M_t$.

A particular function which satisfies the partial integral equation is

$f(s, t) = \kappa(s, t)$, for $\kappa(s, t)$ satisfies the given conditions in t; and so the function

$$g(s, t) = \int_c^d G(s, x)\, \kappa(x, t)\, dx$$

satisfies them also. $g(s, t)$ is therefore a solution of the equation

$$L_s(u) + \kappa(s, t) = 0$$

which satisfies the given linear conditions in both s and t.

Operating on this equation with M_t, we have

$$M_t L_s(u) = -M_t\, \kappa(s, t) = -L_s\, \kappa(s, t);$$

therefore $$L_s[M_t(u) + \kappa(s, t)] = 0.$$

Now $M_t(u) + \kappa(s, t)$ satisfies the given linear conditions in s, since both u and $\kappa(s, t)$ do so, and we see from the above that it also satisfies the equation $L_s(v) = 0$; accordingly, it must be identically zero; for we know from the Green's formula that the solution of the equation

$$L_s(v) + f(s) = 0$$

is given by $$v = \int_c^d G(s, x)\, f(x)\, dx,$$

and, if $f(x)$ is zero, v is also zero.

Putting, then, $$M_t(u) + \kappa(s, t) = 0,$$

we have, since u is a function which satisfies the given linear conditions in t,

$$g(s, t) = u = \int_a^b H(x, t)\, \kappa(s, x)\, dx,$$

which gives the required relation

$$\int_c^d G(s, x)\, \kappa(x, t)\, dx = \int_a^b H(x, t)\, \kappa(s, x)\, dx.$$

Now this relation is of some interest in connection with the original integral equations

$$f(s) = \int_a^b \kappa(s, t)\, \phi(t)\, dt,$$

$$\psi(t) = \int_c^d \kappa(s, t)\, \chi(s)\, ds;$$

for we can show that, if f and ϕ are two functions connected by the first relation,

$$f_1(s) = \int_c^d G(s, x)\, f(x)\, dx$$

and $$\phi_1(s) = \int_a^b H(s, t)\, \phi(t)\, dt$$

is another pair.

Substituting the given value of $f(x)$, we have

$$f_1(s) = \int_c^d \int_a^b G(s, x)\, \kappa(x, t)\, \phi(t)\, dx\, dt$$
$$= \int_a^b \int_a^b \kappa(s, x)\, H(x, t)\, \phi(t)\, dx\, dt$$
$$= \int_a^b \kappa(s, x)\, \phi_1(x)\, dx.$$

Similarly, if ψ and χ are one pair of functions connected by the second relation, the functions

$$\psi_1(t) = \int_a^b H(x, t)\, \psi(x)\, dx$$

and

$$\chi_1(t) = \int_c^d G(x, t)\, \chi(x)\, dx$$

are another pair; for, on substitution, we get

$$\psi_1(t) = \int_a^b \int_c^d H(x, t)\, \kappa(s, x)\, \chi(s)\, ds\, dx$$
$$= \int_c^d \int_c^d G(s, x)\, \kappa(x, t)\, \chi(s)\, ds\, dx$$
$$= \int_c^d \kappa(x, t)\, \chi_1(x)\, dx.$$

We shall complete this series of propositions concerning the partial integral equation (52) by remarking that the equation corresponding to the adjoint partial differential equation

$$\overline{L}_s(u) = \overline{M}_t(u)$$

is no other than

$$\int_c^d f(s, x)\, G(x, t)\, dx = \int_a^b H(s, x)\, f(x, t)\, dx.$$

This result follows at once from the fact that, when we interchange the arguments in a Green's function of a linear differential equation, we obtain the corresponding Green's function for the adjoint equation.

10. *Investigations on the Existence Theorem.*

In a former paper* we attempted to define a class of functions $f(s)$ which could be represented by definite integrals of the form

$$\int_a^b \kappa(s, t)\, \phi(t)\, dt$$

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* *Supra*, pp. 103–106.

to any required degree of approximation. The conditions laid down, however, were not sufficient to ensure that the function $\phi(t)$ would always tend to a finite limit when the approximation was used to obtain an exact representation. It is clear that, if the function $\phi(t)$ is restricted to remain finite within the range of integration, the definite integral is only capable of representing a much narrower class of functions. The investigation can, however, be made more satisfactory when this assumption is made and an existence theorem stated more precisely.

The method which we adopted is analogous to that used in solving a linear differential equation by means of a definite integral and depends upon the possibility of constructing a relation of the form

$$\int_a^b \kappa(s, t)\, F(t, x)\, dt = \frac{d}{dx} H(s, x). \tag{53}$$

A relation of this type may be constructed in many ways; the one, however, which adapts itself best to our requirements is obtained as follows :—

Let (c, d) be the range of values of s for which the representation is required, and $h(s, t)$ a convenient function which is finite and integrable for values of s and t within the ranges (c, d) and (a, b) respectively. Further, let

$$\left.\begin{aligned} g_0(t) &= \int_c^d h(s, t)\, f(s)\, ds \\ f_n(s) &= \int_a^b \kappa(s, t)\, g_{n-1}(t)\, dt \\ g_n(t) &= \int_c^d h(s, t)\, f_n(s)\, ds \\ F(t, x) &= x g_1(t) - \frac{x^3}{1\,!}\, g_3(t) + \frac{x^5}{2\,!}\, g_5(t) - \dots \\ 2H(s, x) &= -f(s) + \frac{x^2}{1\,!}\, f_2(s) - \frac{x^4}{2\,!}\, f_4(s) + \dots \end{aligned}\right\}; \tag{54}$$

then it is easily seen that we have the relation

$$\int_a^b \kappa(s, t)\, F(t, x)\, dt = \frac{d}{dx} H(s, x).$$

If now we write $$\phi(t) = 2\int_0^\infty F(t, x)\, dx,$$

we shall have $$\int_a^b \kappa(s, t)\, \phi(t)\, dt = 2\int_0^\infty \frac{d}{dx} H(s, x)\, dx = f(s),$$

provided

(1) The integral $\int_0^\infty F(t, x)\, dx$ has a meaning;

(2) We can change the order of integration in the double integral

$$\int_a^b \int_0^\infty F(t, x)\, \kappa(s, t)\, dt\, dx\,;$$

(3) $$H(s, \infty) = 0.$$

The function $h(s, t)$ is at our disposal. In the previous account of the method we took it to be the same as $\kappa(s, t)$, but it is clearly advantageous to leave it undefined, as this adds to the elasticity of the method.

We shall assume that all the functions we are dealing with are finite and integrable for the given range of values of s and t. The series which represent the functions $F(t, x)$ and $H(s, x)$ will then be absolutely and uniformly convergent for all finite values of x; for, if h, κ, and f are the maximum values of the moduli of $h(s, t)$, $\kappa(s, t)$, and f respectively, it is easy to see that

$$|g_0(s)| \leqslant |d-c|\, hf, \qquad |f_n(s)| \leqslant |b-a|^n\, |d-c|^{n-1} \kappa^n h^{n-1} f,$$
$$|g_n(s)| \leqslant |b-a|^n\, |d-c|^n \kappa^n h^n f;$$

so that the series can be compared with exponential series.

Now write

$$\left.\begin{aligned} P(s, t) &= \int_a^b \kappa(s, r)\, h(t, r)\, dr \\ Q(s, t) &= \int_c^d h(r, s)\, \kappa(r, t)\, dr \end{aligned}\right\}, \tag{55}$$

and let $\psi_m(s)$, $\chi_m(t)$ be the series of functions for which the homogeneous integral equations

$$\left.\begin{aligned} \psi_m(s) &= \lambda_m \int_c^d P(s, t)\, \psi_m(t)\, dt \\ \chi_m(s) &= \lambda_m \int_a^b Q(s, t)\, \chi_m(t)\, dt \end{aligned}\right\} \tag{56}$$

can be satisfied. It should be noticed that the values of λ_m for which these equations possess solutions different from zero are the same; for, if we calculate the determinantal equations of which the quantities λ_m are the roots, we shall find that they are identical.

[*Note added December 26th.*—In what follows it will be supposed that these values of λ_m are all real. This is certainly true if $h(s, t)$ and $\kappa(s, t)$ are the same; for then $P(s, t)$ and $Q(s, t)$ are symmetrical functions. The choice of the function $h(s, t)$ is thus not entirely arbitrary.]

It is easy to see that, if

$$\theta_m(s) = \int_a^b \kappa(s, t)\, \chi_m(t)\, dt,$$

then

$$\theta_m(s) = \lambda_m \int_c^d P(s, t)\, \theta_m(t)\, dt;$$

so that $\psi_m(s)$ may be taken to be equal to $\theta_m(s)$ and can be defined by the above equation. We then have the further relation

$$\chi_m(t) = \lambda_m \int_c^d h(s, t)\, \psi_m(s)\, ds.$$

The existence theorem which we shall now prove is that, *if the function $f(s)$ can be expanded in a convergent series of the form $\Sigma a_m \psi_m(s)$ which is such that the derived series $\Sigma |\lambda_m a_m \psi_m|$ is convergent, ψ_m being the maximum value of $|\psi_m(s)|$ within the range (c, d), then a function $\phi(t)$ exists for which*

$$f(s) = \int^b \kappa(s, t)\, \phi(t)\, dt,$$

and this function may be determined by the formula

$$\phi(t) = 2\int_0^\infty F(t, x)\, dx.$$

If we write

$$\omega(s) = \Sigma \lambda_m a_m \psi_m(s), \tag{57}$$

we have

$$\phi(t) = \int_c^d h(s, t)\, \omega(s)\, ds = \Sigma a_m \chi_m(t),$$

and this series is absolutely and uniformly convergent for

$$|a_m \chi_m(t)| \leqslant |\lambda_m a_m (d-c)\, h \psi_m|$$

where h and ψ_m are the maximum values of the moduli of $h(s, t)$ and $\psi_m(s)$ within the given ranges, and the series $|\lambda_m a_m \psi_m|$ is convergent by hypothesis.

This series, for $\phi(t)$, may be integrated term by term, and we obtain

$$\int_a^b \kappa(s, t)\, \phi(t) = \Sigma a_m \psi_m(s) = f(s)$$

as required.

We have now to prove that this series for $\phi(t)$ may be obtained from the formula

$$\phi(t) = 2\int_0^\infty F(t, x)\, dx. \tag{58}$$

Calculating the functions $g_n(t)$ and $f_{n+1}(s)$ in turn, we have *

$$\left.\begin{aligned} g_0(t) &= \Sigma \frac{a_m}{\lambda_m} \chi_m(t) \\ f_n(s) &= \Sigma \frac{a_m}{\lambda_m^n} \psi_m(s) \\ g_n(t) &= \Sigma \frac{a_m}{\lambda_m^{n+1}} \chi_m(t) \end{aligned}\right\}. \tag{59}$$

The series for $F(t, x)$ may now be transformed into

$$F(t, x) = \sum_0^\infty x \frac{a_m}{\lambda_m^2} e^{-x^2/\lambda_m^2} \chi_m(t),$$

and it is clear that the integral $2\int_0^\infty F(t, x)\,dx$ will give the series for $\phi(t)$, provided the integration term by term is legitimate.

Now a sufficient set of conditions for the integration term by term of a series

$$s(x) = \sum_1^\infty u_n(x)$$

is the following :—

(1) The series $\sum_1^\infty u_n(x)$ should be uniformly convergent in an arbitrary interval ;

(2) $\int u_n(x)\,dx$ should exist for all values of n ;

(3) $\sum_{n=1}^\infty \int_a^\infty u_n\,dx$ should converge for all values of a between 0 and ∞ ;

(4) A number p independent of r should exist for which

$$\left| \sum_{n=1}^r \int_\kappa^\infty u_n\,dx \right| < \epsilon \text{ for all } \kappa\text{'s} > p.$$

The first and third conditions are clearly satisfied, since the series $\Sigma a_m \chi_m(t)$ is absolutely and uniformly convergent; the fourth condition will be satisfied if p can be chosen so that

$$\left| \sum_1^r a_m e^{-\kappa^2/\lambda_m^2} \chi_m(t) \right| < \epsilon \text{ for } \kappa > p.$$

Now this can clearly be done; for, if m_1 is a number such that

$$\sum_{m_1}^r |a_m \chi_m(t)| < \frac{\epsilon}{2} \text{ for all } r\text{'s} > m_1,$$

* These series will all be absolutely and uniformly convergent, since the quantities λ_m increase indefinitely in magnitude, being the zeroes of a whole function.

and p is chosen so large that

$$\left|\sum_1^m a_n e^{-\kappa\lambda_n^2}\psi_n(t)\right| < \frac{\epsilon}{2} \text{ for } \kappa > p,$$

we have

$$\left|\sum_1^r (\ldots)\right| \leqq \left|\sum_1^m (\ldots)\right| + \left|\sum_m^r (\ldots)\right| \leqq \frac{\epsilon}{2} + \sum_m^r \left|a_n\chi_n(t)\right| < \epsilon.$$

The theorem we have just proved does not tell us anything about the uniqueness of the solution of an integral equation, and it does not give a value of $\phi(t)$ different from zero for which

$$0 = \int_a^b \kappa(s, t)\,\phi(t)\,dt$$

when such a value exists. It is by no means certain, however, that the solution which is obtained by using one function $h(s, t)$ is the same as that which would be obtained if we used another. If the two values of $\phi(t)$ thus obtained were different, their difference would be a solution corresponding to $f(s) = 0$.

In general, the function $\phi(t)$ will take a simpler form when the range (c, d) is bounded by two points at which the function $\kappa(s, t)$ is discontinuous than if it is taken arbitrarily. It often happens that the solution in the first case is unique, but not so in the second case, unless we impose additional restrictions upon the function ϕ. Examples of this phenomenon may be obtained by considering the problems in which we require to find the distribution of electricity over a closed surface when the value of the potential function is given (1) over the whole surface, (2) over a portion of the surface.

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